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The Cauchy Problem for BBGKY Hierarchy of Quantum Kinetic Equations with Coulomb Potential

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The existence of a unique solution, in term of initial data of the BBGKY hierarchy of quantum kinetic equations with coulomb potential, is proved. This is based on non-relativistic quantum mechanics and utilizing a semigroup method.

Keywords: Coulomb potential, quantum kinetic equations.

Mathematics Subject Classification: 82C10.

1 Introduction

In the last a few years we can see an increasing interest in the Bogolyubov-Born-Green-Kirkwood-Yvon's (BBGKY) hierarchy. This interest is quite natural, since this hierarchy is related to the equation of Liouville, which is describing the evolution of a system interacting many particles with the Boltzmann [1,7] and Vlasov equations. Last equations describe the evolution of one particle and they are fundamental equations, describing the evolution of particles in solids, semiconductors, in gas and in plasma. Unlike Liouville's equation, the structure of the BBGKY's hierarchy permits the generalization of the physical results for one particle to system of many particles.

Since the time, when it was formulated in 1946, the BBGKY's hierarchy was the object of investigation for physicists as well as mathematicians [2–8, 11–13, 16, 18–22, 24–28].

Well known, that charged particles interact via the coulomb potential. Until present, there is no solution of the BBGKY's hierarchy of quantum kinetic equations in the case when the particles interact via a coulomb potential. This is an important problem for many researchers. The present paper addresses the solution of this problem.

2 Formulation of the Problem

We consider the hierarchy BBGKY of quantum kinetic equations, which describes the evolution of a system of identical particles with mass m and charge q interacting via a coulomb potential [10, 14, 15] $\phi(x_i, x_j) = q^2/|x_i - x_j|$, which depends on the distance between particles

$$|x_i - x_j| = \Big((x_i^1 - x_j^1)^2 + (x_i^2 - x_j^2)^2 + (x_i^3 - x_j^3)^2\Big)^{1/2}$$

and charges q. We assume that the charge is a real constant.

The BBGKY's hierarchy is given by [2–5]

$$\begin{aligned} i\frac{\partial \rho_s(t, x_1, \ldots, x_s; x_1', \ldots, x_s')}{\partial t} &= [H_s, \rho_s](t, x_1, \ldots, x_s; x_1', \ldots, x_s') \\ &+ \frac{N}{V}\Big(1 - \frac{s}{N}\Big) Tr_{x_{s+1}} \sum_{1 \le i \le s} \Big(\phi_{i,s+1}(|x_i - x_{s+1}|) - \phi_{i,s+1}(|x_i' - x_{s+1}|)\Big) \\ &\times \rho_{s+1}(t, x_1, \ldots, x_s, x_{s+1}; x_1', \ldots, x_s', x_{s+1}), \end{aligned} \tag{2.1}$$

with the initial condition

$$\rho_s(t, x_1, \ldots, x_s; x_1', \ldots, x_s')|_{t=0} = \rho_s(0, x_1, \ldots, x_s; x_1', \ldots, x_s'). \tag{2.2}$$

In the problem given by equation (2.1) and (2.2) the vector represented by x_i gives the position of ith particle in the 3-dimensional Euclidean space R^3, $x_i = (x_i^1, x_i^2, x_i^3)$, $i = 1, 2, \ldots, s$, and $x_i^\alpha, \alpha = 1, 2, 3$ are coordinates of a vector x_i. The length of the vector x_i is denoted by

$$|x_i| = \big((x_i^1)^2 + (x_i^2)^2 + (x_i^3)^2\big)^{1/2}.$$

The reduced statistical operator of s particles is $\rho_s(x_1, \ldots, x_s; x_1', \ldots, x_s')$ related by positive symmetric density matrix of N particles by [2–5]

$$\begin{aligned} &\rho_s(x_1, \ldots, x_s; x_1', \ldots, x_s') \\ &\quad = V^s Tr_{x_{s+1}, \ldots, x_N} D(x_1, \ldots, x_s, x_{s+1}, \ldots, x_N; x_1', \ldots, x_s', x_{s+1}, \ldots, x_N), \end{aligned}$$

where $s \in N$, N is the number of particles, V the volume of the system of particles. The trace is defined in terms of the kernel $\rho(x, x')$ by the formula

$$Tr_x \rho = \int \rho(x, x) dx.$$

In equation (2.1) $\hbar = 1$ is the Planck constant and [,] denotes the Poisson bracket.

The Hamiltonian of system is defined as

$$H_s = \sum_{1 \le i \le s} T_i + \sum_{1 \le i < j \le s} \phi_{i,j},$$

where

$$T_i = -\frac{\hbar^2 \triangle_i}{2m}, \quad \phi_{i,j} = \frac{q^2}{|x_i - x_j|},$$

and $\triangle_i$ is the Laplacian

$$\triangle_i = \frac{\partial^2}{\partial(x_i^1)^2} + \frac{\partial^2}{\partial(x_i^2)^2} + \frac{\partial^2}{\partial(x_i^3)^2},$$

The operator given by

$$(\Phi\rho_s)(x_1,\dots,x_s;x_1',\dots,x_s') = \sum_{1\le i<j\le s} \phi_{i,j}(|x_i - x_j|)\rho_s(x_1,\dots,x_s;x_1',\dots,x_s')$$

is symmetric.

In the present work, the Cauchy problem (2.1)-(2.2) is solved for a quantum system finite number paricles contained in the finite bounded region (vessel) with volume $V = |\Lambda|$.

A state of this system is described by a density matrix $\rho_s^\Lambda(t, x_1,\dots,x_s;x_1',\dots,x_s')$ that satisfies the Cauchy problem

$$\begin{aligned} i\frac{\partial\rho_s^\Lambda(t,x_1,\dots,x_s;x_1',\dots,x_s')}{\partial t} &= [H_s^\Lambda,\rho_s^\Lambda](t,x_1,\dots,x_s;x_1',\dots,x_s') \\ &+ \frac{N}{V}\Big(1-\frac{s}{N}\Big)Tr_{x_{s+1}}\sum_{1\le i\le s}\Big(\phi_{i,s+1}(|x_i - x_{s+1}|) - \phi_{i,s+1}(|x_i' - x_{s+1}|)\Big) \\ &\times \rho_{s+1}^\Lambda(t,x_1,\dots,x_s,x_{s+1};x_1',\dots,x_s',x_{s+1}), \end{aligned} \tag{2.3}$$

with the initial condition

$$\rho_s^\Lambda(t,x_1,\dots,x_s;x_1',\dots,x_s')|_{t=0} = \rho_s^\Lambda(0,x_1,\dots,x_s;x_1',\dots,x_s'). \tag{2.4}$$

In (2.3) a Hamiltonian of a system is defined as

$$H_s^\Lambda(x_1,\dots,x_s) = \sum_{1\le i\le s}\Big(-\frac{1}{2m}\triangle_{x_i} + u^\Lambda(x_i)\Big) + \sum_{1\le i<j\le s}\phi_{i,j}(|x_i - x_j|),$$

where $u^\Lambda(x)$ is an external field which keeps the system in the region Λ ($u^\Lambda(x) = 0$ if $x\in\Lambda$ and $u^\Lambda(x) = +\infty$ if $x\notin\Lambda$).

The trace is defined in the region by the formula

$$Tr_x\rho = \int_\Lambda \rho(x,x)dx.$$

Introducing the notation

$$\begin{aligned} &\left(\mathcal{H}^\Lambda\rho^\Lambda\right)_s(t,x_1,\dots,x_s;x_1',\dots,x_s') = \left[H_s^\Lambda,\rho_s^\Lambda\right](t,x_1,\dots,x_s;x_1',\dots,x_s'); \\ &\left(\mathcal{D}_{x_{s+1}}^\Lambda\rho^\Lambda\right)_s(x_1,\dots,x_s;x_1',\dots,x_s') = \rho_{s+1}^\Lambda\left(x_1,\cdots x_s,x_{s+1};x_1',\cdots,x_s',x_{s+1}\right); \end{aligned}$$

$$(\mathcal{A}^{\Lambda}_{x_{s+1}}\rho^{\Lambda})_s(t,x_1,\dots,x_s;x'_1,\dots,x'_s)$$
$$= \frac{N}{V}\Big(1-\frac{s}{N}\Big)\sum_{1\le i\le s}\Big(\phi_{i,s+1}(|x_i-x_{s+1}|)-\phi_{i,s+1}(|x'_i-x_{s+1}|)\Big)$$
$$\cdot\,\rho^{\Lambda}_s(t,x_1,\dots,x_s;x'_1,\dots,x'_s);$$
$$\rho^{\Lambda}(t)=\{\rho^{\Lambda}_1(t,x_1;x'_1),\dots,\rho^{\Lambda}_s(t,x_1,\cdots,x_s:x'_1,\dots,x'_s),\dots\},\quad s=1,2,\dots,$$

we can cast (2.3) and (2.4) in the form

$$i\frac{\partial}{\partial t}\rho^{\Lambda}_s(t,x_1,\dots,x_s;x'_1,\dots,x'_s)=\left(\mathcal{H}^{\Lambda}\rho^{\Lambda}\right)_s(t,x_1,\dots,x_s;x'_1,\dots,x'_s)$$
$$+\int_{\Lambda}\Big(\mathcal{A}^{\Lambda}_{x_{x+1}}\mathcal{D}^{\Lambda}_{x_{x+1}}\rho^{\Lambda}\Big)_s(t,x_1,\dots,x_s,x_{s+1};x'_1,\dots,x'_s,x_{s+1})dx_{x+1},\qquad (2.5)$$
$$\rho^{\Lambda}_s(t,x_1,\dots,x_s;x'_1,\dots,x'_s)|_{t=0}=\rho^{\Lambda}_s(0,x_1,\dots,x_s;x'_1,\dots,x'_s)$$
$$\equiv\rho^{\Lambda}_s(x_1,\dots,x_s;x'_1,\dots,x'_s).\qquad (2.6)$$

3 Solution of the Cauchy Problem for BBGKY Hierarchy of Quantum Kinetic Equations with Coulomb Potential

To obtain the solution of the Cauchy problem defined by (2.3) and (2.4) or by the reduced form in (2.5) and (2.6), we use a semigroup method [9, 17, 19–21].

Let $L^s_2(\Lambda)$ be the Hilbert space of functions $\psi^{\Lambda}_s(x_1,\dots,x_s)$, $x_i\in R^3(\Lambda)$, and B^{Λ}_s be the Banach space of positive-definite, self-adjoint nuclear operators $\rho^{\Lambda}_s(x_1,\dots,x_s;x'_1,\dots,x'_s)$ on $L^s_2(\Lambda)$

$$(\rho^{\Lambda}_s\psi^{\Lambda}_s)(x_1,\dots,x_s)=\int_{\Lambda}\rho^{\Lambda}_s(x_1,\dots,x_s;x'_1,\dots,x'_s)\psi^{\Lambda}_s(x'_1,\dots,x'_s)dx'_1\dots,dx'_s$$

with norm

$$\left|\rho^{\Lambda}_s\right|_1=\sup\sum_{1\le i\le\infty}\left|(\rho^{\Lambda}_s\psi^s_i,\varphi^s_i)\right|,$$

where the upper bound is taken over all orthonormalized systems of finite, twice differentiable functions with compact support $\{\psi^s_i\}$ and $\{\varphi^s_i\}$ in $L^s_2(\Lambda)$, $s\ge1$ and $\left|\rho^{\Lambda}_0\right|_1=\left|\rho^{\Lambda}_0\right|$.

We'll suppose that the operators $\rho^{\Lambda}_s(t)$ and H^{Λ}_s act in the space $L^s_2(\Lambda)$ with zero boundary conditions.

Let B^{Λ} be the Banach space of sequences of nuclear operators

$$\rho^{\Lambda}=\{\rho^{\Lambda}_0,\rho^{\Lambda}_1(x_1;x'_1),\dots,\rho^{\Lambda}_s(x_1,\dots,x_s;x'_1,\dots,x'_s),\dots\},$$

where ρ^{Λ}_0 are complex numbers, $\rho^{\Lambda}_s\subset B^{\Lambda}_s$,

$$\rho^{\Lambda}_s(x_1,\dots,x_s;x'_1,\dots,x'_s)=0,\qquad\text{when } s>s_0,$$

s_0 is finite and the norm is

$$\left|\rho^\Lambda\right|_1 = \sum_{s=0}^{\infty} \left|\rho_s^\Lambda\right|_1 .$$

The coulomb potential $\phi_{i,j} = q^2/|r_{i,j}|$ can be represented as

$$\phi_{i,j} = \phi_{i,j}^1 + \phi_{i,j}^2,$$

where

$$\phi_{i,j}^1 = \frac{q^2}{|r_{i,j}|}\Big(\frac{1}{1+|r_{i,j}|}\Big) \subset L_2(R^3), \qquad \phi_{i,j}^2 = \frac{q^2}{1+|r_{i,j}|} \subset L_\infty(R^3),$$
$$r_{i,j} = \big((x_i^1 - x_j^1)^2 + (x_i^2 - x_j^2)^2 + (x_i^3 - x_j^3)^2\big)^{1/2} .$$

Therefore the coulomb potential satisfies the conditions of Theorem X.15 in [23] and the Hamiltonian with coulomb potential

$$H_s^C(x_i, x_j) = -\sum_{1\le i\le s} \frac{1}{2} \triangle_{x_i} + \sum_{1\le i<j\le s} \frac{q^2}{|x_i - x_j|}$$

is a self-adjoint operator on the set $D(H_s^C)$ of finite, twice differentiable functions with compact support [10].

Let $\tilde{B}_s^\Lambda$ be a dense set of "good" elements of B_s^Λ of type $B_s^\Lambda \cap D(H_s^C) \otimes D(H_s^C)$, where $D(H_s^C)$ is the domain of the operator H_s^C [10] and $\otimes$ denote the algebraic tensor product.

Consider the operators

$$\left(\omega^\Lambda(t)\rho^\Lambda\right)_s (x_1, \dots, x_s; x_1', \dots, x_s') = \left(e^{-iH_s^\Lambda t}\rho^\Lambda e^{iH_s^\Lambda t}\right)_s (x_1, \dots, x_s; x_1', \dots, x_s'),$$

on $\rho_s^\Lambda(x_1, \dots, x_s; x_1', \dots, x_s') \subset B_s^\Lambda$.

Theorem 3.1. *The operators $\omega^\Lambda(t)$ define a strongly continuous group of isometries on B^Λ the generators of which coincides with $-i\mathcal{H}^\Lambda$ on $\tilde{B}^\Lambda$ everywhere dense in B^Λ.*

Proof. The prove is summarized in the following four steps:

Step 1. The operator $\omega^\Lambda(t)$ is an isometry in the nuclear norm on B^Λ:

$$\begin{aligned}
\left|\omega^\Lambda(t)\rho^\Lambda\right|_1 &= \left|\exp(-iH^\Lambda t)\rho^\Lambda \exp(iH^\Lambda t)\right|_1 \\
&= \sup \sum_{1\le i\le\infty} \left|(e^{-iH^\Lambda t}\rho^\Lambda e^{iH^\Lambda t}\varphi_i^\Lambda, \psi_i^\Lambda)\right| \\
&= \sup \sum_{1\le i\le\infty} \left|\rho^\Lambda e^{iH^\Lambda t}\varphi_i, e^{iH^\Lambda t}\psi_i\right| = |\rho^\Lambda|_1,
\end{aligned}$$

where the upper bound is taken over all orthonormalized systems of finite, twice differentiable functions with compact support $\{\psi_i^s\}$ and $\{\varphi_i^s\}$ in $L_2^s(\Lambda)$.

Step 2. Operator $\omega^\Lambda(t)$ is strongly continuous on t in the nuclear norm on B^Λ:

The strong continuity of $\omega^\Lambda(t)$ on B^Λ follows from the relations

$$\begin{aligned}\Big|\exp(-iH^\Lambda t)\rho^\Lambda\exp(iH^\Lambda t)-\rho^\Lambda\Big|_1 &= \Big|\exp(-iH^\Lambda t)\rho^\Lambda\exp(iH^\Lambda t)-\rho^\Lambda \\ &\quad+\exp(-iH^\Lambda t)\rho_n^\Lambda\exp(iH^\Lambda t)-\rho_n^\Lambda-\exp(-iH^\Lambda t)\rho_n^\Lambda\exp(iH^\Lambda t)+\rho_n^\Lambda\Big|_1 \\ &\le \Big(\Big|\exp(-iH^\Lambda t)(\rho^\Lambda-\rho_n^\Lambda)\exp(iH^\Lambda t)\Big|_1 \\ &\quad+\Big|\rho^\Lambda-\rho_n^\Lambda\Big|_1+\Big|\exp(-iH^\Lambda t)\rho_n^\Lambda exp(iH^\Lambda t)-\rho_n^\Lambda\Big|_1\Big) \\ &= 2\Big|\rho^\Lambda-\rho_n^\Lambda\Big|_1+\Big|\exp(-iH^\Lambda t)\rho_n^\Lambda\exp(iH^\Lambda t)-\rho_n^\Lambda\Big|_1. \end{aligned} \tag{3.1}$$

The term $2|\rho^\Lambda-\rho_n^\Lambda|_1$ in (3.1) can be made as small as desired because the ρ_n^Λ is dense in the space of nuclear operators [19, 21]. Therefore

$$\Big|\exp(-iH^\Lambda t)\rho^\Lambda\exp(iH^\Lambda t)-\rho^\Lambda\Big|_1 \le \Big|\exp(-iH^\Lambda t)\rho_n^\Lambda\exp(iH^\Lambda t)-\rho_n^\Lambda\Big|_1. \tag{3.2}$$

It follows from (3.2) that

$$\begin{aligned}\lim_{t\to0}\Big|\omega^\Lambda(t)\rho^\Lambda-\rho^\Lambda\Big|_1 &= \lim_{t\to0}\Big|\exp(-iH^\Lambda t)\rho^\Lambda\exp(iH^\Lambda t)-\rho^\Lambda\Big|_1 \\ &\le \lim_{t\to0}\Big|\exp(-iH^\Lambda t)\rho_n^\Lambda\exp(iH^\Lambda t)-\rho_n^\Lambda\Big|_1 \\ &\le \lim_{t\to0}\Big|\exp(-iH^\Lambda t)\rho_n^\Lambda(\exp(iH^\Lambda t)-I)\Big|_1 \\ &\quad+\lim_{t\to0}\Big|(\exp(-iH^\Lambda t)-I)\rho_n^\Lambda\Big|_1 \\ &\le \lim_{t\to0}\Big(\sum_{1\le i\le n}\lambda_i\|\psi_i\|\|(\exp(iH^\Lambda t)-I)\varphi_i\| \\ &\quad+\sum_{1\le i\le n}\lambda_i\|(\exp(-iH^\Lambda t)-I)\psi_i\|\|\varphi_i\|\Big)=0. \end{aligned} \tag{3.3}$$

It follows from the strong continuity of the group $\exp(\mp iH^\Lambda t)$ that

$$\lim_{t\to0}\Big\|(\exp(iH^\Lambda t)-I)\varphi_i\Big\|=0,$$

$$\lim_{t\to0}\Big\|(\exp(-iH^\Lambda t)-I)\psi_i\Big\|=0.$$

In (3.3) we used

$$\Big|\rho_n^\Lambda\Big|_1\le\sum_{1\le i\le n}\lambda_i\|\psi_i\|\|\varphi_i\|,$$

$$\Big|\exp(-iH^\Lambda t)\rho_n^\Lambda(\exp(iH^\Lambda t)-I)\Big|_1\le\sum_{1\le i\le n}\lambda_i\|\psi_i\|\Big\|\Big(\exp(iH^\Lambda t)-I\Big)\varphi_i\Big\|,$$

$$\Big|(\exp(-iH^\Lambda t)-I)\rho_n^\Lambda(t_0)\Big|_1\le\sum_{1\le i\le n}\lambda_i\Big\|\Big(\exp(-iH^\Lambda t)-I\Big)\psi_i\Big\|\|\varphi_i\|,$$

where $\{\varphi_i\}$ and $\{\psi_i\}$ from $L_2^s(\Lambda)$ and $i = 1, 2, \ldots$ are systems of finite, twice differentiable functions with compact support.

Step 3. The operator $\omega^\Lambda(t)$ satisfies the group property:

$$\omega^\Lambda(t_1)\omega^\Lambda(t_2)\rho^\Lambda = \omega^\Lambda(t_1)e^{-iH^\Lambda t_2}\rho^\Lambda e^{iH^\Lambda t_2} = e^{iH^\Lambda t_1}e^{iH^\Lambda t_2}\rho^\Lambda e^{iH^\Lambda t_2}e^{iH^\Lambda t_1}$$
$$= e^{-iH^\Lambda(t_1+t_2)}\rho^\Lambda e^{iH^\Lambda(t_1+t_2)} = \omega^\Lambda(t_1+t_2)\rho^\Lambda.$$

Analogously,

$$\omega^\Lambda(t_2)\omega^\Lambda(t_1)\rho^\Lambda = \omega^\Lambda(t_2+t_1)\rho^\Lambda.$$

Step 4. The generator of the group $\omega^\Lambda(t)$ is defined on B^Λ coincides with $-i\mathcal{H}$ on $\tilde{B}^\Lambda$:

$$\lim_{t\to 0}\Big|\frac{\omega^\Lambda(t)\rho^\Lambda-\rho^\Lambda}{t}\Big|_1 = \lim_{t\to 0}\sup\sum_{1\le i\le\infty}\Big|\Big(e^{-iH^\Lambda t}\rho^\Lambda\frac{e^{iH^\Lambda t}-I}{t}\varphi_i+\frac{e^{iH^\Lambda t}-I}{t}\rho^\Lambda e^{iH^\Lambda t}\varphi_i,\psi_i\Big)\Big|$$
$$= \sup\sum_{1\le i\le\infty}\Big|\Big((\rho^\Lambda iH^\Lambda - iH^\Lambda\rho^\Lambda)\varphi_i,\psi_i\Big)\Big| = \Big|-i[H^\Lambda,\rho^\Lambda]\Big|_1,$$

where the upper bound is taken over all orthonormalized systems of finite, twice differentiable functions with compact support $\{\psi_i^s\}$ and $\{\varphi_i^s\}$ in $L_2^s(\Lambda)$. □

We introduce the operator $\Omega(\Lambda)$ on the space B^Λ by

$$\Big(\Omega(\Lambda)\rho^\Lambda\Big)_s(x_1,\ldots,x_s;x'_1,\ldots,x'_s)$$
$$=\frac{N}{V}\Big(1-\frac{s}{N}\Big)\int_\Lambda\sum_i\rho^\Lambda_{s+1}(x_1,\ldots,x_s,x_{s+1};x'_1,\ldots,x'_s,x_{s+1})g_i^1(x_{s+1})\tilde{g}_i^1(x_{s+1})dx_{s+1},$$

where $g_i^1(x_{s+1})$ is a complete orthonormal system of vectors in the one-particle space $L_2(\Lambda)$.

We also introduce the operator $U^\Lambda(t)$ on B_s^Λ by the formula

$$\rho_s^\Lambda(t,x_1,\ldots,x_s;x'_1,\ldots,x'_s) = (U^\Lambda(t)\rho^\Lambda)_s(x_1,\ldots,x_s;x'_1,\ldots,x'_s)$$
$$= (e^{\Omega(\Lambda)}e^{-iH^\Lambda t}e^{-\Omega(\Lambda)}\rho^\Lambda e^{iH^\Lambda t})_s(x_1,\ldots,x_s;x'_1,\ldots,x'_s). \quad (3.4)$$

The history of derivation of this formula for the case of other bounded potentials is given in [19].

Theorem 3.2. *The operator $U^\Lambda(t)$ generates a strongly continuous group of bounded operators on B^Λ, the generators of which coincide with the operator $\mathcal{H} + Tr_x\mathcal{A}_x\mathcal{D}_x$ on $\tilde{B}^\Lambda$ everywhere dense in B^Λ.*

Proof. The proof is summarized in the following four steps:

Step 1. Let us show that the operator $U^\Lambda(t)$ is bounded on B^Λ. We begin by evaluating the operator Ω^Λ [19, 21].

$$\begin{aligned}
|\Omega(\Lambda)\rho^\Lambda|_1 &= \frac{1}{|\rho^\Lambda|_1}\sum_{s=1}^{\infty}\Big|(\Omega(\Lambda)\rho^\Lambda)_s\Big|_1 \\
&= \frac{1}{|\rho^\Lambda|_1}\sum_{s=1}^{\infty}\sup\sum_i\Big|(\psi_i^s,(\Omega(\Lambda)\rho^\Lambda)_s\varphi_i^s)\Big|_1 \\
&= \frac{1}{|\rho^\Lambda|_1}\sum_{s=1}^{\infty}\sup\sum_i\Big|\frac{N}{V}\left(1-\frac{s}{N}\right)\Big|\Big|(\psi_i^{s+1},\rho_{s+1}^\Lambda\varphi_i^{s+1})\Big|_1 \\
&\le \frac{\max_s|\frac{N}{V}\left(1-\frac{s}{N}\right)|}{|\rho^\Lambda|_1}\sum_{s=1}^{s_0}|\rho_s^\Lambda|_1 \\
&= \max_s\Big|\frac{N}{V}\left(1-\frac{s}{N}\right)\Big|\frac{|\rho^\Lambda|_1}{|\rho^\Lambda|_1} = \max_s\Big|\frac{N}{V}\left(1-\frac{s}{N}\right)\Big| \\
&= \frac{1}{v(\Lambda)} = \text{constant},
\end{aligned} \tag{3.5}$$

where the upper bound is taken over all orthonormal system of vectors $\{\psi_i^s\}$ and $\{\varphi_i^s\}$ in the s-particle space $L_2^s(\Lambda)$ and $\psi_i^{s+1} = g_i\psi_i^s$, $\varphi_i^{s+1} = g_i\varphi_i^s$, $s \ge 1$ and is taken into account that $\rho_s^\Lambda(x_1,\dots,x_s;x_1',\dots,x_s') = 0$, when $s > s_0$, where s_0 is finite.

From the boundedness of the operator $\Omega(\Lambda)$ (3.5), it follows that $e^{\Omega(\Lambda)}$ is bounded $|e^{\pm\Omega(\Lambda)}|_1 \le e^{1/v(\Lambda)}$.

The operator $U^\Lambda(t)$, as a product of the bounded operators of $e^{\pm\Omega(\Lambda)}$ and the unitary operators $e^{\mp iH_s^\Lambda t}$, is bounded and satisfies the estimate $U^\Lambda(t) \le e^{2/v(\Lambda)}$ on B^Λ.

Step 2. Strong continuity of the operator $U^\Lambda(t)$ on B^Λ follows from boundedness of the operator $e^{\pm\Omega(\Lambda)}$ and the strong continuity of the operator $\omega^\Lambda(t)$ on B^Λ [19].

$$\lim_{t\to 0}\Big|e^{\Omega(\Lambda)}\exp(-iH^\Lambda t)e^{-\Omega(\Lambda)}\rho^\Lambda(t_0)exp(iH^\Lambda t) - \rho^\Lambda(t_0)\Big|_1 = 0.$$

Proof is analogously to (3.1)-(3.3).

Step 3. The operator $U^\Lambda(t)$ satisfies the group property on B^Λ:

$$\begin{aligned}
U^\Lambda(t_1)U^\Lambda(t_2)\rho^\Lambda &= U^\Lambda(t_1)e^{\Omega(\Lambda)}e^{-iH^\Lambda t_2}\left(e^{-\Omega(\Lambda)}\rho^\Lambda\right)e^{iH^\Lambda t_2} \\
&= e^{iH^\Lambda t_1}e^{\Omega(\Lambda)}e^{iH^\Lambda t_2}e^{-\Omega(\Lambda)}\rho^\Lambda e^{iH^\Lambda t_2}e^{iH^\Lambda t_1} \\
&= e^{\Omega(\Lambda)}\left(e^{-iH^\Lambda(t_1+t_2)}\left(e^{-\Omega(\Lambda)}\rho^\Lambda\right)e^{iH^\Lambda(t_1+t_2)}\right) = U^\Lambda(t_1+t_2)\rho^\Lambda.
\end{aligned}$$

Analogously,

$$U^\Lambda(t_2)U^\Lambda(t_1)\rho^\Lambda = U^\Lambda(t_1+t_2)\rho^\Lambda.$$

Step 4. The generator of the operator $U^{\Lambda}(t)$ is defined on B^{Λ} consides with $-i(\mathcal{H}^{\Lambda} + Tr_x\mathcal{A}_x^{\Lambda}\mathcal{D}_x^{\Lambda})$ on $\tilde{B}^{\Lambda}$:

The infinitesimal generator of the group $U^{\Lambda}(t)$ is defined on the set of finite sequences of nuclear operators

$$\rho^{\Lambda} = \{\rho_0^{\Lambda}, \rho_1^{\Lambda}(x_1,;x_1'), \ldots, \rho_s^{\Lambda}(x_1,\ldots,x_s;x_1',\ldots,x_s'),\ldots\},$$
$$\rho_s^{\Lambda}(x_1,\ldots,x_s;x_1',\ldots,x_s') = 0, \qquad s > s_0,$$

with the property that the commutator $[H_s^{\Lambda}, \rho_s^{\Lambda}]$ belongs to B_s^{Λ} together with ρ_s^{Λ}. This set is everywhere dence in B^{Λ} and belongs to $D(-i(\mathcal{H}^{\Lambda} + Tr_x\mathcal{A}_x^{\Lambda}\mathcal{D}_x^{\Lambda}))$.

$$\begin{aligned}
\lim_{t\to 0}\Big|\frac{U^{\Lambda}(t)\rho^{\Lambda}-\rho^{\Lambda}}{t}\Big|_1 &= \lim_{t\to 0}\Big|\frac{1}{t}(\omega^{\Lambda}(t)\rho^{\Lambda} - \rho^{\Lambda} + \Omega(\Lambda)\omega^{\Lambda}(t)\rho^{\Lambda} - \omega^{\Lambda}(t)\Omega(\Lambda)\rho^{\Lambda} \\
&\quad + \sum_{n=2}^{N-s}\sum_{k=0}^{n}\frac{(-1)^k}{k!(n-k)!}\Omega^{n-k}(\Lambda)\omega^{\Lambda}(t)\Omega^k(\Lambda)\rho^{\Lambda})\Big|_1 \\
&= \lim_{t\to 0}\Big|(\frac{1}{t}(\omega^{\Lambda}(t)\rho^{\Lambda} - \rho^{\Lambda} + \Omega(\Lambda)\left(\omega^{\Lambda}(t)\rho^{\Lambda} - \rho^{\Lambda}\right) - \left(\omega^{\Lambda}(t)\rho^{\Lambda} - \rho^{\Lambda}\right)\Omega(\Lambda)\rho^{\Lambda} \\
&\quad + \sum_{n=2}^{N-s}\sum_{k=0}^{n}\frac{(-1)^k}{k!(n-k)!}\Omega^{n-k}(\Lambda)\left(\omega^{\Lambda}(t)\rho^{\Lambda} - \rho^{\Lambda}\right)\Omega^k(\Lambda)\rho^{\Lambda})\Big|_1 \\
&= \Big| -i\left(\mathcal{H}^{\Lambda} + Tr_x\mathcal{A}_x^{\Lambda}\mathcal{D}_x^{\Lambda}\right)\rho^{\Lambda} + \sum_{n=2}^{N-s}\sum_{k=0}^{n}\frac{(-1)^k}{k!(n-k)!}\Omega^{n-k}(\Lambda)\mathcal{H}^{\Lambda}\Omega^k(\Lambda)\rho^{\Lambda}\Big|_1 \\
&= \sum_{s=1}^{s=s_0}\Big| -i(\mathcal{H}^{\Lambda} + Tr_{x_{s+1}}\mathcal{A}_x^{\Lambda}\mathcal{D}_x^{\Lambda})\rho_s^{\Lambda} \\
&\quad + \sum_{n=2}^{N-s}\sum_{k=0}^{n}\frac{(-1)^k}{k!(n-k)!}\Omega^{n-k}(\Lambda)[H_s^{\Lambda} + H_{n-k}^{\Lambda} + H_{s,n-k}^{\Lambda}, \Omega^k(\Lambda)\rho_{s+n}^{\Lambda}]\Big|_1 \\
&= \sum_{s=1}^{s=s_0}\Big| -i\left(\mathcal{H}^{\Lambda} + Tr_{x_{s+1}}\mathcal{A}_x^{\Lambda}\mathcal{D}_x^{\Lambda}\right)\rho_s^{\Lambda} + \sum_{n=2}^{N-s}\sum_{k=0}^{n}\frac{(-1)^k}{k!(n-k)!}\Omega^{n}(\Lambda)[H_{s,n-k}^{\Lambda}, \rho_{s+n}^{\Lambda}]\Big|_1 \\
&= \sum_{s=1}^{s=s_0}\Big| -i\left(\mathcal{H}^{\Lambda} + Tr_{x_{s+1}}\mathcal{A}_x^{\Lambda}\mathcal{D}_x^{\Lambda}\right)\rho_s^{\Lambda} + \sum_{n=2}^{N-s}\sum_{k=0}^{n}\frac{(-1)^k(n-k)}{k!(n-k)!}\Omega^{n}(\Lambda)[H_{s,1}^{\Lambda}, \rho_{s+n}^{\Lambda}]\Big|_1 \\
&= \sum_{s=1}^{s=s_0}\Big| -i\left(\mathcal{H}^{\Lambda} + Tr_{x_{s+1}}\mathcal{A}_x^{\Lambda}\mathcal{D}_x^{\Lambda}\right)\rho_s^{\Lambda}\Big|_1 = \Big| -i(\mathcal{H}^{\Lambda} + Tr_x\mathcal{A}_x^{\Lambda}\mathcal{D}_x^{\Lambda})\rho^{\Lambda}\Big|_1.
\end{aligned} \tag{3.6}$$

In (3.6) we took into account that $\rho_s^{\Lambda}(x_1,\ldots,x_s;x_1',\ldots,x_s') = 0$, when $s > s_0$, where s_0 is finite and we used the following identities:

$$\sum_{k=0}^{n}\frac{(-1)^k}{k!(n-k)!}\Omega^{n-k}(\Lambda)\left[H_s, \Omega^k(\Lambda)\rho_{s+n}^{\Lambda}\right] = \sum_{k=0}^{n}\frac{(-1)^k}{k!(n-k)!}\Omega^{n}(\Lambda)\left[H_s, \rho_{s+n}^{\Lambda}\right] = 0,$$
$$\sum_{k=0}^{n}\frac{(-1)^k}{k!(n-k)!} = 0,$$

and

$$\Omega^{n-k}(\Lambda)\left[H^{\Lambda}_{s,n-k}, \Omega^k(\Lambda)\rho^{\Lambda}_{s+n}\right] = 0,$$

since the operators H^{Λ}_{n-k} and $\Omega^k(\Lambda)$ under the sign of the trace commute.

Here

$$\sum_{k=0}^{n} \frac{(-1)^k}{k!(n-k)!}\Omega^{n-k}(\Lambda)\left[H^{\Lambda}_{s,n-k}, \Omega^k(\Lambda)\rho^{\Lambda}_{s+n}\right] = \sum_{k=0}^{n} \frac{(-1)^k}{k!(n-k)!}\Omega^{n}(\Lambda)\left[H^{\Lambda}_{s,n-k}, \rho^{\Lambda}_{s+n}\right]$$

and from identity of particles, we have

$$\sum_{k=0}^{n} \frac{(-1)^k(n-k)}{k!(n-k)!}\Omega^{n}(\Lambda)\left[H^{\Lambda}_{1,n-k}, \rho_{s+n}\right] = 0.$$

So,

$$\begin{aligned}\lim_{t\to 0}\Big|\Big(\frac{U^{\Lambda}(t)\rho^{\Lambda}-\rho^{\Lambda}}{t}\Big)_s(x_1,\dots,x_s;x_1',\dots,x_s') - \Big(-i\left[H^{\Lambda}_s,\rho^{\Lambda}_s\right](x_1,\dots,x_s;x_1',\dots,x_s')\\ +\frac{N}{V}\Big(1-\frac{s}{N}\Big)Tr_{x_{s+1}}\sum_{1\le i\le s}\Big(\phi_{i,s+1}(|x_i-x_{s+1}|)-\phi_{i,s+1}(|x_i'-x_{s+1}|)\Big)\\ \cdot\rho^{\Lambda}_{s+1}(x_1,\dots,x_s,x_{s+1};x_1',\dots,x_s',x_{s+1})\Big)\Big|_1 = 0.\end{aligned}$$

This implies that the infinitesimal operator of the group $U^{\Lambda}(t)$ on B^{Λ}_s by i concides with the operator

$$\left[H^{\Lambda}_s,\right] + \frac{N}{V}\Big(1-\frac{s}{N}\Big)Tr_{x_{s+1}}\sum_{1\le i\le s}\Big(\phi_{i,s+1}(|x_i-x_{s+1}|)-\phi_{i,s+1}(|x_i'-x_{s+1}|)\Big) \quad (3.7)$$

on the right-hand side of the BBGKY hierarchy of quantum kinetic equations on $\tilde{B}^{\Lambda}_s$.

According to [10] and Theorem 2.2 of Chapter XIX of reference [9], since $U^{\Lambda}(t)$ is a strongly continuous semigroup on B^{Λ} with generator on $\tilde{B}^{\Lambda}_s$ which is dense in B^{Λ}_s, the abstract Cauchy problem (2.3), (2.4) associated with operator (3.7) has the unique solution

$$\begin{aligned}\rho^{\Lambda}_s(t,x_1,\dots,x_s;x_1',\dots,x_s') &= \Big(U^{\Lambda}(t)\rho^{\Lambda}\Big)_s(x_1,\dots,x_s;x_1',\dots,x_s')\\ &= \Big(e^{\Omega(\Lambda)}e^{-iH^{\Lambda}t}e^{-\Omega(\Lambda)}\rho^{\Lambda}e^{iH^{\Lambda}t}\Big)_s(x_1,\dots,x_s;x_1',\dots,x_s') \quad (3.8)\end{aligned}$$

for each $\rho^{\Lambda}_s(x_1,\dots,x_s;x_1',\dots,x_s') \subset \tilde{B}^{\Lambda}_s$. For the initial data ρ^{Λ}_s belonging to a certain subset of B^{Λ}_s (to the domain of definition of $D(-i(\mathcal{H}+Tr_{x_{s+1}}\mathcal{A}_x\mathcal{D}_x)_s)$ of the operator $-i(\mathcal{H}+Tr_{x_{s+1}}\mathcal{A}_x\mathcal{D}_x)_s$, which is everewhere dense in B^{Λ}_s, (3.8) is strong solution of Cauchy problem (2.3)-(2.4).

The proof is completed. □

4 Summary

In this paper we have proved the existance of a unique solution for BBGKY's hierarchy of quantum kinetic equations with coulomb potential.

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References

[1] O. Benedetto, F. Castella, R. Esposito, and M. Pulvirenti, From the N-body Schrödinger equation to the quantum Boltzmann equation: a term-by-term convergence result in the weak coupling regime, *Comm. in Math. Phys.* **277** (2008), 1–44.

[2] N. N. Bogolyubov, *Problems of a Dynamical Theory in Statistical Physics*, Moscow 1946; in: *Studies in Statistical Mechanics*, **1** (J. de Boer and G. E. Uhlenbeck, editors), North-Holland, Amsterdam, 1962.

[3] N. N. Bogolyubov, *Lectures on Quantum Statistics*, Gordon & Breach, New York, 1970.

[4] N. N. Bogolyubov, *Selected Works*, [in Russian], **2** Naukova Dumka, Kiev, 1970.

[5] N. N. Bogolyubov and N. N. (Jn.) Bogolyubov, *Introduction to Quantum Statistical Mechanics*, Nauka, Moscow, 1984.

[6] M. Born and H. S. Green, *A General Kinetic Theory of Fluids*, Cambridje University Press, Cambridge, 1949.

[7] C. Cercignani, *Theory and Application of the Boltzmann Equation*, Scottish Academic Press, Edinburgh/Elsevier, New York, 1975.

[8] E. G. D. Cohen, *The generalization of the Boltzmann Equation to Higher densities*, in: Statistical Mechanics at the turn of the decade, Cohen, E.G.D., Marcel Dekker, Inc., New York, 1971.

[9] I. Gohberg, S. Goldberg, and M. A. Kaashoek, *Classes of Linear Operators*, **1**, Birkhäuser Verlag, Basel, 1990.

[10] T. Kato, *Perturbation Theory for Linear Operators*, Springer-Verlag, Berlin, 1966.

[11] F. King, *BBGKY Hierarchy for Positive Potentials*, in: Ph.D. Thesis, Department of Mathematics, University of California at Berkeley, 1975.

[12] J. G. Kirkwood, The statistical mechanical theory of transport processes 1: general theory, *J. Chem. Phys.* **14**, (1946), 180.

[13] J. G. Kirkwood, The statistical mechanical theory of transport processes 2: transport in gazes, *J. Chem. Phys.* **15**, (1947), 72.

[14] E. Lieb, *The Stability of Matter: from atoms to star*, in: Bulletin (New Series) of the American Mathematical Society, **22**, 1990, pp. 1–49.

[15] E. H. Lieb and J. L. Lebowitz, Lectures on the thermodynamic limit for Coulomb systems, in: Statistical Mechanics and Mathematical Problems, Battelle 1971 Recontres, Springer Lecture Notes in Physics **20** (1973), 136–161.

[16] O. Lanford III, *The Evolution of Large Classical Systems, Dynamical Systems, Theory and Applications*, Springer, Berlin, 1975.

[17] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, New York, 1983.

[18] D. Ya. Petrina, On solution of the Bogolyubov kinetic equations: quantum statistics, *Theoret. and Math. Phys.* **13**, (1972), 391–404.

[19] D. Ya. Petrina, *Mathematical Foundation of Quantum Statistical Mechanics, Continuous Systems*, Kluwer Acadec Publishers, Dordrecht, 1995.

[20] D. Ya. Petrina and A. K. Vidibida, Cauchy problem for the Bogolyubov kinetic equations, *Trudi MI ANUSSR*, **136** (1975), 370–378.

[21] M. Yu. Rasulova, Cauchy problem for the Bogolyubov kinetic Equations: Quantum Case. in: Preprint ITP-44R, Kiev (1975); *Docl. Acad. Nauk Uzbek SSR*, **2** (1976), 248–254.

[22] M. Yu. Rasulova, On evolution of perturbations of stationary solutions of Bogolyubov's kinetic equations, *Theoret. and Math. Phys.* **42** (1980), 124–132.

[23] M. Reed and B. Simon, *Methods of Modern Mathematical Physics*, **2**, Academic Press, New York, 1975.

[24] Ya. Sinai and Yu. M. Suhov, On existence theorem for solution of Bogolyubov's chain of equations, *Theoret. and Math. Phys.* **19** (1974), 344–363.

[25] H. Spohn, Kinetic equations from hamiltonian dynamics, *Rev. Mod. Phys.* **52** (1980), 569–615.

[26] H. Spohn, *On the Integrated Form of the BBGKY Hierarchy for Hard Spheres, Large Scale Dynamics of Interacting Particles*, Springer-Verlag, Heydelberg, 1991; math-ph\0605068v1 25 May (2006).

[27] Yu. M. Suhov, Steady solutions of the BBGKY hierarchy and first integrals of the motion of a system of classical particles One-dimensional case, *Theoret. and Math. Phys.* **55** (1983), 78–87.

[28] J. Yvon, *La Theorie Statistique des Fluides*, Actualites Scientifiques et Industrielles, Herrman, Paris, 1935.

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