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Common Fixed Point Theorem in Partially Ordered Metric Spaces

Vishal Gupta[1,*], *Raman Deep*[2] *and Seema Devi*[3]

[1] Department of Mathematics, Maharishi Markandeshwar University, Mullana-133207, Ambala, Haryana, India
[2] Maharishi Markandeshwar University, Mullana-133207, Ambala, Haryana, India
[3] Green Valley College of Education, Sahapur, Jind, Haryana, India

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Abstract: In this paper, we prove a common tripled fixed point theorem for mapping having mixed monotone property and satisfying a contractive condition in partially ordered metric spaces.

Keywords: Tripled fixed point, mixed monotone property, partially ordered set.

1 Introduction and Preliminaries

The Banach contraction principle is the most celebrated fixed point theorem. Many authors extended the Banach contraction principle to the case of nonlinear contraction mappings. Very first in 2004, Ran and Recuring [1] proved some fixed point Theorems for contraction type mappings in partially ordered metric spaces. Partial ordered metric spaces play an important role in constructing models in the field of computational and domain theory. Recently, Luong and Thuan [2] presented some coupled fixed point theorems for a mixed monotone mapping in a partially ordered metric space, which are generalizations of the results of Bhaskar and Lakshmikantham [3]. Berinde and Borcut [4] introduced the concept of tripled fixed points and proved a fixed point result in partial metric spaces. Some other results in partially ordered metric spaces are mentioned in [5]-[10]. Here, our aim is to prove a common tripled fixed point theorem in partially ordered metric spaces.

Definition 1 *Let $(X, \leq)$ be a partially ordered set. The mapping $F : X^3 \mapsto X$ is said to have the mixed monotone property if for any $x, y, z \in X$ and*

$$\begin{aligned} x_1, x_2 \in X, x_1 \leq x_2 &\implies F(x_1, y, z) \leq F(x_2, y, z), \\ y_1, y_2 \in X, y_1 \leq y_2 &\implies F(x, y_1, z) \geq F(x, y_2, z), \\ z_1, z_2 \in X, z_1 \leq z_2 &\implies F(x, y, z_1) \leq F(x, y, z_2). \end{aligned}$$

Definition 2 *An element $(x, y, z) \in X^3$ is called a tripled fixed point of F, if $F(x, y, z) = x, F(y, x, z) = y$ and $F(z, y, x) = z$.*

2 Main Result

Theorem 1 *Let $(X, \leqslant)$ be a partially ordered set and (X, d) is a complete metric space. Let $F : X \times X \times X \mapsto X$ be a mapping having the mixed monotone property on X such that $\exists\ x_0, y_0, z_0 \in X$ with $x_0 \leqslant F(x_0, y_0, z_0)$, $y_0 \geqslant F(y_0, z_0, y_0)$ and $z_0 \leqslant F(z_0, y_0, z_0)$.*

Suppose there exist non-negative real numbers a_1, a_2, a_3 and a_4 with $a_1 + a_2 + a_3 < 1$ such that

$$\begin{aligned} &d\left(F\left(x,y,z\right),F\left(u,v,w\right)\right) \\ &\quad \leqslant a_1 d\left(x,u\right) \\ &\qquad + a_2 d\left(y,v\right) + a_3 d\left(z,w\right) \\ &\qquad + a_4 \min\left\{\begin{array}{l} d\left(F\left(x,y,z\right),u\right), d\left(F\left(u,v,w\right),x\right), \\ d\left(F\left(x,y,z\right),v\right) d\left(F\left(u,v,w\right),y\right), \\ d\left(F\left(x,y,z\right),w\right), d\left(F\left(u,v,w\right),z\right) \end{array}\right\} \end{aligned} \quad (1)$$

for all $x, y, z, u, v, w \in X$ with $x \geq u, y \leq v$ and $z \geq w$. Also suppose:

(i) F is continuous; or

* Corresponding author e-mail: vishal.gmn@gmail.com

(ii) X has the following properties:
(a) If a non-decreasing sequence $\{x_n\} \mapsto x$, then $x_n \leqslant x$ for all n;
(b) If a non-increasing sequence $\{y_n\} \mapsto y$, then $y_n \geqslant y$ for all n;
(c) If a non-decreasing sequence $\{z_n\} \mapsto z$, then $z_n \leqslant z$ for all n.
Then there exist $x, y, z \in X$ such that $F(x, y, z) = x, F(y, x, y) = y, F(z, y, x) = z$

Proof. Let $x_0, y_0, z_0 \in X$ such that

$$x_0 \leq F(x_0, y_0, z_0), y_0 \geq F(y_0, x_0, y_0) \text{ and } z_0 \leq F(z_0, y_0, x_0). \quad (2)$$

We can choose $x, y_1, z_1 \in X$ such that

$$x_1 = F(x_0, y_0, z_0), y_1 = F(y_0, x_0, y_0) \text{ and } z_1 = F(z_0, y_0, x_0). \quad (3)$$

In this way, we can construct sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ in X such that

$$x_{n+1} = F(x_n, y_n, z_n), y_{n+1} = F(y_n, x_n, y_n) \text{ and } z_{n+1} = F(z_n, y_n, x_n). \quad (4)$$

By induction, we shall show that

$$x_n \leq x_{n+1}, y_{n+1} \leq y_n \quad \text{and} \quad z_n \leq z_{n+1}. \quad (5)$$

For $n = 0$, using (1) and (2), we get $x_0 \leq x_1, y_0 \geq y_1$ and $z_0 \leq z_1$. Thus (5) holds for $n = 0$.

Again consider, (5) holds for some fixed $n \geq 0$. Then since $x_n \leq x_{n+1}, y_{n+1} \leq y_n$ and $z_n \leq z_{n+1}$ and by mixed monotone property of F, we have,

$$\begin{aligned} x_{n+2} &= F(x_{n+1}, y_{n+1}, z_{n+1}) \\ &\geqslant F(x_{n+1}, y_n, z_{n+1}) \\ &\geqslant F(x_{n+1}, y_n, z_n) \\ &\geqslant F(x_n, y_n, z_n) \\ &= x_{n+1}, \end{aligned}$$

$$\begin{aligned} y_{n+2} &= F(y_{n+1}, x_{n+1}, y_{n+1}) \\ &\leqslant F(y_{n+1}, x_n, y_{n+1}) \\ &\leqslant F(y_n, x_n, y_{n+1}) \\ &\leqslant F(y_n, x_n, y_n) \\ &= y_{n+1} \end{aligned}$$

and

$$\begin{aligned} z_{n+2} &= F(z_{n+1}, y_{n+1}, x_{n+1}) \\ &\geqslant F(z_{n+1}, y_{n+1}, x_n) \\ &\geqslant F(z_{n+1}, y_n, x_n) \\ &\geqslant F(z_n, y_n, x_n) \\ &= z_{n+1}. \end{aligned}$$

Hence (5) is true for any $n \in N$. Therefore,

$$\begin{aligned} &x_0 \leqslant x_1 \leqslant x_2 .... \leqslant x_n \leqslant x_{n+1}, \\ &y_0 \geqslant y_1 \geqslant y_2 .... \geqslant y_n \geqslant y_{n+1}, \\ &z_0 \leqslant z_1 \leqslant z_2 .... \leqslant z_n \leqslant z_{n+1}. \end{aligned} \quad (6)$$

Since $x_n \geqslant x_{n-1}, y_n \leqslant y_{n-1}$ and $z_n \geqslant z_{n-1}$, therefore from (1), we get,

$$\begin{aligned} &d(x_{n+1}, x_n) \\ &= d(F(x_n, y_n, z_n), F(x_{n-1}, y_{n-1}, z_{n-1})) \\ &\leqslant a_1 d(x_n, x_{n-1}) + a_2 d(y_n, y_{n-1}) + a_3 d(z_n, z_{n-1}) \\ &+ a_4 \min \left\{ \begin{array}{l} d(F(x_n, y_n, z_n), x_{n-1}), \\ d(F(x_{n-1}, y_{n-1}, z_{n-1}), x_n), \\ d(F(x_n, y_n, z_n), y_{n-1}) \\ d(F(x_{n-1}, y_{n-1}, z_{n-1}), y_n), \\ d(F(x_n, y_n, z_n), z_{n-1}), \\ d(F(x_{n-1}, y_{n-1}, z_{n-1}), z_n) \end{array} \right\} \end{aligned}$$

or

$$d(x_{n+1}, x_n) \leqslant a_1 d(x_n, x_{n-1}) + a_2 d(y_n, y_{n-1}) + a_3 d(z_n, z_{n-1}). \quad (7)$$

Similarly, since $y_{n-1} \geqslant y_n, x_{n-1} \leqslant x_n$ and $z_n \geqslant z_{n-1}$, again using (1), we obtain

$$\begin{aligned} &d(y_n, y_{n+1}) \\ &= d(F(y_{n-1}, x_{n-1}, z_{n-1}), F(y_n, x_n, z_n)) \\ &\leqslant a_1 d(y_{n-1}, y_n) + a_2 d(x_{n-1}, x_n) + a_3 d(z_{n-1}, z_n) \\ &+ a_4 \min \left\{ \begin{array}{l} d(F(y_{n-1}, x_{n-1}, z_{n-1}), y_n), \\ d(F(y_n, x_n, z_n), y_{n-1}), \\ d(F(y_{n-1}, x_{n-1}, z_{n-1}), x_n), \\ d(F(y_n, x_n, z_n), x_{n-1}) \\ d(F(y_{n-1}, x_{n-1}, z_{n-1}), z_n), \\ d(F(y_n, x_n, z_n), z_{n-1}) \end{array} \right\} \end{aligned}$$

or

$$d(y_n, y_{n+1}) \leqslant a_1 d(y_{n-1}, y_n) + a_2 d(x_{n-1}, x_n) + a_3 d(z_{n-1}, z_n). \quad (8)$$

Again, as $z_n \geqslant z_{n-1}, y_n \leqslant y_{n-1}$ and $x_n \geq x_{n-1}$, using (1), we have

$$\begin{aligned} &d(z_{n+1}, z_n) \\ &\quad = d(F(z_n, y_n, x_n), F(z_{n-1}, y_{n-1}, x_{n-1})) \\ &\quad \leqslant a_1 d(z_n, z_{n-1}) + a_2 d(y_n, y_{n-1}) + a_3 d(x_n, x_{n-1}) \\ &\quad + a_4 \min \left\{ \begin{array}{l} d(F(z_n, y_n, x_n), z_{n-1}), \\ d(F(z_{n-1}, y_{n-1}, x_{n-1}), z_n), \\ d(F(z_n, y_n, x_n), y_{n-1}), \\ d(F(z_{n-1}, y_{n-1}, x_{n-1}), y_n), \\ d(F(z_n, y_n, x_n), x_{n-1}) \\ d(F(z_{n-1}, y_{n-1}, x_{n-1}), x_n) \end{array} \right\} \end{aligned}$$

or

$$\begin{aligned} d(z_{n+1}, z_n) &\leqslant a_1 d(z_n, z_{n-1}) + a_2 d(y_n, y_{n-1}) \\ &\quad + a_3 d(x_n, x_{n-1}). \end{aligned} \tag{9}$$

By adding (7), (8) and (9), we deduce,

$$\begin{aligned} &d(x_{n+1}, x_n) + d(y_{n+1}, y_n) + d(z_{n+1}, z_n) \\ &\quad \leqslant (a_1 + a_2 + a_3) \\ &\quad [d(x_n, x_{n-1}) + d(y_n, y_{n-1}) + d(z_n, z_{n-1})]. \end{aligned} \tag{10}$$

On setting,

$$d_n = d(x_{n+1}, x_n) + d(y_{n+1}, y_n) + d(z_{n+1}, z_n)$$

and

$$\delta = a_1 + a_2 + a_3 < 1.$$

From (10), we have,

$$d_n \leqslant \delta d_{n-1} \leqslant \delta^2 d_{n-2} \cdots \leqslant \delta^n d_0. \tag{11}$$

Now, for each $m \geqslant n$, we have,

$$\begin{aligned} d(x_m, x_n) &\leqslant d(x_m, x_{m-1}) \\ &\quad + d(x_{m-1}, x_{m-2}) + \cdots + d(x_{n+1}, x_n) \\ d(y_m, y_n) &\leqslant d(y_m, y_{m-1}) \\ &\quad + d(y_{m-1}, y_{m-2}) + \cdots + d(y_{n+1}, y_n) \end{aligned}$$

and

$$\begin{aligned} d(z_m, z_n) &\leqslant d(z_m, z_{m-1}) \\ &\quad + d(z_{m-1}, z_{m-2}) + \cdots + d(z_{n+1}, z_n) \end{aligned}$$

Therefore, by using (11), we obtain,

$$\begin{aligned} &d(x_m, x_n) + d(y_m, y_n) + d(z_m, z_n) \\ &\quad \leqslant d_{m-1} + d_{m-2} + \cdots + d_n \\ &\quad \leqslant \left(\delta^{m-1} + \delta^{m-2} + \cdots + \delta^n\right) d_0 \\ &\quad \leqslant \frac{\delta^n}{1-\delta} d_0. \end{aligned} \tag{12}$$

Implies

$$\lim_{n,m\to\infty} [d(x_m, x_n) + d(y_m, y_n) + d(z_m, z_n)] = 0.$$

Therefore, $\{x_n\}, \{y_n\}$ and $\{z_n\}$ are Cauchy sequences in X. Since X is a complete metric space. Therefore there exists $x, y, z \in X$ such that

$$\lim_{n\to\infty} x_n = x, \lim_{n\to\infty} y_n = y \quad \text{and} \quad \lim_{n\to\infty} z_n = z \tag{13}$$

Suppose (i) hold, then F is continuous mapping, we have,

$$\begin{aligned} x &= \lim_{n\to\infty} x_n \\ &= \lim_{n\to\infty} F(x_{n-1}, y_{n-1}, z_{n-1}) \\ &= F\left(\lim_{n\to\infty} x_{n-1}, \lim_{n\to\infty} y_{n-1}, \lim_{n\to\infty} z_{n-1},\right) \\ &= F(x, y, z), \\ y &= \lim_{n\to\infty} y_n \\ &= \lim_{n\to\infty} F(y_{n-1}, x_{n-1}, y_{n-1}) \\ &= F\left(\lim_{n\to\infty} y_{n-1}, \lim_{n\to\infty} x_{n-1}, \lim_{n\to\infty} y_{n-1}\right) \\ &= F(y, x, y) \end{aligned}$$

and

$$\begin{aligned} z &= \lim_{n\to\infty} z_n \\ &= \lim_{n\to\infty} F(z_{n-1}, y_{n-1}, x_{n-1}) \\ &= F\left(\lim_{n\to\infty} z_{n-1}, \lim_{n\to\infty} y_{n-1}, \lim_{n\to\infty} x_{n-1}\right) \\ &= F(z, y, x). \end{aligned}$$

Hence, F has a tripled fixed point in X.

Now, suppose (ii) holds.

Using (6) and (13), $\{x_n\}$ is non-decreasing sequence and $\{x_n\} \mapsto x$, $\{y_n\}$ is non-increasing sequence and $\{y_n\} \mapsto y$ and $\{z_n\}$ is a non-decreasing sequence and $\{z_n\} \mapsto z$ as $n \to \infty$.

Hence, by assumption (ii) we have for all $n \geqslant 0$,

$$x_n \leqslant x, y_n \geqslant y \quad \text{and} \quad z_n \leqslant z. \tag{14}$$

Now, we have

$$\begin{aligned} &d(F(x_n, y_n, z_n), F(x, y, z)) \\ &\quad \leqslant a_1 d(x_n, x) + a_2 d(y_n, y) + a_3 d(z_n, z) \\ &\quad + a_4 \min \left\{ \begin{array}{l} d(F(x_n, y_n, z_n), x), \\ d(F(x, y, z), x_n), \\ d(F(x_n, y_n, z_n), y) \\ d(F(x, y, z), y_n), \\ d(F(x_n, y_n, z_n), z), \\ d(F(x, y, z), z_n) \end{array} \right\} \end{aligned} \tag{15}$$

Take $n \to \infty$ in (15), we get

$$d(x, F(x, y, z)) < 0$$

which implies $F(x, y, z) = x$.

Using the same process, we get $F(y, x, y) = y$ and $F(z, y, x) = z$.

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Vishal Gupta is Associate Professor in the Department of Mathematics, Maharishi Markandeshwar University, India. He received his Ph.D degree in 2010. He is also having various other degrees like M.Phil. (Mathematics), M.Ed. and M.Tech(I.T). He has published one book and more than fifty research articles in reputed national and international journals of mathematical and engineering sciences. He has presented more than thirty research papers in prestigious national and international conferences. His main research fields include fixed point theory, Fuzzy Set Theory and Fuzzy Mappings, Topology, Linear Algebra, Operational Research, Differential and integral equations.

Raman Deep is a Research Scholar of Mathematics at Department of Mathematics, Maharishi Markandeshwar University, Mullana, Haryana, India. She has done M.Sc in Mathematics from Kurukshetra University, Kurukshetra, India. She has published many papers in journals and proceedings of the conferences.

Seema Devi is working as Assistant Professor in Department of Mathematics, Green Valley College of Education, Jind, Haryana, India. She has done M.Sc in Mathematics from Maharishi Dayanand University, Rohtak, India and M.Ed from Kurukshetra University, Kurukshetra, India. She has also qualified Net(JRF) conduct by council of Scientific and Industrial Research(CSIR). She has published many papers in journals and proceedings of the conferences.