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Fixed Point Results in G-Metric Space

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Abstract: The purpose of this paper is to study common fixed point theorems for six mappings and sequences of mappings that satisfy certain contractive conditions on a nonsymmetric and noncomplete G-metric space where the completeness is replaced with weaker condition. Our results improve, extend and generalize the corresponding results given by many authors.

Keywords: G-Metric spaces, Symmetric G-Metric spaces, G-convergent and G-Cauchy sequence, Common fixed point, weakly compatible maps.

1 Introduction

The metric space theory plays a major role in mathematics (geometry, topology, analysis ...), computer sciences and applied sciences, such that optimization, economic theories.
In 2005, Zead Mustafa and Brailey Sims introduced a new structure of generalized metric spaces ([6]), which are called G-metric spaces as generalization of metric space(X,d). Many authors in [3-8] proved several fixed point theorems for one map satisfying various contractive conditions on complete G-metric spaces. Abbas et al. in [1] prove a fixed point theorem for one map and several fixed point theorems for two maps in G-metric spaces. The main object of this paper is to prove common fixed point theorems for six mappings and sequences of mappings in$G$-metric spaces where the completeness is replaced with weaker condition. Our results improve, extend and generalize the corresponding results given by many authors.
Definition 1.1[6] Let X be a nonempty set,R^+, the set of all nonnegative real numbers, and let $G : X x X x X \to R^+$ be a function satisfies the following properties:
(1) $G(x,y,z) = 0$ if $x = y = z$,
(2)$G(x,x,y) > 0,\ \forall x,y \in X, x \neq y$,
(3)$G(x,x,y) \leq G(x,y,z),\ \forall x,y,z \in X, z \neq y$,
(4)$G(x,y,z) = G(x,z,y) = G(y,x,z) = \ldots$, (Symmetry in all three variables),
(5) $G(x,y,z) \leq G(x,a,a) + G(a,y,z)$for all$x,y,z,a \in X$, (rectangle inequality).
Then the function G is called a generalized metric, or, more specifically a G-metric on$X$, and the pair (X,G) is called a G-metric space.
Definition 1.2[6] A G-metric space is said to be symmetric if $G(x,y,y) = G(y,x,x)$ for all $x,y \in X$.
Definition 1.3[6] Let (X,G) be a G-metric space, let $\{x_n\}$ be a sequence of points of X, a point $x \in X$ is said to be the limit of the sequence $\{x_n\}$, we say that $\{x_n\}$is G-convergent to x if $\lim_{n,m\to\infty} G(x,x_n,x_m) = 0$.

Thus if $x_n \to x$ in a G-metric space(X,G), then for any$\varepsilon > 0$, there exists $N \in \mathbf{N}$ such that$G(x,x_n,x_m) < \varepsilon$, for all$n,m \geq N$, (through this paper we mean by **N** the set of all natural numbers**)**.

Definition 1.4[6] Let (X,G) be a G-metric space, a sequence $\{x_n\}$ is called G-Cauchy if given$\varepsilon > 0$, there is $N \in \mathbf{N}$ such that $G(x_n,x_m,x_l) < \varepsilon$ for all $n,m,l \geq N$ that is if $G(x_n,x_m,x_l) \to 0$ as $n,m,l \to \infty$.
Definition 1.5[6] A G-metric space(X,G) is said to be G-complete (or complete$G$-metric) if every G-Cauchy sequence in (X,G) is G-convergent in(X,G).
Definition 1.6[6] A G-metric space(X,G) is called symmetric G-metric space if $G(x,y,y) = G(x,x,y)$ for all$x,y \in X$.
Definition 1.7[6] Let (X,G) and (X',G') be G-metric spaces and let $f : X \to X'$ be a function, then f is said to be G-continuous at a point $a \in X$if and only if, given$\varepsilon > 0$, there exists $\delta > 0$ such that$x,y \in X$; and $G(a,x,y) < \delta$ implies$G'(f(a),f(x),f(y)) < \varepsilon$. A function

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f is G-continuous at X if and only if it is G-continuous at all $a \in X$.

Definition 1.8[2] The mappings $A, B : X \to X$ are weakly compatible if they commute at coincidence points. i.e. for each point u in X such that $Au = Bu$, we have $ABu = BAu$.

2 Main Results

Here we start our work with the following theorem.

Theorem 2.1 Let (X,G) be a G-metric space and $A,B,C,I,J,K : X \to X$ be mappings such that
(i) $A(X) \subseteq J(X)$, $B(X) \subseteq I(X)$ and $C(X) \subseteq K(X)$
(ii) $G(Ax,By,Cz)$

$$\leq a\,G(Kx,Jy,Iz) + b\,G(Kx,Jy,By)$$
$$+c\,G(Jy,Iz,Cz) + d\,G(Iz,Kx,Ax),$$

for all x,y and z in X and $0 \leq a+b+c+d < 1$,
(iii) the pairs $\{A,K\}$, $\{C,I\}$ and $\{B,J\}$ are weakly compatible.
Suppose that one of the maps $A(X)$, $B(X)$, $C(X)$, $I(X)$, $J(X)$ and $K(X)$ is complete subspace of X. Then A, B, C, I, J and K have a unique common fixed point u in X.

Proof Let $x_0 \in X$ be an arbitrary point. By (i) there exists $x_1, x_2, x_3 \in X$ such that
$Ax_0 = Jx_1 = y_0$, $Bx_1 = Ix_2 = y_1$ and $Cx_2 = Kx_3 = y_2$.
Consequently, we can define a sequence $\{y_n\}$ in X such that
$y_{3n} = Ax_{3n} = Jx_{3n+1}$, $y_{3n+1} = Bx_{3n+1} = Ix_{3n+2}$ and $y_{3n+2} = Cx_{3n+2} = Kx_{3n+3}$, for all $n = 0,1,2,\ldots$.
Now, we prove $\{y_n\}$ is a G-Cauchy sequence.
Let $G_m = G(y_m, y_{m+1}, y_{m+2})$ and by (ii), we obtain

$$G_{3n} = G(y_{3n}, y_{3n+1}, y_{3n+2}) = G(Ax_{3n}, Bx_{3n+1}, Cx_{3n+2})$$
$$\leq aG(Kx_{3n}, Jx_{3n+1}, Ix_{3n+2}) + bG(Kx_{3n}, Jx_{3n+1}, Bx_{3n+1})$$
$$+cG(Jx_{3n+1}, Ix_{3n+2}, Cx_{3n+2}) + d\,G(Ix_{3n+2}, Kx_{3n}, Ax_{3n})$$
$$\leq aG(y_{3n-1}, y_{3n}, y_{3n+1}) + bG(y_{3n-1}, y_{3n}, y_{3n+1})$$
$$+cG(y_{3n}, y_{3n+1}, y_{3n+2}) + d\,G(y_{3n+1}, y_{3n-1}, y_{3n})$$
$$\leq (a+b+d)\,G_{3n-1} + cG_{3n},$$

which implies, $G_{3n} \leq \alpha\, G_{3n-1}$, where $\alpha = \frac{a+b+d}{1-c} < 1$, since $a+b+c+d < 1$.
From above inequality and by (3), we obtain

$$G(y_n, y_n, y_{n+1}) \leq G(y_n, y_{n+1}, y_{n+2})$$
$$\leq \alpha G(y_{n-1}, y_n y_{n+1}) \leq \ldots \leq \alpha^n G(y_0, y_1, y_2).$$

Then, for all $n,m \in \mathbf{N}$, $n < m$ and above inequality, we obtain that

$$G(y_n, y_m, y_m)$$
$$\leq G(y_n, y_{n+1}, y_{n+1}) + G(y_{n+1}, y_{n+2}, y_{n+2})$$
$$+G(y_{n+2}, y_{n+3}, y_{n+3}) + \ldots + G(y_{m-1}, y_m, y_m)$$
$$\leq \left(\alpha^n + \alpha^{n+1} + \ldots + \alpha^{m-1}\right) G(y_0, y_1, y_2)$$

$\leq \frac{\alpha^n}{1-\alpha} G(y_0, y_1, y_2) \to 0$, as $n, m \to \infty$.
For $n,m,l \in \mathbf{N}$, above inequality and by (5) implies that $G(y_n, y_m, y_l) \leq G(y_n, y_m, y_m) + G(y_l, y_m, y_m) \to 0$, as $n, m, l \to \infty$. So, $\{y_n\}$ is a G-Cauchy sequence. Then the subsequence $\{y_{3n}\} = \{Jx_{3n+1}\} \subseteq J(X)$ is a G-Cauchy sequence in $J(X)$. Suppose that $J(X)$ is complete, therefore by the above, the sequence $\{Jx_{3n+1}\}$ is G-Cauchy and hence $Jx_{3n+1} \to u = Jv \in J(X)$ for some $v \in X$. Hence, the sequence $\{y_n\}$ converges also to u and the subsequence $\{Ax_{3n}\}, \{Bx_{3n+1}\}, \{Cx_{3n+2}\}, \{Kx_{3n}\}$ and $\{Ix_{3n+2}\}$ converge to u.
We shall prove that $Bv = Jv = u$. On using (ii), we obtain that

$$G(Ax_{3n}, Bv, Cx_{3n+2})$$
$$\leq aG(Kx_{3n}, Jv, Ix_{3n+2}) + bG(Kx_{3n}, Jv, Bv).$$
$$+cG(Jv, Ix_{3n+2}, Cx_{3n+2})$$
$$+dG(Ix_{3n+2}, Kx_{3n}, Ax_{3n}).$$

As, $n \to \infty$, we have $G(u, Bv, u) \leq b\,G(u, Bv, u)$ is a contradiction. Thus $Bv = Jv = u$.
Since $\{B,J\}$ is weakly compatible, thus, $BJv = JBv$. Hence, $Bu = Ju$.
Now, we prove that $Bu = u$, if $Bu \neq u$, then

$$G(Ax_{3n}, Bu, Cx_{3n+2})$$
$$\leq aG(Kx_{3n}, Ju, Ix_{3n+2}) + bG(Kx_{3n}, Ju, Bu)$$
$$+cG(Ju, Ix_{3n+2}, Cx_{3n+2})$$
$$+d\,G(Ix_{3n+2}, Kx_{3n}, Ax_{3n})$$

As, $n \to \infty$, we have $G(u, Bu, u) \leq b\,G(u, Bu, u)$ is a contradiction. Thus, $Bu = Ju = u$, that is, u is a common fixed point of B, J.
Since $u = Bu \in B(X) \subseteq I(X)$, hence there exists $w \in X$ such that $Iw = u$. We prove that $Cw = u$. On using (ii), we obtain that

$$G(Ax_{3n}, Bu, Cw)$$
$$\leq aG(Kx_{3n}, Ju, Iw) + bG(Kx_{3n}, Ju, Bu)$$
$$+cG(Ju, Iw, Cw)$$
$$+dG(Iw, Kx_{3n}, Ax_{3n})$$

As, $n \to \infty$, we have $G(u, u, Cw) \leq c\,G(u, u, Cw)$ is a contradiction. Thus, $Cw = Iw = u$, by the weak compatibility of the pair $\{C, I\}$, we have $ICw = ICw$, and so, $Iu = Cu$.
Now, we prove that $Cu = u$, if $Cu \neq u$, then

$$G(Ax_{3n}, u, Cu) = G(Ax_{3n}, Bu, Cu)$$
$$\leq aG(Kx_{3n}, Ju, Iu) + bG(Kx_{3n}, Ju, Bu).$$
$$+cG(Ju, Iu, Cu) + dG(Iu, Kx_{3n}, Ax_{3n})$$

As, $n \to \infty$, we have $G(u,u,Cu) \leq c\,G(u,u,Cu)$ is a contradiction. Thus, $Cu = Iu = u$, that is, u is a common fixed point of C, I.
Similarly, $u = Cu \in C(X) \subseteq K(X)$, hence there exists $p \in X$ such that $Kp = u$. We prove that $Ap = u$. On using (ii), we obtain that

$$G(Ap, u, u) = G(Ap, Bu, Cu)$$
$$\leq aG(Kp, Ju, Iu) + bG(Kp, Ju, Bu)$$
$$+cG(Ju, Iu, Cu) + dG(Iu, Kp, Ap)$$
$$\leq d\,G(Ap, u, u),$$

is a contradiction. Thus,$Ap = Kp = u$, by the weak compatibility of the pair $\{A, K\}$, we have $AKp = KAp$, and so, $Au = Ku$.
Now, we prove that$Au = u$, if $Au \neq u$, then

$$G(Au, u, u) = G(Au, Bu, Cu)$$
$$\leq aG(Ku, Ju, Iu) + bG(Ku, Ju, Bu)$$
$$+cG(Ju, Iu, Cu) + dG(Iu, Ku, Au)$$
$$\leq dG(u, u, Au),$$

is a contradiction. Thus, $Au = Ku = u$, that is, u is a common fixed point of A, K. Then

$$Au = Bu = Cu = Iu = Ju = Ku = u$$

Now, we prove the uniqueness. To see the point $u$is unique, suppose that w is another common fixed point of A, B, C, K, J and $I$with$w \neq u$.

$$G(u, u, w) = G(Au, Bu, Cw)$$
$$\leq aG(Ku, Ju, Iw) + bG(Ku, Ju, Bu)$$
$$+cG(Ju, Iw, Cw) + d\,G(Iw, Ku, Au)$$
$$\leq aG(u, u, w) + cG(u, w, w) + d\,G(w, u, u)$$

By using (5), we have $G(u,u,w) \leq (a+2c+d)G(u,u,w)$ a contradiction. Therefore, $w = u$ is the unique common fixed point of maps A, B, C, I, J and$K$.

If we put $a = b = c = d = q$ in Theorem 2.1, we obtain the following theorem
Theorem 2.2 Let (X, G) be a G-metric space and $A, B, C, I, J, K : X \to X$ be mappings such that
(i)$A(X) \subseteq J(X)$, $B(X) \subseteq I(X)$ and $C(X) \subseteq K(X)$
(ii)$G(Ax, By, Cz)$

$$\leq q \begin{bmatrix} G(Kx, Jy, Iz) + G(Kx, Jy, By) \\ +G(Jy, Iz, Cz) + G(Iz, Kx, Ax) \end{bmatrix}$$

for all x, y and z in $X$and $0 \leq q < {}^1\!/_4$,
(iii) the pairs $\{A, K\}, \{C, I\}$ and $\{B, J\}$ are weakly compatible.
Suppose that one of the maps $A(X), B(X), C(X), I(X), J(X)$ and $K(X)$ is complete subspace of$X$. Then A, B, C, I, J and K have a unique common fixed point u in$X$.

If we put $K = J = I = i$(the identity mapping) in Theorem 2.1, we obtain a common fixed point theorem for three mappings as the following
Theorem 2.3 Let (X, G) be a G-metric space and $A, B, C : X \to X$ be mappings such that

$$G(Ax, By, Cz)$$
$$\leq aG(x, y, z) + bG(x, y, By)$$
$$+cG(y, z, Cz) + d\,G(z, x, Ax),$$

for all x, y and z in $X$and $0 \leq a + b + c + d < 1$. Suppose that one of the mappings $A(X), B(X)$ and $C(X)$ is complete subspace of$X$. Then A, B and Chave a unique common fixed point u in$X$.

In the following theorem, we have a common fixed point results for two mappings
Theorem 2.4 Let (X, G) be a G-metric space, suppose mappings $f, g : X \to X$ satisfy one of the following condition

$$1. G(fx, fy, fz)$$
$$\leq aG(gx, gy, gz) + bG(gx, gy, fy)$$
$$+cG(gy, gz, fz) + dG(gz, gx, fx)$$

or

$$2. G(fx, fy, fz)$$
$$\leq aG(x, gy, gz) + bG(x, gy, fy)$$
$$+cG(gy, gz, fz) + dG(gz, x, fx),$$

or

$$3. G(fx, fy, fz)$$
$$\leq aG(gx, y, gz) + bG(gx, y, fy)$$
$$+cG(y, gz, fz) + dG(gz, gx, fx),$$

or

$$4. G(fx, fy, fz)$$
$$\leq aG(gx, gy, z) + bG(gx, gy, fy)$$
$$+cG(gy, z, fz) + dG(z, gx, fx),$$

or

$$5. G(fx, fy, fz)$$
$$\leq aG(x, y, gz) + bG(x, y, fy)$$
$$+cG(y, gz, fz) + d\,G(gz, x, fx),$$

or

$$6. G(fx, fy, fz)$$
$$\leq aG(x, gy, z) + bG(x, gy, fy)$$
$$+cG(gy, z, fz) + dG(z, x, fx),$$

or

$$7. G(fx, fy, fz)$$
$$\leq aG(gx, y, z) + bG(gx, y, fy)$$

$$+cG(y,z,fz)+dG(z,gx,fx),$$

for all x,y and z in X and $0\leq a+b+c+d<1$. If $f(X)\subseteq g(X)$, f and g are weakly compatible and $f(X)$ or $g(X)$ is complete subspace of X. Then f and g have a unique common fixed point u in X.

Proof To prove that f and g have a unique common fixed point u in X

1. Setting $A=B=C=f$ and $K=J=I=g$ in Theorem 2.1.
2. Setting $A=B=C=f$, $J=I=g$ and $K=i$ (the identity mapping) in Theorem 2.1.
3. Setting $A=B=C=f$, $K=I=g$ and $J=i$ (the identity mapping) in Theorem 2.1.
4. Setting $A=B=C=f$, $K=J=g$ and $I=i$ (the identity mapping) in Theorem 2.1.
5. Setting $A=B=C=f$, $I=g$ and $K=J=i$ (the identity mapping) in Theorem 2.1.
6. Setting $A=B=C=f$, $J=g$ and $K=I=i$ (the identity mapping) in Theorem 2.1.
7. Setting $A=B=C=f$, $K=g$ and $J=I=i$ (the identity mapping) in Theorem 2.1.

Corollary 2.5 The condition 1 in Theorem 2.3

$$G(fx,fy,fz)$$

$$\leq aG(gx,gy,gz)+bG(gx,gy,fy)$$

$$+cG(gy,gz,fz)+dG(gz,gx,fx)$$

improves and is weaker than the conditions of Theorems 2.3-2.6 of [1].

Corollary 2.6 Let (X,G) be a G-metric space, suppose mappings $f,g:X\to X$ satisfy one of the following condition

$$G(fx,fy,fz)$$

$$\leq aG(gx,gy,gz)+b\begin{bmatrix}G(gx,fy,fy)\\+G(gy,fy,fy)\end{bmatrix}$$

$$+c\begin{bmatrix}G(gy,fz,fz)\\+G(gz,fz,fz)\end{bmatrix}+d\begin{bmatrix}G(gz,fx,fx)\\+G(gx,fx,fx)\end{bmatrix},$$

or

$$G(fx,fy,fz),$$

$$\leq aG(gx,gy,gz)+b\begin{bmatrix}G(gx,gy,gy)\\+G(gy,gy,fy)\end{bmatrix}$$

$$+c\begin{bmatrix}G(gy,gz,gz)\\+G(gz,gz,fz)\end{bmatrix}+d\begin{bmatrix}G(gz,gx,gx)\\+G(gx,gx,fx)\end{bmatrix}$$

for all x,y and z in X and $0\leq a+b+c+d<1$. If $f(X)\subseteq g(X)$, f and g are weakly compatible and $f(X)$ or $g(X)$ is complete subspace of X. Then f and g have a unique common fixed point u in X.

Theorem 2.7 Let (X,G) be a G-metric space, suppose mappings $A,I:X\to X$ satisfy one of the following conditions

$$1.G(A^nx,A^ny,A^nz)$$

$$\leq aG(I^mx,I^my,I^mz)+bG(I^mx,I^my,A^ny)$$

$$+cG(I^my,I^mz,A^nz)+dG(I^mz,I^mx,A^nx),$$

or

$$2.G(A^nx,A^ny,A^nz)$$

$$\leq aG(x,I^my,I^mz)+bG(x,I^my,A^ny)$$

$$+cG(I^my,I^mz,A^nz)+dG(I^mz,x,A^nx),$$

or

$$3.G(A^nx,A^ny,A^nz)$$

$$\leq aG(I^mx,y,I^mz)+bG(I^mx,y,A^ny)$$

$$+cG(y,I^mz,A^nz)+dG(I^mz,I^mx,A^nx),$$

or

$$4.G(A^nx,A^ny,A^nz)$$

$$\leq aG(I^mx,I^my,z)+bG(I^mx,I^my,A^ny)$$

$$+cG(I^my,z,A^nz)+dG(z,I^mx,A^nx),$$

or

$$5.G(A^nx,A^ny,A^nz)$$

$$\leq aG(x,y,I^mz)+bG(x,y,A^ny)$$

$$+cG(y,I^mz,A^nz),+dG(I^mz,x,A^nx),$$

or

$$6.G(A^nx,A^ny,A^nz)$$

$$\leq aG(x,I^my,z)+bG(x,I^my,A^ny)$$

$$+cG(I^my,z,A^nz)+dG(z,x,A^nx),$$

or

$$7.G(A^nx,A^ny,A^nz)$$

$$\leq aG(I^mx,y,z)+bG(I^mx,y,A^ny)$$

$$+cG(y,z,A^nz)+dG(z,I^mx,A^nx),$$

for all x,y and z in X and $0\leq a+b+c+d<1$. If $A^n(X)\subseteq I^m(X)$, the pairs $\{A^n,I^m\}$ are weakly compatible and one of the maps $A^n(X)$ or $I^m(X)$ is a complete subspace of X. Then A and I have a unique common fixed point u in X.

Proof To prove that A^n and I^m have a unique common fixed point u in X

1. Setting $A=B=C=A^n$ and $K=J=I=I^m$ in Theorem 2.1.
2. Setting $A=B=C=A^n$, $J=I=I^m$ and $k=i$ (the identity mapping) in Theorem 2.1.
3. Setting $A=B=C=A^n$, $K=I=I^m$ and $J=i$ (the identity mapping) in Theorem 2.1.
4. Setting $A=B=C=A^n$, $K=J=I^m$ and $I=i$ (the identity mapping) in Theorem 2.1.
5. Setting $A=B=C=A^n$, $I=I^m$ and $K=J=i$ (the identity mapping) in Theorem 2.1.
6. Setting $A=B=C=A^n$, $J=I^m$ and $K=I=i$ (the identity mapping) in Theorem 2.1.
7. Setting $A=B=C=A^n$, $K=I^m$ and $J=I=i$ (the identity mapping) in Theorem 2.1.

That is, there exists $u \in X$ such that $A^n u = I^m u = u$.
Since $A^n(Au) = A(A^n u) = Au$, it follows that Au is a fixed point of A^n and I^m and hence
$Au = u$. Similarly, we have $Iu = u$.

Theorem 2.8 Let (X,G) be a G-metric space, suppose $f : X \to X$ satisfy one of the following conditions

$$1.G(fx, fy, fz) \leq aG(x,y,z) + bG(x,y,fy) + cG(y,z,fz) + dG(z,x,fx),$$

or

$$2.G(fx, fy, fz) \leq aG(fx,y,z) + bG(fx,y,fy) + cG(y,z,fz) + dG(z,fx,fx),$$

or

$$3.G(fx, fy, fz) \leq aG(x,fy,z) + bG(x,fy,fy) , + cG(fy,z,fz) + dG(z,x,fx)$$

or

$$4.G(fx, fy, fz) \leq aG(x,y,fz) + bG(x,y,fy) + cG(y,fz,fz) + dG(fz,x,fx),$$

or

$$5.G(fx, fy, fz) \leq aG(fx,fy,z) + bG(fx,fy,fy) + cG(fy,z,fz) + dG(z,fx,fx),$$

or

$$6.G(fx, fy, fz) \leq aG(fx,y,fz) + bG(fx,y,fy) + cG(y,fz,fz) + dG(fz,fx,fx),$$

or

$$7.G(fx, fy, fz) \leq aG(x,fy,fz) + bG(x,fy,fy) + cG(fy,fz,fz) + dG(fz,x,fx),$$

for all x,y and z in X and $0 \leq \alpha < 1$. If $f(X)$ is a complete subspace of X. Then f has a unique common fixed point u in X and f is G continuous at u.

Proof To prove that f has a unique common fixed point u in X

1.Setting $A = B = C = f$ and $K = J = I = i$(the identity mapping) in Theorem 2.1.
2.Setting $A = B = C = K = f$ and $J = I = i$(the identity mapping) in Theorem 2.1.
3.Setting $A = B = C = J = f$ and $K = I = i$(the identity mapping) in Theorem 2.1.
4.Setting $A = B = C = I = f$ and $K = J = i$(the identity mapping) in Theorem 2.1.
5.Setting $A = B = C = K = J = f$ and $I = i$ (the identity mapping) in Theorem 2.1.
6.Setting $A = B = C = K = I = f$ and $J = i$(the identity mapping) in Theorem 2.1.
7.Setting $A = B = C = J = I = f$ and $K = i$(the identity mapping) in Theorem 2.1.

To show that f is G continuous at u, let $\{x_n\} \subseteq X$ be a sequence such that $\lim_{n\to\infty} x_n = u$.
By using 7, we obtain

$$G(fx_n, fu, fx_n) \leq aG(x_n, fu, fx_n) + bG(x_n, fu, fu) + cG(fu, fx_n, fx_n) + dG(fx_n, x_n, fx_n)$$

Since $fu = u$, we deduce that

$$G(fx_n, u, fx_n) \leq aG(x_n, u, fx_n) + bG(x_n, u, u). + cG(u, fx_n, fx_n) + dG(fx_n, x_n, fx_n)$$

But (5) implies that

$$G(x_n, u, fx_n) \leq G(x_n, fx_n, fx_n) + G(fx_n, u, fx_n) \leq G(x_n, u, u) + 2G(fx_n, u, fx_n),$$

$$G(x_n, fx_n, fx_n) \leq G(x_n, u, u) + G(fx_n, u, fx_n)$$

Thus, we obtain that

$$G(fx_n, u, fx_n) \leq \frac{a+b+d}{1-(2a+c+d)} G(x_n, u, u) \to 0,$$

as, $n \to \infty$.
Then, $fx_n \to u = fu$, i.e. f is G continuous at u.

Remarks 2.9

1. Theorem 2.8 improves the Theorem 2.1 of [1]
2.Theorem 2.8 improves and generalizes the results of [3-8]

Theorem 2.10 Let (X,G) be a G-metric space and $A_t, B_j, C_k, I, J, K : X \to X$, for all $t, j, k \in \mathbf{N}$ be mappings such that
(i) there exists $t_0,\ j_0,\ k_0 \in \mathbf{N}$ such that $A_{t_0}(X) \subseteq J(X),\ B_{j_0}(X) \subseteq I(X)$ and $C_{k_0}(X) \subseteq K(X)$
(ii) $G(A_t x, B_j y, C_k z)$

$$\leq aG(Kx, Jy, Iz) + bG(Kx, Jy, B_j y) + cG(Jy, Iz, C_k z) + dG(Iz, Kx, A_t x)$$

for all x, y and z in X and $0 \leq a+b+c+d < 1$,
(iii) the pairs $\{A_{t_0}, K\}$, $\{C_{k_0}, I\}$ and $\{B_{j_0}, J\}$ are weakly compatible.
Suppose that one of the maps $I(X), J(X)$ and $K(X)$ is complete subspace of X. Then A_t, B_j, C_k, I, J and K have a unique common fixed point u in X.
Proof By Theorem 2.1, the mappings $A_{t_0}, B_{j_0}, C_{k_0}, I, J$ and K for some $t_0, j_0, k_0 \in \mathbf{N}$ have a unique common fixed point in X. That is, there exists a unique point $u \in X$ such that

$$A_{t_0}u = B_{j_0}u = C_{k_0}u = Iu = Ju = Ku = u$$

Suppose that there exists $t \in \mathbf{N}$ such that $t \neq t_0$. Then by using (ii), we have

$$G(A_t u, u, u) = G(A_t x, B_{j_0}u, C_{k_0}u)$$
$$\leq aG(Ku, Ju, Iu) + bG(Ku, Ju, B_{j_0}u)$$
$$+cG(Ju, Iu, C_{k_0}u) + dG(Iu, Ku, A_t u)$$
$$\leq dG(u, u, A_t u)$$

is a contradiction. Hence for every $t \in \mathbf{N}$, we have $A_t u = u$. Similarly $B_j u = u$ and $C_k u = u$. Therefore for every $t, j, k \in \mathbf{N}$, we have

$$A_t u = B_j u = C_k u = Iu = Ju = Ku = u.$$

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