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A General Class of Weighted Banach Function Spaces

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Abstract: In this paper, we introduce a general class of analytic functions which extend the generalized Hardy space. Moreover, investigate the continuity of the point evaluations on this space.

Keywords: Weighted Bergman spaces, Hardy spaces

1 Introduction

Let $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk in the complex plane $\mathbb{C}$, $\partial\Delta$ its boundary and $H(\Delta)$ the space of all analytic function on the unit disk. For an analytic function f on the unit disk and $0 < r < 1$, we define the delay function f_r by $f_r(e^{i\theta}) = f(re^{i\theta})$. It is easy to see that the functions f_r are continuous on $\partial\Delta$ for each r.
The theory of harmonic functions motivates the following classes of analytic functions, determined by their limiting behavior as their arguments approach to the boundary $\partial\Delta$. For $0 < p < \infty$, the Hardy space H^p is defined as the set of analytic functions $f : \Delta \to \mathbb{C}$ such that

$$\|f\|_{H^p}^p = \sup_{0<r<1} \int_0^{2\pi} |f_r(e^{i\theta})|^p \frac{d\theta}{2\pi} < \infty.$$

By the Littlewood Subordination Theorem (see [7]), we see that the supremum in the above definition of H^p is actually a limit, that is,

$$\|f\|_{H^p}^p = \lim_{r\to 1} \int_0^{2\pi} |f_r(e^{i\theta})|^p \frac{d\theta}{2\pi} < \infty.$$

It should be mentioned that the function $\|.\|_{H^p}^p : H^p \to \mathbb{R}^+$ is a norm on H^p, and makes H^p into a Banach space for $1 \le p < \infty$ (see [8]). For more studies on Hardy space, we refer to [8, 11, 13].
Recently Fatehi [10], introduced the following definition

Definition 1. *Let $F : H(\Delta) \to H(\Delta)$ be a linear operator such that $F(f) = 0$ if and only if $f = 0$, that is, F is $1-1$. For $1 \le p < \infty$, the generalized Hardy space $H_{F,p}(\Delta) = H_{F,p}$ is defined to be the collection of all analytic functions f on Δ for which*

$$\sup_{0<r<1} \int_0^{2\pi} |(F(f))_r(e^{i\theta})|^p \frac{d\theta}{2\pi} < \infty.$$

Denote the pth root of this supremum by $\|f\|_{H_{F,p}}$. Since, $|F(f)|^p$ is a subharmonic function, so by [7], we have

$$\|f\|_{H_{F,p}}^p = \lim_{r\to 1^-} \int_0^{2\pi} |F(f)_r(e^{i\theta})|^p \frac{d\theta}{2\pi} < \infty.$$

Therefore, $f \in H_{F,p}$ if and only if $F(f) \in H^p$ and

$$\|F(f)\|_p^p = \|f\|_{H_{F,p}}^p = \lim_{r\to 1^-} \int_0^{2\pi} |F(f)_r(e^{i\theta})|^p \frac{d\theta}{2\pi}.$$

It is easy to see that $H_{F,p}$ is a normed space with the norm $\|.\|_{H_{F,p}}$.
For $0 < p < \infty$, the Bergman space A^p is the set of all $f \in H(\Delta)$ such that

$$\int_\Delta |f(z)|^p dA(z) < \infty,$$

where $dA(z) = dxdy = rdrd\theta$ is the Lebegue area measure. We mention [9] as general reference for the theory of Bergman spaces.

We assume from now on that $K : [0,\infty) \to [0,\infty)$ to appear in this paper is right-continuous and nondecreasing functions such that the integral

$$\int_0^{1/e} K(\log(1/\rho))\rho d\rho = \int_1^\infty K(t)e^{-2t}dt < \infty.$$

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We can define an auxiliary function as follows:

$$\varphi_K(s)=\sup_{0<t\le 1}\frac{K(st)}{K(t)},\ 0<s<\infty,$$

we assume that

$$\int_0^1 \varphi_K(s)\frac{ds}{s}<\infty, \tag{1}$$

and

$$\int_1^\infty \varphi_K(s)\frac{ds}{s}<\infty. \tag{2}$$

From now on we suppose that the above weight function K satisfies the following properties:
(a) K is nondecreasing on $[0,\infty)$,
(b) K is twice differentiable on $(0,1)$,
(c) $\int_0^{\frac{1}{e}} K(\log\frac{1}{r})rdr<\infty$,
(d)$K(t)=K(1)>0, t\ge 1$ *and*
(e)$K(st)\approx K(t),\ t\ge 0$.

We will need the following result in the sequel.

Theorem 1. *([16]) If K satisfies condition (2), then for any $\alpha\ge 1$ and $0\le\beta<1$, we have*

$$\int_0^1 r^{\alpha-1}(\log\frac{1}{r})^{-\beta}K(\log\frac{1}{r})dr$$
$$\approx C(\beta)\left(\frac{1-\beta}{\alpha}\right)^{1-\beta}\Phi\left(\frac{1-\beta}{\alpha}\right), \tag{3}$$

where $C(\beta)$ is a constant depending only on β.

An important tool our study is the auxiliary function Ψ_{ω_1} defined by

$$\Psi_{\omega_1}(s)=\sup_{0<t<1}\frac{\omega_1(st)}{\omega_1(t)},\quad 0<s<1.$$

Lemma 1. *(see [4]) If ω_1 satisfies, the following condition*

$$\int_1^{\frac{1}{t}}\Psi_{\omega_1}(s)\frac{ds}{s^2}<\infty.$$

$$\omega^*(t)=t\int_t^1\frac{\omega_1(s)}{s^2}ds\quad (where,\ 0<t<1),$$

has the following properties :

(A) ω^* is nondecreasing on $(0,1)$.

(B) $\omega^*(t)/t$ is nonincreasing on $(0,1)$.

(C) $\omega^*(t)\ge\omega_1(t)$ for all $t\in(0,1)$.

(D) $\omega^*\lesssim\omega_1$ on $(0,1)$.

If $\omega_1(t)=\omega_1(1)$ for $t\ge 1$, then we also have

(E) $\omega^*(t)=\omega^*(1)=\omega_1(1)$ for $t\ge 1$, so $\omega^*\approx\omega_1$ on $(0,1)$.

Throughout this work, P denotes the set of all analytic polynomials and for a function F, R_F denotes the range of F. We assume also, $\Phi(r)=\frac{K(\log\frac{1}{r})}{\omega_1(1-r)}$, where ω_1 is a given reasonable function $\omega_1:(0,1]\to(0,\infty)$ with $\omega_1\ne 0$, for more properties of the reasonable function ω_1, we refer to [4,14] and [15].
For $p,q\in(0,\infty)$, the weighted Bergman space $A^p_{\Phi,q}$ is the set of all $f\in H(\Delta)$ such that

$$\|f\|_{A^p_{\Phi,q}}$$
$$=\sup_{0<\rho<1}\int_0^1\int_0^{2\pi}|f_\rho(e^{i\theta})|^p\Phi(r)r\,d\theta\,dr<\infty. \tag{4}$$

The above formula defines a norm that turns $A^2_{\Phi,q}$ into a Hilbert space whose inner product is given by

$$\langle f,g\rangle_{A^2_{\Phi,q}}$$
$$=\sum_{n=0}^\infty\widehat{f}(n)\overline{\widehat{g}(n)}=\int_0^{2\pi}\left(f_r(e^{i\theta})\right)\left(\overline{g_r(e^{i\theta})}\right)r\,dr\,d\theta \tag{5}$$

for each $f,g\in A^2_{\Phi,q}$.

Remark. By using known technique, it is easy to prove that $(A^p_{\Phi,q},\|.\|_{A^p_{\Phi,q}})$ is a Banach space, that is, the norm $\|.\|_{A^p_{\Phi,q}}$ is complete.

2 The generalized space

Definition 2. *Let $F:H(\Delta)\to H(\Delta)$ be a linear operator such the $F(f)=0$ if and only if $f=0$, that is,F is $1-1$. Suppose that $\Phi(r)=\frac{K(\log\frac{1}{r})}{\omega_1(1-r)}$ is a nondecreasing and right-continuous function. For $p,q\in(0,\infty)$, the (F,Φ)-Bergman space $A^p_{F,\Phi,q}(\Delta)=A^p_{F,\Phi,q}$ is defined to be the collection of all analytic function f on Δ for which*

$$\|f\|_{A^p_{F,\Phi,q}}$$
$$=\sup_{0<\rho<1}\int_0^1\int_0^{2\pi}|F(f_\rho(e^{i\theta}))|^p\Phi(r)rdrd\theta<\infty. \tag{6}$$

The importance of this definition is that it contains some known classes of analytic function spaces like Bergman and Hardy classes as we mention in the following remark:

Remark. We note that if $\int_0^1\Phi(r)r\,dr=1$, then we obtain the generalized Hardy space as defined and studied in [10]. Also, if $\Phi(r)=1, q=0$, and $F(f_\rho(e^{i\theta}))=f(z)$, then we obtain the Bergman space A^p.

Theorem 2. *Let $p,q\in(0,\infty)$ and $P\subseteq R_F$. Then $A^p_{\Phi,q}$ is a subspace of R_F if and only if $A^p_{F,\Phi,q}$ is a Banach space.*

Proof. Suppose that $A^p_{\Phi,q}\subseteq R_F$. Since $A^p_{F,\Phi,q}$ is a normed space, it suffices to show that it is complete. Let $\{f_n\}$ be Cauchy sequence in $A^p_{F,\Phi,q}$ and set $F(f_n)=g_n$. Then $\{g_n\}$

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is a Cauchy sequence in $A^p_{\Phi,q}$. Since $A^p_{\Phi,q}$ is complete, there is a $g \in A^p_{\Phi,q}$ such that

$$\|g_n - g\|_{A^p_{\Phi,q}} \to 0, \ as n \to \infty$$

Since $A^p_{\Phi,q} \subseteq R_F$, there is an $f \in A(\Delta)$ such that $F(f) = g$. Now we show that this f is the $A^p_{F,\Phi,q}$-limit of $\{f_n\}$. We have

$$\|f_n - f\|_{A^p_{F,\Phi,q}} = \|g_n - g\|^p_{\Phi,q} \to 0, \ as n \to \infty$$

Hence $f_n \to f \in A^p_{F,\Phi,q}$ for sufficiently large positive integer n, which implies that $f \in A^p_{F,\Phi,q}$. So $f_n \to f$ in $A^p_{F,\Phi,q}$ as $n \to \infty$.
Conversely, suppose that $A^p_{F,\Phi,q}$ is a Banach space. If $A^p_{\Phi,q} \subseteq R_F$, then there is a $g \in A^p_{\Phi,q}$ such that g is not in R_f. Since the polynomials are dense in $A^p_{\Phi,q}$, there is a sequence $\{p_n\}$ in P such that $\|p_n - g\|_{A^p_{\Phi,q}} \to 0$ as $n \to \infty$. Let $q_n = F^{-1}(p_n)$. Then $\{q_n\}$ is a Cauchy sequence in $A^p_{F,\Phi,q}$ and so there is a $q \in A^p_{F,\Phi,q}$ such that $\|q_n - q\|_{A^p_{F,\Phi,q}} \to 0$ as $n \to \infty$. Hence $\|F(q_n) - F(q)\|_{A^p_{\Phi,q}} \to 0$ as $n \to \infty$. On the other hand, $\|F(q_n) - g\|_{A^p_{\Phi,q}} \to 0$ as $n \to \infty$. This shows that $g = F(q)$ which is a contradiction.

Proposition 1. *If $A^2_{\Phi,q} \subseteq R_F$, and suppose that*

$$\mathscr{J}(\Phi,q) = \int_0^1 \Phi(r) r\, dr < \infty, \tag{7}$$

then $A^2_{F,\Phi,q}$ is a Hilbert space.

Proof. We define the scalar product on $A^2_{F,\Phi,q}$ by

$$\begin{aligned}
&\langle f,g\rangle_{A^2_{F,\Phi,q}} \\
&= \int_0^1 \int_0^{2\pi} F(f_\rho(e^{i\theta}))\overline{F(g_r(e^{i\theta}))}\Phi(r) r\, dr d\theta \\
&\leq C \int_0^{2\pi} F(f_\rho(e^{i\theta}))\overline{F(g_r(e^{i\theta}))}\, d\theta \\
&= \langle F(f), F(g)\rangle_{H^2}.
\end{aligned}$$

It is easy to show that this scalar product defines an inner product on $A^2_{F,\Phi,2}$.

There is a Banach space $A^p_{\Phi,q}$, such that it does not satisfy the condition of Theorem 2. For example, let $1 \leq p,q < \infty$, $F(f) = zf$ for each $f \in H(\Delta)$. Then $1 \not\exists R_F$. By the following proposition, we see that although $A^p_{\Phi,q} \subseteq R_F$, $A^p_{F,\Phi,q}$ is a Banach space.

Proposition 2. *Suppose that $1 \leq p < \infty$, $0 < q < \infty$, $h \in H(\Delta)$, $h \neq 0$, and $F(f) = fh$ for every $f \in H(\Delta)$. Then $A^p_{F,\Phi,q}$ is a Banach space.*

Proof. If $A^p_{\Phi,p} \subseteq R_F$, then by Theorem 2.1, the proposition holds. Otherwise, let f_n be a Cauchy sequence in $A^p_{F,K,q}$. Setting $F(f_n) = g_n$, so $\{g_n\}$ is a Cauchy sequence in $A^p_{\Phi,q}$. Therefore, there is a $g \in A^p_{\Phi,q}$ such that $\|g_n - g\|_{A^p_{\Phi,q}} \to 0$ as $n \to \infty$. If $g \in R_F$, then the proof is similar to the proof of Theorem 2.
Now suppose that g is not in R_F. Then there are $z_0 \in \Delta$, $m_1 \geq 0$, and $m_2 > m_1$ such that

$$g(z) = (z - z_0)^{m_1} g_0(z),$$

$$h(z) = (z - z_0)^{m_2} h_0(z),$$

where $h_0, g_0 \in H(\Delta), g_0(z_0) \neq 0$, and $h_0(z_0) \neq 0$. Therefore, we have

$$\begin{aligned}
&\|g_n - g\|_{A^p_{\Phi,q}} = \|h f_n - g\|_{A^p_{\Phi,q}} \\
&= \int_0^1 \int_0^{2\pi} |T(\rho,\theta)|^p \Phi(r) r\, dr d\theta,
\end{aligned}$$

where

$$\begin{aligned}
&T(\rho,\theta) \\
&= (\rho e^{i\theta} - z_0)^{m_2} h_0(\rho e^{i\theta}) f_n - (\rho e^{i\theta} - z_0)^{m_1} g_0(\rho e^{i\theta}).
\end{aligned}$$

Since $\|g_n - g\|_{A^p_{\Phi,q}} \to 0$ as $n \to \infty$, we have

$$\lim_{n\to\infty} \int_0^1 \int_0^{2\pi} |T(\rho,\theta)|^p \Phi(r) r\, dr d\theta = 0. \tag{8}$$

Hence, $\|(z - z_0)^{m_2} h_0 f_n - (z - z_0)^{m_1} g_0\|_{A^p_{\Phi,q}} \to 0$ as $n \to \infty$. Since the point evaluation at z_0 is a bounded linear functional on $A^p_{\Phi,q}$, we have

$$(z_0 - z_0)^{m_2} h_0 f_n(z_0) - (z_0 - z_0)^{m_1} g_0(z_0) \to 0, \ n \to \infty. \tag{9}$$

So $g_0(z_0) = 0$, which is a contradiction.

In the following proposition, we will find a dense subset in $A^p_{F,\Phi,q}$, whenever $P \subseteq R_F$.

Proposition 3. *Suppose that $1 \leq p < \infty$, $0 < q < \infty$, and $P \subseteq R_F$. Then $\overline{\{F^{-1}(p) : p \in P\}} = A^p_{F,\Phi,q}$.*

Proof. It is clear that $\{F^{-1}(p) : p \in P\} \subseteq A^p_{F,\Phi,q}$. Suppose that $f \in A^p_{F,\Phi,q}$. Then there is a sequence $\{h_n\}$ in P such that $\|h_n - F(f)\|_{A^p_{\Phi,q}} \to 0$ as $n \to \infty$. Setting $f_n = F^{-1}(h_n)$, we have

$$\|f_n - f\|_{A^p_{F,\Phi,q}} = \|h_n - F(f)\|_{A^p_{\Phi,q}}, \tag{10}$$

so the result follows.

Corollary 1. *Suppose that $1 \leq p < \infty$, $\underline{0 < q < \infty}$,, $P \subseteq R_F$, and $F^{-1}(p) \in P$ for each $p \in P$. Then $\overline{P \cap A^p_{F,\Phi,q}} = A^p_{F,\Phi,q}$.*

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3 Point Evaluations

Let e_ω be the point evaluation at ω, that is, $e_\omega(f) = f(\omega)$. It is well known that point evaluations at the point of Δ are all continuous on $A^2_{K,q}$.
Let $\omega \in \Delta$ and H be a Hilbert space of analytic functions on Δ. If e_ω is a bounded linear functional on H, then the Riesz Representation Theorem implies that there is a function (which is usually called K_ω) in H that induces this linear functional, that is, $e_\omega(f) = \langle f, K_\omega\rangle$.

In this section, we investigate the continuity of the point evaluations on $A^p_{F,\Phi,q}$.
Next, we prove that an analytic function f on the unit disk with Hadamard gaps, that is, $f(z)$ satisfying $\frac{n_{k+1}}{n_k} \geq c > 1$ for all $k \in \mathbb{N}$ belongs to the space $A^p_{F,\Phi,q}$.

Theorem 3. *If $\Phi(r) = \frac{K(\log\frac{1}{r})}{\omega_1(1-r)}$ and*

$$f(z) = \sum_{j=1}^{\infty} b_j z^{n_j-1}, \tag{11}$$

is in the Hadamard gap class, then $f \in A^p_{F,\Phi,q}$ if

$$\sum_{j=1}^{\infty} |b_j|^p \Phi\left(\frac{1}{n_j}\right) < \infty. \tag{12}$$

Proof. First assume that condition (12) holds. We write $z = re^{i\theta}$ in polar form and observe that

$$|f(z)| \leq \sum_{j=1}^{\infty} |b_j| r^{n_j-1}.$$

Then, by Theorem 2.1 and Lemma 1, let $F(f) = g$, we obtain

$$\begin{aligned}
&\|f\|_{A^p_{F,\phi,q}} \\
&= \int_0^1\int_0^{2\pi} |F(f(re^{i\theta}))|^p \Phi(r) r dr d\theta \\
&= \int_0^1\int_0^{2\pi} |g(re^{i\theta})|^p \Phi(r) r dr d\theta \\
&= \int_0^1\int_0^{2\pi} \left(\sum_{j=1}^{\infty} |b_j| r^{n_j-1}\right)^p \Phi(r) r dr d\theta \\
&= 2\pi \int_0^1 r^{-p+1}\left[\sum_{j=1}^{\infty} |b_j| r^{n_j}\right]^p \Phi(r)\, dr
\end{aligned}$$

Using the Cauchy-Schwarz inequality to produce

$$\begin{aligned}
&\left[\sum_{j=1}^{\infty} |b_j| r^{n_j}\right]^p \\
&= \left[\sum_{n=0}^{\infty}\sum_{n_j\in I_n} |b_j| r^{n_j}\right]^p \leq \left[\sum_{n=0}^{\infty}\sum_{n_j\in I_n} |b_j| r^{2^n}\right]^p \\
&\leq \left[\sum_{n=0}^{\infty} (2^{n/2} r^{2^n})^{1-1/p} (r^{2^n} 2^{(1-p)n/2})^{1/p} \sum_{n_j\in I_n} |b_j|\right]^p \\
&\leq \left[\sum_{n=0}^{\infty} r^{2^n} 2^{((1-p)/2)n}\left(\sum_{n_j\in I_n} |b_j|\right)^p\right]\left[\sum_{n=0}^{\infty} 2^{n/2} r^{2^n}\right]^{p-1} \\
&\leq C\left(\log\frac{1}{r}\right)^{-(p-1)/2}\sum_{n=0}^{\infty} r^{2^n} 2^{((1-p)/2)n}\left(\sum_{n_j\in I_n} |b_j|\right)^p
\end{aligned}$$

where $I_n = \{j : 2^n \leq j < 2^{n+1}, j \in \mathbb{N}\}$. To this end, we combine the elementary estimates:

$$\begin{aligned}
\sum_{n=0}^{\infty} 2^{\frac{n}{2}} r^{2^n} &= \sqrt{2}\sum_{n=0}^{\infty}\int_{2^n}^{2^{n+1}} t^{-\frac{1}{2}} r^{\frac{t}{2}} dt \\
&\leq \sqrt{2}\int_0^{\infty} t^{-\frac{1}{2}} r^{\frac{t}{2}} dt \\
&\leq 2\Gamma(\frac{1}{2})\left(\log\frac{1}{r}\right)^{-\frac{1}{2}}
\end{aligned}$$

This very useful tool can now be applied to the calculation above to obtain

$$\begin{aligned}
&\|f\|_{A^p_{F,\Phi,q}} \leq C\sum_{n=0}^{\infty} (2^n)^{\frac{1-p}{2}}\left[\sum_{n_j\in I_n} |b_j|\right]^p \\
&\times \int_0^1 r^{2^n-p+1}\left(\log\frac{1}{r}\right)^{\frac{2q-p-3}{2}} \Phi(r) dr
\end{aligned} \tag{13}$$

where $(1-r^2) \leq 2\log\frac{1}{r}$. This together with (13) and Theorem 1.2 for $\alpha = 2^n - p + 2, \beta = \frac{2q-p-3}{2}$, we obtain

$$\begin{aligned}
&\|f\|_{A^p_{F,\Phi,q}} \\
&\leq C\sum_{n=0}^{\infty}\left[\sum_{n_j\in I_n} |b_j|\right]^p\left(\frac{1}{2^n}\right)^{\frac{p-1}{2}}\left(\frac{5+p-2q}{2^{n+1}-2(p-2)}\right)^{\frac{5+p-2q}{2}} \\
&\times \Phi\left(\frac{5+p-2q}{2^{n+1}-2(p-2)}\right) \\
&\leq C\sum_{n=0}^{\infty}\left[\sum_{n_j\in I_n} |b_j|\right]^p\left(\frac{1}{2^n}\right)^{\frac{p-1}{2}}\left(\frac{1}{2^n}\right)^{\frac{5+p-2q}{2}}\Phi\left(\frac{1}{2^n}\right)
\end{aligned}$$

$$\leq C\sum_{n=0}^{\infty}\left[\sum_{n_j\in I_n} |b_j|\right]^p\left(\frac{1}{2^n}\right)^{p-q+2}\Phi\left(\frac{1}{2^n}\right) \tag{14}$$

If $n_j \in I_n$, then $n_j < 2^n < 2^{n+1}$. It follows from the monotonicity of k, Lemma 1 and $K(2t) \leq CK(t)$ for all $0 \leq 2t \leq 1$, such that

$$\left(\frac{1}{2^n}\right)^{p-q+2}\Phi\left(\frac{1}{2^n}\right) < n_j^{(p-q+2)}\Phi\left(\frac{1}{n_j}\right).$$

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Combining this with (14), we obtain

$$\|f\|_{A^p_{F,\Phi,q}} \lesssim \sum_{n=0}^{\infty}\left[\sum_{n_j\in I_n}|b_j|\right]^p n_j^{p-q+2}\Phi\left(\frac{1}{n_j}\right). \qquad (15)$$

Since f is in the Hadamard gap class, there exists a constant c such that $n_{j+1}\geq cn_j$ for all $j\in\mathbb{N}$. Hence, the Taylor series of $f(z)$ has at most $([\log_c 2]+1)$ terms $a_jz^{n_j}$ such that $n_j\in I_n$. By (15) and Hölder's inequality,

$$\|f\|_{A^p_{F,\Phi,q}} \lesssim (\log_c 2+1)^{p-q+2}\sum_{n=0}^{\infty}\sum_{n_j\in I_n}|b_j|^p\Phi\left(\frac{1}{n_j}\right)$$

Then, $f\in A^p_{F,\Phi,q}$

Lemma 2. *If $f\in A^p_{\Phi,q}(0<p,q<\infty)$, then*

$$\lim_{\rho\to1}\int_0^1\int_0^{2\pi}|F(f(\rho e^{i\theta}))|^p\Phi(r)r\,drd\theta = \int_0^1\int_0^{2\pi}|F(f(e^{i\theta}))|^p\Phi(r)r\,drd\theta$$

and

$$\lim_{\rho\to1}\int_0^1\int_0^{2\pi}|F(f(\rho e^{i\theta}))-F(f(e^{i\theta}))|^p\Phi(r)rdrd\theta=0.$$

Proof. First let us prove

$$\lim_{\rho\to1}\int_0^1\int_0^{2\pi}|F(f_\rho(e^{i\theta}))-F(f(\rho e^{i\theta}))|^p\Phi(r)rdrd\theta=0$$

for $p=2$. If $F(f(z))=\sum b_j^p\Phi\left(\frac{1}{n_j}\right)(f(z))^n$ is in $A^2_{F,\Phi,q}$, then $\sum_{j=1}^{\infty}|b_j|^p\Phi\left(\frac{1}{n_j}\right)<\infty$.

But by Fatou's lemma, we have

$$\int_0^1\int_0^{2\pi}|F(f_\rho(e^{i\theta}))-F(f(\rho e^{i\theta}))|^2\Phi(r)r\,dr\,d\theta$$
$$\leq \liminf_{\rho\to1}\int_0^1\int_0^{2\pi}|F(f_\rho(e^{i\theta}))-F(f(\rho e^{i\theta}))|^2\Phi(r)\,dr\,d\theta$$
$$=\sum_{n=1}^{\infty}\int_0^1\int_0^{2\pi}\left|b_j\Phi\left(\frac{1}{n_j}\right)^{\frac12}f(\rho e^{i\theta})-b_j\Phi\left(\frac{1}{n_j}\right)^{\frac12}f(e^{i\theta})\right|^2 \times\Phi(r)r\,dr\,d\theta$$
$$=\sum_{n=1}^{\infty}|b_j|^2\Phi\left(\frac{1}{n_j}\right)\int_0^1\int_0^{2\pi}|f(\rho e^{i\theta})-f(\rho e^{i\theta})|^2 \times\Phi(r)rdrd\theta$$

which tends to zero as $\rho\to1$. Now, we proof

$$\lim_{\rho\to1}\int_0^1\int_0^{2\pi}|F(f(\rho e^{i\theta}))|^p\Phi(r)r\,drd\theta = \int_0^1\int_0^{2\pi}|F(f(e^{i\theta}))|^p\Phi(r)r\,drd\theta$$

in the case $p=2$, If $f\in A^p_{F,\Phi,q}$ $(0<p,q<\infty)$, we use the factorization $f=Bg$ where $B(z)$ is a Blaschke product and g(z) is an $A^p_{F,\Phi,q}$. Since $(g(z))^{p/2}\in A^2_{F,\Phi,q}$, it follows from what we have just proved that

$$\int_0^1\int_0^{2\pi}|F(f(\rho e^{i\theta}))|^pK(\log\frac{1}{r})rdrd\theta$$
$$\leq\int_0^1\int_0^{2\pi}|F(g(\rho e^{i\theta}))|^p\Phi(r)r\,dr\,d\theta\to$$
$$\int_0^1\int_0^{2\pi}|F(g(e^{i\theta}))|^pK(\log\frac{1}{r})rdr\,d\theta$$
$$=\int_0^1\int_0^{2\pi}|F(f(e^{i\theta}))|^p\Phi(r)\,dr\,d\theta.$$

This together with Fatou's lemma complete the proof.

Theorem 4. *If $\Phi(r)=\frac{K(\log\frac1r)}{\omega_1(1-r)}$ and $A^p_{\Phi,q}\subseteq R_F$. For $1\leq p<2$, $0<q<\infty$ and $\sum_{j=0}^{\infty}\overline{F^{-1}(z^j)(\omega)}\,z^j\in H^\infty$. If for each $0<\rho<1$, $f\in A^1_{F,\Phi,q}$, and $(F(f))_\rho=F(f_\rho)$, then e_ω is continuous on $A^p_{F,\Phi,q}$.*

Proof. Let $f\in A^1_{F,\Phi,q}$. Then for each $0<\rho<1$, $f_\rho\in A^2_{F,\Phi,q}$ and then

$$f_\rho(\omega)=\langle f_\rho,K_\omega\rangle_{A^2_{F,\Phi,q}}$$
$$=\langle F(f_\rho),F(K_\omega)\rangle_{A^2_{\Phi,q}}$$
$$=\int_0^1\int_0^{2\pi}F(f_\rho(e^{i\theta}))\overline{F(K_\omega(\rho e^{i\theta}))}\Phi(r)r\,dr\,d\theta.$$

Also by Lemma 3.1, we have $\|(F(f))_\rho-F(f)\|_{A^1_{F,\Phi,q}}\to0$ as $r\to1$.

Hence, using Hölder's inequality and the fact that $F(K_\omega)=\sum_{j=0}^{\infty}\overline{F^{-1}(z^j)(\omega)}\,z^j$, we obtain

$$\left|\int_0^1\int_0^{2\pi}(F((f))_\rho-F(f))(\rho e^{i\theta})\overline{F(K_\omega)}(\rho e^{i\theta})\Phi(r)rdrd\theta\right|$$
$$\leq\|F(K_\omega)\|_\infty\int_0^1\int_0^{2\pi}F(f_\rho(e^{i\theta}))-F(f(\rho e^{i\theta}))\,\Phi(r)r\,dr\,d\theta$$
$$\leq\|F(K_\omega)\|_\infty\|(F(f))_\rho-F(f)\|_{A^1_{F,\Phi,q}}\to0 \text{ as } \rho\to1,$$

so we obtain

$$f(\omega)=\lim_{\rho\to1}f_\rho(\omega)$$
$$=\int_0^1\int_0^{2\pi}F(\lim_{\rho\to1}f_\rho(\rho e^{i\theta}))\overline{F(K_\omega)}(\rho e^{i\theta})\Phi(r)rdrd\theta$$
$$=\int_0^1\int_0^{2\pi}F(f(e^{i\theta}))\overline{F(K_\omega)}(re^{i\theta})\Phi(r)rdrd\theta.$$

Hence,

$$|f(\omega)|=\left|\int_0^1\int_0^{2\pi}F(f(e^{i\theta}))\Phi(r)rdrd\theta\right|$$
$$\leq\|F(K_\omega)\|_\infty\|f\|_{A^1_{F,\Phi,q}}$$

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for each $f \in A^1_{F,\Phi,q}$. Now let $1 \le p < 2$. If $f \in A^p_{F,\Phi,q}$, then

$$|f(w)| \le \|F(K_\omega)\|_\infty \|f\|_{A^1_{F,\Phi,q}} \le \|F(K_\omega)\|_\infty \|f\|_{A^p_{F,\Phi,q}},$$

so, the result follows.

Theorem 5. *Let $\Phi : [0,\infty) \to [0,\infty)$ be a non-decreasing and right-continuous function satisfying (7) and let $1 \le p < \infty$, $0 < q < \infty$, $\omega \in \Delta$, $h \in H(\Delta)$, $h \ne 0$. For each $f \in H(\Delta)$, $F(f) = fh$. Then e_ω is continuous on $A^p_{F,\Phi,q}$.*

Proof. We break the proof in to two parts.
(1) Let $h(w) \ne 0$. If $|\omega| < \rho < 1$ and Γ_ρ is the circle of radius ρ with center at the origin, then the Cauchy formula shows that for any f in $A^p_{F,\Phi,q}$,

$$\begin{aligned} f(\omega)h(\omega) &= \frac{1}{2\pi i}\int_{\Gamma_\rho} \frac{f(\zeta)h(\zeta)}{\zeta-\omega} d\zeta \\ &= \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(\rho e^{i\theta})h(\rho e^{i\theta})}{\rho e^{i\theta}-\omega}\rho i e^{i\theta} d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} f(\rho e^{i\theta})h(\rho e^{i\theta})\frac{\rho}{\rho-\omega e^{-i\theta}} d\theta, \end{aligned}$$

Then,

$$\int_0^1 f(\omega)h(\omega)\Phi(r)rdr$$

$$= \frac{1}{2\pi}\int_0^1\int_0^{2\pi} \frac{f(re^{i\theta})h(\rho e^{i\theta})}{\rho-\omega e^{-i\theta}}\Phi(\rho)r\rho\, dr d\theta.$$

By Hölder's inequality, it follows that

$$|f(\omega)||h(\omega)|\int_0^1 \Phi(r)r\,dr$$

$$\le \frac{1}{2\pi}\|(fh)_\rho\|_{A^p_{\phi,q}} \left\|\frac{\rho}{\rho-\omega e^{-i\theta}}\right\|_{p^*} \qquad (16)$$

where $\frac{1}{p}+\frac{1}{p^*}=1$. Now if r tends to 1, $\left|\frac{\rho}{(\rho-\omega e^{-i\theta})}\right|$ converges uniformly to the bounded function $|1-\omega e^{i\theta}|^{-1}$ and

$$\|(fh)_\rho\|_{A^p_{\Phi,q}} \le \|fh\|_{A^p_{\Phi,q}}.$$

Hence there in an $M = \frac{\|\rho/(\rho-\omega e^{-i\theta})\|}{2\pi\mathscr{J}(\Phi,q)} < \infty$ such that

$$|f(\omega)| \le \frac{M}{|h(\omega)|}\|f\|_{A^p_{F,\Phi,q}},$$

and the result follows.
(2) Let $h(\omega)=0$. Then $h(z)=(z-\omega)^m h_0(z)$, where $m \in \mathbb{N}$, $h_0 \in H(\Delta)$, and $h_0(\omega) \ne 0$.
Let $F_1(f) = fh_0$ for each $f \in H(\Delta)$, it is easy to see that $A^p_{F,\Phi,q} \subseteq A^p_{F_1,\Phi,q}$. Then by the preceding part, there is a constant $0 < C < \infty$ such that

$$|f(\omega)|^p \le C\|fh_0\|_{A^p_{\Phi,q}}$$

$$= C\int_0^1\int_0^{2\pi} \left|f(\rho e^{i\theta})\right|^p \left|h_0(\rho e^{i\theta})\right|^p E(\rho)\Phi(r)rdrd\theta$$

$$\le \frac{C}{(1-|\omega|)^{mp}}\int_0^1\int_0^{2\pi} \left|f(\rho e^{i\theta})\right|^p \left|h(\rho e^{i\theta})\right|^p \Phi(r)rdrd\theta$$

$$= \frac{C}{(1-|\omega|)^{mp}}\|f\|_{A^p_{F,\Phi,q}}$$

for each $f \in A^p_{F,\Phi,q}$. So e_ω is continuous on $A^p_{F,\Phi,q}$, where $E(\rho) = \frac{|\rho e^{i\theta}-\omega|^{mp}}{|\rho e^{i\theta}-\omega|^{mp}}$.

Remark. It should be remarked that our results in this paper generalize and improve the recent results in [3,10]. It is still an open problem to extend these results to Clifford Analysis. For more information on studies of function spaces in Clifford analysis, we refer to [1,2,5,6,12] and others.

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