Information Sciences Letters
An International Journal

http://dx.doi.org/10.12785/isl/030201

An Approximate Algorithm for TSP with Four Vertices and Three Lines Inequality

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Received: 5 Dec. 2013, Revised: 16 Mar. 2014, Accepted: 17 Mar. 2014
Published online: 1 May 2014

Abstract: If the distances of TSP satisfy the triangle inequality, the minimum-cost-spanning tree (MST) heuristics produces a tour whose length is guaranteed to be less than 2 times the optimum tour length and Christofides' heuristics generates the 3/2 times the optimum tour length. Otherwise, the quality of the approximation is hard to evaluate. Here a four vertices and three lines inequality is used to construct an approximation of the optimum tour instead of the triangle inequality. The performance ratio of the heuristics may not be a constant for all kinds of TSP. But it is determined for a concrete TSP.

Keywords: TSP, Approximate algorithm, Four vertices and three lines inequality

1 Introduction

TSP has been proven to be NP-complete [1] in most cases. The number of the Hamiltonian cycles (HC) increases in proportion to the factorial of the number of cities. It is a great challenge to find the optimum Hamiltonian cycle (OHC) in such a large space without heuristics. Due to the theoretical and practical values, TSP has been widely studied in the fields of combinatorial optimization, operation research, computer science etc. in order to resolve it within a reasonable computation time. Unfortunately, $P = NP$ is still one of the great unanswered questions in mathematics [2].

Given an algorithm A, it produces the worst HC whose length is noted as $l(\mathrm{HC}^A)$. Let $l(\mathrm{OHC})$ represent the length of the OHC. The performance of the algorithm A is evaluated by its performance ratio which is defined as $l(\mathrm{HC}^A)/l(\mathrm{OHC})$. If the distances of TSP conform to the triangle inequality, such as the Euclidean TSP, we can find an approximation no less than 2 or 3/2 times the OHC with the MST [3] and Christofides' heuristics [4]. This kind of TSP is called the △-TSP. Many researchers focus on the △-TSP and a lot of achievements has been obtained. Bollig and Capelle [5] integrated the ordered binary decision diagrams (OBDD) into the MST to design the approximate algorithm for metric TSP. It is found that the number of OBDDs changes exponentially according to the scale of TSP.

The approximate ratio $0.999\log_2 n$ is introduced with a cycle cover algorithm for asymmetrical △-TSP [6] and it is improved to $0.842\log_2 n$ by Kaplan et al in 2005 [7]. Bazgan et, al. [8] defined two objective functions and they designed an algorithm for Max TSP. If the two objective functions fulfill the triangle inequality, they obtain the performance ratio 0.41. Otherwise, a performance ratio 0.27 is approximated. The parameterized △-TSP is studied by Thomas and Hans [9] and found that the approximate ratio changes according to a defined triangle factor. In worst case, the approximate ratio becomes infinite if the triangle factor is bigger than 1 with the MST and Christofides' heuristics. Bkenhauer et, al. [10] merged the precedence constraints into the approximate algorithm for parameterized △-TSP and found the performance ratio is determined by the number of ordered vertices and triangle factor. In total, the quality of the approximations is hard to guarantee for the other kind of TSP. Thats to say, a constant performance ratio of an algorithm does not exist for TSP of general distances, unless $P = NP$ [11].

Here the four vertices and three lines inequality [12] is used to compute an approximation of the OHC. The four point conditions for symmetrical TSP are summarized by Deineko, Klinz and Woeginger [13] in 2006. Under their defined restrictions, the OHC of some kinds of TSP will be found within a polynomial computation time. But the others obey the four point

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conditions are still NP-complete. Therefore, it is useful to find an approximation of the hard TSP under the four vertices and three lines inequality. The four vertices and three lines inequality is a universal characteristic of four adjacent vertices whether they obey the triangular inequality or not. The paths obey the inequality illustrate the restrictions of the underlying distance matrices in a weighted graph. Given four adjacent vertices *h*, *i*, *j* and *k*, they will combine 4!/2 paths for symmetrical TSP and half of them satisfy the four vertices and thee lines inequality. In this paper, we will design an approximate algorithm for TSP based on the four vertices and three lines inequality. With the limitative weights of edges, we believe that the performance ratio of the approximate algorithm is able to compute for an arbitrary kind of TSP.

The paper is organized as follows. The four vertices and three lines inequality is introduced in section 2. The performance ratio of the heuristics is computed in section 3. Section 4 gives the summary of the approximate method.

2 The four vertices and three lines inequality

Given an undirected graph G=(V,E), the length of an edge e_{ij} connecting two vertices i and j is noted as $l_{ij}>0$, where $e_{ij}\in E$ and i, $j\in V$. For arbitrary three vertices i, j and k $\in V$, the following inequality (1) holds.

$$l_{ij}+l_{jk}\geq l_{ik} \tag{1}$$

This kind of TSP is called $\triangle$-TSP, such as the Euclidean TSP. Except the $\triangle$-TSP, most of the other TSP are non-$\triangle$-TSP, i.e. not all of the three adjacent vertices i, j and k obey the inequality (1).

Similarly, four vertices h, i, j and $k\in V$ are given. For TSP with more than 4 vertices, they will combine 12 paths shown in Figure 1. These paths are arranged in two columns. The paths in the left column are noted with odd numbers and the paths of right column are noted with even numbers. For the two paths in the same line, such as the 1st and 2nd paths, their two end vertices are identical whereas the two middle vertices are exchanged. Whatever the vertices h, i, j and k obey the triangle inequality, one of the two paths is shorter than the other. For example, if the 5th path is shorter than the 6th path, the inequality (2) holds. This is the four vertices and three lines inequality. In view of Figure 1, total 6 paths satisfy inequality (2).

$$l_{hi}+l_{ij}+l_{jk}\leq l_{hj}+l_{ji}+l_{ik} \tag{2}$$

For symmetrical TSP, the inequality is simplified as l_{hi} + l_{jk} $\leq$ l_{hj} + l_{ik} which is one restriction of the four point conditions [9].

The OHC is composed of n edges and it is also taken as the combinations of n paths composed of four vertices. The four-vertex paths are noted as P^4=(h, i, j, k) (1 $\leq$ h, i, j, k $\leq$ n). All the P^4s in the OHC must satisfy the four

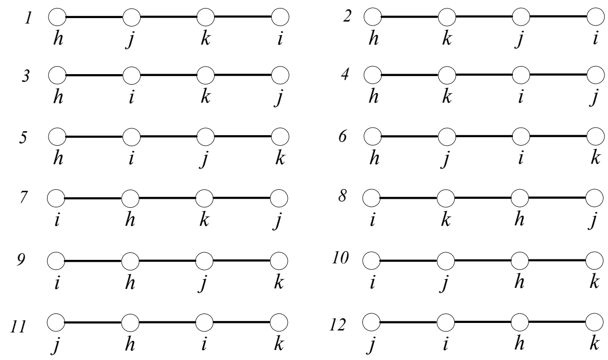

Fig. 1: 12 paths combined with 4 vertices

vertices and three lines inequality although some of them violate the triangle inequality. The P^4 obeys the four vertices and three lines inequality is noted as OP^4. Otherwise, it is represented as CP^4. A CP^4 will become an OP^4 after the two middle vertices are exchanged, such as the pair of the 1st and 2nd paths in Figure 1. Therefore, the CP^4 sets S_1 and the OP^4 sets S_2 are mapped one to one and the function is the four vertices and three lines inequality, i.e., f_4: $CP^4\rightarrow OP^4$, where $CP^4\in S_1$, $OP^4\in S_2$ and f_4 represents the inequality (2). An arbitrary CP^4 in S_1 has one reflection OP^4 in S_2.

When a CP^4 is mapped into its reflection OP^4, we are interested in the error between the CP^4 and the OP^4. Two simple Euclidean paths CP^4=(h, j, i, k) and OP^4=(h, i, j, k) is used to illustrate the error err. They are illustrated in Figure 2, where o is the intersection of edges e_{hj} and e_{ik}. For symmetrical TSP, they own the same middle edge. The error between the CP^4 and OP^4 is defined as formula (3).

$$err=\frac{l_{hj}+l_{ik}-(l_{hi}+l_{jk})}{l_{hi}+l_{jk}}\times 100\% \tag{3}$$

Due to the diversity of TSP, we only give the conclusion that the error err is bigger than 0. When the four vertices and three lines inequality is used to construct the HCs, it will always generate the shorter HCs with the OP^4s. It is also used to improve the HCs by changing the CP^4s into the OP^4s. The middle edges in the CP^4s are neglected in formula (3) and they can represent any paths with more than two vertices. At this time, the function of the four vertices and three lines inequality is the same as that of the 2-opt move [14].

In Figure 2, o is the intersection of edges e_{hj} and e_{ik}. It is clear that the inequality l_{hi} + l_{jk} $\leq$ l_{hj} + l_{ik} holds due to the two triangle inequality l_{hi} $\leq$ l_{oh} + l_{oi} and l_{jk} $\leq$ l_{oj} + l_{ok}. In general, it is not the OHC if one HC include two intersecting edges for Euclidean TSP.

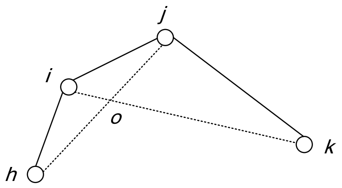

Fig. 2: The illustration of error err with two simple paths

3 The heuristics based on four vertices and three lines inequality

A heuristics is designed based on the four vertices and three lines inequality. It is shown as follows.

Step 1. Input an initial OP^4= (h, j, i, k) into the head of a vacant HC.
Step 2. While(the HC is not full)
Step 3. Compute the next OP^4=(j, i, k, x) with a new vertex x.
Step 4. Input the vertex x into the HC behind and adjacent to k.
Step 5. j:=i, i:=k, k:=x.
Step 6. End

The heuristic algorithm will produce a HC composed of OP^4s. We want to know the performance ratio of the approximate algorithm in worst case.

Total 6 OP^4s are combined with four vertices h, i, j and k. The length of the 6 OP^4s are different. The shortest or the other OP^4s may belong to the OHC. The shortest OP^4 is noted as SOP^4 and the longest OP^4 is noted as LOP^4. In worst case, the SOP^4 belong to the OHC whereas the LOP^4 is generated. The ratio between the LOP^4 and the SOP^4 is defined as formula (4).

$$r = \frac{l(LOP^4)}{l(SOP^4)} \tag{4}$$

For symmetrical TSP with n cities (vertices), the number of the OP^4s is computed as $n(n\text{-}1)(n\text{-}2)(n\text{-}3)/4$. The number of the SOP^4s and LOP^4s is equal to $n(n\text{-}1)(n\text{-}2)(n\text{-}3)/24$, respectively. With the $n(n\text{-}1)(n\text{-}2)(n\text{-}3)/24$ pairs of the SOP^4s and LOP^4s with the same four vertices, $n(n\text{-}1)(n\text{-}2)(n\text{-}3)/24$ ratios will be computed $r_1, r_2, \cdots, r_{\frac{n(n-1)(n-2)(n-3)}{24}}$. The maximal value is noted as r_{max}=max$\{r_1, r_2, \cdots, r_{\frac{n(n-1)(n-2)(n-3)}{24}}\}$. As we know, the OHC is composed of n OP^4s. With the four vertices and three lines inequality, we can compute the r_{max}-approximation in worst case. The approximate ratio r_{max} is not a constant for all kinds of TSP whereas it is determined for a concrete TSP.

We will end the paper with a simple example. The regular square with edge of length 1 is shown in Figure 3. The length of the OHC is 4. The longest LOP^4s are (h, i, k, j), (h, k, i, j), (i, h, j, k) and (i, j, h, k) whose length is equal to 2+$\sqrt{2}$. The shortest SOP^4s are (h, k, j, i), (h, i, j,k), (i, h, k, j) and (j, i, h, k) whose length is equal to 3. r_{max} is computed as $\frac{2+\sqrt{2}}{3}$. In worst case, MST heuristics is guaranteed to produce the 2×4 approximation, Christofides heuristics is guaranteed to produce $3/2\times4$ approximation and our method is guaranteed to produce $\frac{2+\sqrt{2}}{3}\times 4$ approximation. This simple example can not explain this method is better than MST heuristics and Christofides' heuristics. It only proves that an approximation with the worst ratio will be found with the heuristics of four vertices and three lines inequality.

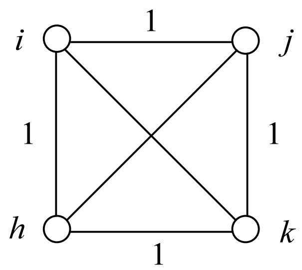

Fig. 3: A regular square with edge of length 1

4 Conclusion

The four vertices and three lines inequality is the characteristic of four adjacent vertices in a weighted graph. The OP^4s illuminate the restrictions of the underlying distance matrices. A heuristics based on the four vertices and three lines inequality is designed to find an approximation of TSP. The performance ratio may be not a constant for all kinds of TSP. However, it is determined for a concrete TSP in worst case. In the future, the experiments will be done to show the r_{max} of TSP instances and the heuristics will be improved to reduce the r_{max}.

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Yong Wang received the Ph.D. degree in aeronautics and astronautics at Beihang University, Beijing, China. His research intrests focus on the graph theory, combinatorial mathematics, data structure, algorithms and their applications, such as the complex network optimization, path optimization, assembly sequence planning, etc. He has published 20 research papers including 7 journal articles in above areas.