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Some Properties of the Taylor Summability Method in Complete Ultrametric Fields

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Abstract: In this paper, we study some properties such as translativity and consistency of the Taylor method of summability in complete, non-trivially valued, ultrametric fields of characteristic zero and also prove few tauberian theorems on such a method .

Keywords: Ultrametric field, Taylor summability method, Tauberian theorem , Mazur-Orlicz theorem

1 Introduction and Preliminaries

Throughout the present paper, K denotes a complete, non-trivially valued, ultrametric field of characteristic zero (Q_p, the p-adic field for a prime p, is one such field). Infinite matrices, sequences, and series considered in the sequel have entries in K. Given an infinite matrix $A = (a_{nk})$, $a_{nk} \in K$, $n,k = 0,1,2,\dots$ and a sequence $x = \{x_k\}$, $x_k \in K$, $k = 0,1,2,\dots$, by the A-transform of $x = \{x_k\}$, we mean the sequence $Ax = \{(Ax)_n\}$, where

$$(Ax)_n = \sum_{k=0}^{\infty} a_{nk} x_k, \quad n = 0,1,2,\dots,$$

it being assumed that the series on the right converge. If $\{(Ax)_n\}$ converges to S, we say that $x = \{x_k\}$ is summable Aor A-summable to s. If $\lim_{n\to\infty}(Ax)_n = s$ whenever $\lim_{k\to\infty} x_k = s$, we say that A is regular. The following theorem, which gives necessary and sufficient conditions for $A = (a_{nk})$ to be regular in terms of the entries of the matrix, is well known (see [4] for a proof using 'Uniform Boundedness Principle' and [5] for a proof using 'Sliding Hump method').

Theorem 1.$A = (a_{nk})$ *is regular if and only if*

1. $\sup_{n,k} |a_{nk}| < \infty$
2. $\lim_{n\to\infty} a_{nk} = 0, \quad k = 0,1,2,\dots,$

   and
3. $\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{nk} = 1.$

An infinite series $\sum_{k=0}^{\infty} x_k$, $x_k \in K$, $k = 0,1,2,\dots$, is said to be A-summable to s if $\{s_n\}$ is A-summable to s, where $s_n = \sum_{k=0}^{\infty} x_k$, $n = 0,1,2,\dots$.

In the present paper, we prove some interesting properties of the Taylor method of summability introduced earlier by Natarajan [9].

General references for the study of summability methods in the classical case are [3,10], while for analysis in ultrametric fields, see [1].

Definition 1.*Let $r \in K$ be such that $|r| < 1$. The Taylor method of order r or the $[T,r]$ method is given by the infinite matrix $(t^{(r)}_{n,k})$ which is defined as follows:*
If $r \neq 0$,

$$t^{(r)}_{n,k} = \begin{cases} kC_n r^{k-n}(1-r)^{n+1}, & k \geq n \\ 0, & k < n \end{cases}$$

If $r = 0$,

$$t^{(r)}_{n,k} = \begin{cases} 1, & k = n \\ 0, & k \neq n \end{cases}$$

$(t^{(r)}_{n,k})$ is called the $[T,r]$ matrix.

*Remark.*We note that $r \neq 1$, since $|r| < 1$.

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The following results are needed in the sequel.

Theorem 2. *Let $x = sup\{|x|/x \in K, |x| < 1\}$. Let $r \in K$ satisfy $|r| < x^{-\frac{1}{x-1}}$. Then the $[T,r]$ method is regular.*

Theorem 3. *The product of the $[T,r]$ and $[T,s]$ matrices is the matrix $(1-r)(1-s)[E,(1-r)(1-s)]$.*

Corollary 1. *The $[T,r]$ matrix is invertible and its inverse is the $\left[T,-\frac{r}{1-r}\right]$ matrix.*

2 Main Results

In this section, we prove some interesting properties of the Taylor method.

Theorem 4(Limitation theorem). *If $\sum_{k=0}^{\infty} x_k$ is $[T,r]$ summable, then $\{x_k\}$ is bounded.*

Proof. Let $\{\sigma_n^{(r)}\}$ be the $[T,r]$ transform of $\{s_n\}$, where $s_n = \sum_{k=0}^{n} x_k$, $n = 0,1,2,\ldots$, i.e.,

$$\sigma_n^{(r)} = \sum_{k=n}^{\infty} kC_n r^{k-n}(1-r)^{n+1}s_k, \quad n = 0,1,2,\ldots.$$

By hypothesis $\lim_{n\to\infty} \sigma_n = \sigma$ (say). So $\{\sigma_n\}$ is bounded.
i.e., there exists $M > 0$ such that $|\sigma_n| \le M$, $n = 0,1,2,\ldots$.
Note that, in view of Corollary 1,

$$\begin{aligned} s_n &= \sum_{k=n}^{\infty} kC_n \left(-\frac{r}{1-r}\right)^{k-n}\left(1+\frac{r}{1-r}\right)^{n+1}\sigma_k, \quad n = 0,1,2,\ldots \\ &= \sum_{k=n}^{\infty} kC_n(-r)^{k-n}(1-r)^{k+1}\sigma_k, \quad n = 0,1,2,\ldots \\ &\le M.\max_{k\ge n}\{|r|^0|1-r|^{-(n+1)}|r||1-r|^{-(n+2)}\ldots\} \\ &\le M, \end{aligned}$$

since $|kC_n| \le 1$, $|r| < 1$, $|1-r| = \max\{|r|,1\} = 1$.
Consequently,

$$|x_k| = |s_k - s_{k-1}| \le \max\{|s_k|,|s_{k-1}|\} \le M, k = 0,1,2,\ldots,$$

so that $\{x_k\}$ is bounded.

Remark. We recall that the classical Mazur-Orticz theorem says that if a conservative matrix sums a bounded divergent sequence, then it sums an unbounded one. It was pointed out in [8] that the above theorem fails to hold in the ultrametric case, a counter examples being any regular (N,p_n) and $[E,r]$ methods. Theorem 4 shows that any $[T,r]$ method is also a counter example to show that the Mazur-Orticz theorem fails to hold in the ultrametric set up.

Definition 2. *Given a sequence $\{x_k\}$, define the sequence $\{\overline{x}_k\}$ by $\overline{x}_n = 0, \overline{x}_k = x_{k-1}, k \ge n, n = 0,1,2,\ldots$. $A = (a_{nk})$ is said to be left translative if the A-summability of $\{x_k\}$ to s implies the A-summability of $\{\overline{x}_k\}$ to s.*

Theorem 5. *$[T,r]$ is right translative but not left.*

Proof. Let $\{\sigma_n(r)\}$ be the $[T,r]$ transform of $\{x_k\}$ and $\{\tau_n(r)\}$ be the $[T,r]$ transform of $\{\overline{x}_k\}$. We shall now prove that

$$\lim_{n\to\infty} \tau_n(r) = s \Rightarrow \lim_{n\to\infty} \sigma_n(r) = s.$$

Now,

$$\begin{aligned} \sigma_n(r) &= \sum_{k=n}^{\infty} kC_n r^{k-n}(1-r)^{n+1}x_k, \text{ since } x_{k-1} = \overline{x}_k \\ &= \sum_{k=n}^{\infty} kC_n r^{k-n}(1-r)^{n+1}\overline{x}_{k+1}, \text{ since } \overline{x}_n = 0 \\ &= \sum_{j=n+1}^{\infty} j-1C_n r^{j-1-n}(1-r)^{n+1}\overline{x}_j, \text{ put } k = j-1 \\ &= \sum_{j=n+1}^{\infty} j-1C_n r^{j-1-n}(1-r)^{n+1}\left(\sum_{k=j}^{\infty} kC_j\left(-\frac{r}{1-r}\right)^{k-j}\left(1+\frac{r}{1-r}\right)^{j+1}\tau_k(r)\right) \\ &= \sum_{k=n+1}^{\infty} r^{k-1-n}(1-r)^{n-k}\tau_k(r)\left(\sum_{j=n+1}^{k}(-1)^{k-j}kC_j j-1C_n\right) \end{aligned}$$

Using the identity

$$\sum_{k=n+1}^{\infty}\left(\sum_{j=n+1}^{k}(-1)^{k-j}j-1C_n\right)z^k = \sum_{k=n+1}^{\infty} z^k,$$

We note that

$$\sum_{k=n+1}^{\infty}(-1)^{k-j}j-1C_n = 1, \quad k \ge n+1. \tag{1}$$

In view of (1), we have

$$\sigma_n^{(r)} = \sum_{k=n+1}^{\infty} r^{k-1-n}(1-r)^{n-k}\tau_k^{(r)}.$$

Since $|r| < 1$, all the conditions of Theorem 1 are fulfilled and so $\lim_{k\to\infty} \tau_k(r) = s$ implies that $\lim_{k\to\infty} \sigma_n(r) = s$. Thus $[T,r]$ is right translative.
Now,

$$\begin{aligned} \tau_n(r) &= \sum_{k=n}^{\infty} kC_n r^{k-n}(1-r)^{n+1}\overline{x}_k \\ &= \sum_{k=n}^{\infty} kC_n r^{k-n}(1-r)^{n+1}x_{k-1} \\ &= \sum_{j=n-1}^{\infty} j+1C_n r^{j+1-n}(1-r)^{n+1}x_j \text{ put } k = j+1 \\ &= \sum_{j=n-1}^{\infty} j+1C_n r^{j+1-n}(1-r)^{n+1}\sum_{k=j}^{\infty} kC_j\left(-\frac{r}{1-r}\right)^{k-j}\left(1+\frac{r}{1-r}\right)^{j+1}\sigma_k(r) \\ &= \sum_{k=n-1}^{\infty} r^{k+1-n}(1-r)^{n-k}\sigma_k(r)\left(\sum_{j=n-1}^{\infty}(-1)^{k-j}kC_j j-1C_n\right) \end{aligned} \tag{2}$$

We note that

$$\sum_{j=n-1}^{k} (-1)^{k-j} kC_j j + 1C_n \neq 1, \quad k \geq n-1 \tag{3}$$

In view of (3) and $|r| < 1$ (2) does not satisfy all the conditions of Theorem 1.
$[T,r]$ is not left translative.

Definition 3.*The inifnite matrix methods $A = (a_{nk})$, $B = (b_{nk})$ are said to be 'consistent' if no sequence is summable to different values by A and B, i.e., if a sequence $\{x_n\}$ is A-summable to ℓ and B summable to m, then $\ell = m$.*

As in the case of regular (N, p_n) methods (see [11], Theorem 4.1) we have the following result.

Theorem 6.*Any two Taylor methods are consistent.*

*Proof.*Consider the Taylor methods $[T,r]$ and $[T,s]$. We then have $|r|$, $|s| < 1$. Let $\{\sigma_n(r)\}$, $\{\tau_n(s)\}$ be the $[T,r]$, $[T,s]$ transforms of $\{x_n\}$ respectively. Let $\lim_{n\to\infty} \sigma_n(r) = \sigma$ and $\lim_{n\to\infty} \tau_n(s) = \tau$. We claim that $\sigma = \tau$. Now,

$$\sigma_n(r) = [T,r](\{x_n\})$$

and

$$\tau_n(s) = [T,s](\{x_n\})$$

So

$$\begin{aligned}\sigma_n(r) &= [T,r][T,s]^{-1}(\{\tau_n(s)\}) \\ &= [T,r]\left[T, -\frac{s}{1-s}\right](\{\tau_n(s)\}), \quad \text{using Corollary1} \\ &= \left[T, \frac{r-s}{1-s}\right](\{\tau_n(s)\}) \quad \text{[see [9]]}\end{aligned} \tag{4}$$

Note that

$$\left|\frac{r-s}{1-s}\right| = |r-s|, \quad \text{since } |1-s| = 1,$$

using

$$\begin{aligned}|s| &< 1 \\ &= |(1-s)-(1-r)| \\ &\leq \max\{|1-s|, |1-r|\} \\ &< 1\end{aligned}$$

so that $\left[T, \frac{r-s}{1-s}\right]$ is regular, in view of Definition 1 and Theorem 2. Using (4), it follows that $\sigma = \tau$, completing the proof.

We shall now prove a few Tauberian theorems for the method $[T,r]$ modelled on those proved for $[N, p_n]$ and $[E,r]$ methods by Natarajan [7] and Deepa et al. [2] respectively.

Theorem 7.*If $\sum_{k=0}^{\infty} a_k$ is $[T,r]$ summable to σ and if $a_n \to \ell$, $n \to \infty$, then $\sum_{k=0}^{\infty} a_k$ converges to σ.*

*Proof.*In view of Theorem 1 of [7], it suffices to prove that the sequence $\{k\}$ of integers is not $[T,r]$ summable. Let $\{\sigma_n(r)\}$ be the $[T,r]$ transform of $\{k\}$, i.e.,

$$\sigma_n(r) = \sum_{k=n}^{\infty} kC_n r^{k-n}(1-r)^{n+1}k, \quad n = 0,1,2,\ldots.$$

Now,

$$\begin{aligned}\sigma_n(r) - \sigma_{n+1}(r) &= \sum_{k=n}^{\infty} kC_n r^{k-n}(1-r)^{n+1}k - \sum_{k=n+1}^{\infty} kC_{n+1} r^{k-(n+1)}(1-r)^{n+2}k \\ &= (1-r)^{n+1}n + \sum_{k=n+1}^{\infty}\left(kC_n r^{k-n}(1-r)^{n+1} - kC_{n+1} r^{k-(n+1)}(1-r)^{n+2}\right)k \\ &= (1-r)^{n+1}n + \sum_{k=n+1}^{\infty} kC_n r^{k-n}(1-r)^{n+1}k - \sum_{k=n+1}^{\infty} kC_{n+1} r^{k-(n+1)}(1-r)^{n+2}k\end{aligned}$$

Using $|r| < 1$, $|1-r| = 1$, $|k| \leq 1$, $k = 0,1,2,\ldots$, we have,

$$\begin{aligned}\left|\sum_{k=n+1}^{\infty} kC_n r^{k-n}(1-r)^{n+1}k\right| &\leq \underset{k\geq n+1}{Max}\{|n+1C_n||r||1-r|^{n+1}, |n+2C_n||r|^2|1-r|^{n+1}, \ldots\} \\ &< Max\{|r||1-r|^{n+1}, |r|^2|1-r|^{n+1}, \ldots\} \\ &< 1, \quad \text{since } |r| < 1 and |1-r| = 1, |n+kC_n| \leq 1.\end{aligned}$$

Similarly,

$$\left|\sum_{k=n+1}^{\infty} kC_n r^{k-n}(1-r)^{n+1}k\right| < 1$$

$$|(1-r)^{n+1}n| = 1, \quad \because |1-r| = 1, |n| \leq 1$$

so that

$$|\sigma_n(r) - \sigma_{n+1}(r)| = 1, \quad n = 0,1,2,\ldots.$$

Thus $\{\sigma_n(r)\}$ is not a Cauchy sequence and hence diverges, i.e., $\{k\}$ is not $[T,r]$ summable, completing the proof.

Using Theorem **??** of [7], we have,

Theorem 8.*If $\sum_{k=0}^{\infty} a_k$ is $[T,r]$ summable to σ and if $a_{n+1} - a_n \to \ell$, $n \to \infty$, then $\sum_{k=0}^{\infty} a_k$ converges to σ.*

As in the case of regular (N, p_n) method ([7], Theorem 5), we have the following thoerem too.

Theorem 9.*If $\sum_{k=0}^{\infty} a_k$ is $[T,r]$ summable, then the following Tauberian conditions are equivalent:*

(i)$a_n \to \ell$, $n \to \infty$;
(ii)$a_{n+1} - a_n \to \ell'$, $n \to \infty$
If, further, $a_n \neq 0$, $n = 0,1,2,\ldots$, each of

(iii) $\frac{a_{n+1}}{a_n} \to \ell, n \to \infty;$
and
(iv) $\frac{a_{n+2}+a_n}{a_{n+1}} \to 2, n \to \infty$

is a weaker Tauberian condition for the $[T,r]$ *summability of* $\sum_{k=0}^{\infty} a_k$.

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