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Commuting Mappings and Generalization of Darbo's Fixed Point Theorem

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Abstract: In this paper, we present a common fixed point theorem for commuting operators which generalizes Darbo's fixed point theorem and some results in the literature. As an application, we study the existence of common solutions of a class of equations in Banach spaces.

Keywords: fixed point theorem; measure of noncompactness; Darbo's fixed point theorem

1 Introduction and Preliminaries

Fixed point theory is one of the most fruitful and effective tools in mathematics which plays an important role in nonlinear analysis (for example see [3,4]). In this paper, we are interested in the existence of a fixed point for commuting mapping $S,\{T_i\}_{i\in I}$ satisfying the following inequalities:

$$\mu(S(A)) \leq \varphi(\sup_{i\in I}(\mu(T_i(A)))), \qquad (1)$$

or

$$\psi(\mu(S(A)) \leq \psi(\mu(T_i(A))) - \varphi(\mu(T_i(A))), \qquad (2)$$

where μ is a measure of noncompactness on the Banach space E, I is the set of indices, S and T_i for $i\in I$ are continuous functions from a closed bounded and convex subset Ω of E into E and $\psi,\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ are nondecreasing functions such that $\lim_{n\to\infty}\varphi^n(t) = 0$ for each $t \geq 0$ and ψ satisfies some certain conditions, specified later. Equation (1) and (2), in the case T_i is the identity function for $i \in I$ has been studied in [2].

At the beginning we provide some notations, definitions and auxiliary facts which will be needed in the sequel. From now on, assume that E is a given Banach space with the norm $\|.\|$ and zero element θ. Denote by $B(x,r)$ the closed ball in E centered at x and with radius r. We write B_r to denote $B(\theta,r)$. If X is a subset of E then the symbols $\overline{X}$, $ConvX$ stand for the closure and the closed convex hull of X, respectively. The algebraic operations on sets will be denoted by $X+Y$ and λX ($\lambda\in\mathbb{R}$). Moreover, we denote by $\mathfrak{M}_E$ the family of all nonempty bounded subsets of E and by $\mathfrak{N}_E$ its subfamily consisting of all relatively compact sets.

Definition 1([5]). *A mapping $\mu : \mathfrak{M}_E \to \mathbb{R}_+$ is said to be measure of noncompactness in E if it satisfies the following conditions:*

(1)*The family $ker\mu = \{X \in \mathfrak{M}_E : \mu(X) = 0\}$ is nonempty and $ker\mu \subset \mathfrak{N}_E$.*
(2)$X \subset Y \Rightarrow \mu(X) \leq \mu(Y)$.
(3)$\mu(\overline{X}) = \mu(X)$.
(4)$\mu(ConvX) = \mu(X)$.
(5)$\mu(\lambda X + (1-\lambda)Y) \leq \lambda\mu(X) + (1-\lambda)\mu(Y)$ *for* $\lambda \in [0,1]$.
(6)*If (X_n) is a nested sequence of closed sets from $\mathfrak{M}_E$ such that $\lim_{n\to\infty}\mu(X_n) = 0$, then the intersection set $X_\infty = \cap_{n=1}^{\infty}X_n$ is nonempty.*

Observe that the intersection set X_∞ from axiom (6) is a member of the kerμ. In fact, since $\mu(X_\infty) \leq \mu(X_n)$ for any n, we have that $\mu(X_\infty) = 0$. This yields that $X_\infty \in ker\mu$.

Definition 2([7]). *A measure μ is called sublinear if it satisfies the following tow conditions:*

(1)$\mu(\lambda X) = |\lambda|\mu(Y)$ *for* $\lambda\in\mathbb{R}$
(2)$\mu(X+Y) \leq \mu(X) + \mu(Y)$

Where $X,Y \in \mathfrak{M}_E$.

Definition 3([7]). *A measure μ satisfying the condition*

$$\mu(X\cup Y) = max\{\mu(X),\mu(Y)\}$$

will be referred to as a measure with maximum property.

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It is worthwhile mentioning that the Kuratowski and Hausdorff measure of noncompactness have maximum property.

Definition 4([10]). *A mapping T of a convex set M is said to be affine if it satisfies the identity*

$$T(kx+(1-k)y) = kTx+(1-k)Ty$$

whenever $0<k<1$, and $x,y \in M$.

Theorem 1([1]). *Let Ω be a nonempty, bounded, closed and convex subset of a Banach space E. Then each continuous and compact map $F:\Omega \to \Omega$ has at least one fixed point in Ω.*

Obviously the above theorem constitutes the well known Schauder fixed point principle. Its generalization, called the Darbo's fixed point theorem, is formulated below.

Theorem 2([8]). *Let Ω be a nonempty, bounded, closed and convex subset of a Banach space E and let $T:\Omega \to \Omega$ be a continuous mapping. Assume that there exists a constant $k \in [0,1)$ such that*

$$\mu(TX) \le k\mu(X)$$

for any nonempty subset X of Ω, where μ is a measure of noncompactness defined in E. Then T has a fixed point in the set Ω.

Lemma 1([2]). *Let $\varphi:\mathbb{R}_+ \to \mathbb{R}_+$ be a nondecreasing and upper semicontinuous function. Then the following two conditions are equivalent:*

$(1)\lim_{n\to\infty}\varphi^n(t)=0$ *for each* $t \ge 0$.
$(2)\varphi(t)<t$ *for any* $t>0$.

2 Main results

Theorem 3. *Let E be a Banach space, Ω be a convex closed bounded subset of E, I be a set of indices, and $\{T_i\}$, S be continuous functions from Ω into Ω such that*

(i) For any $i \in I$, T_i commutes with S.

(ii) For any $A \subset \Omega$ and $i \in I$, we have $T_i(\overline{Conv}(A)) \subset \overline{Conv}(T_i(A))$ where Conv(A) is the convex hull of A .

(iii) There exists an upper semicontinuous and nondecreasing function $\varphi:\mathbb{R}_+ \to \mathbb{R}_+$ where φ is such that $\lim_{n\to\infty}\varphi^n(t)=0$ for each $t \ge 0$ and for any $A \subset \Omega$

$$\mu(S(A)) \le \varphi(\sup_{i\in I}\mu((T_i(A)), \quad (3)$$

whenever μ is an arbitrary measure of noncompactness on E.

Then, we have

(1) The set $\{x \subset \Omega : S(x)=x\}$ is nonempty and compact.

(2) For any $i \in I$, T_i has a fixed point and the set $\{x \in \Omega : T_i(x)=x\}$ is closed and invariant by S.

(3) If T_i is affine and $\{T_i\}_{i\in I}$ is a commuting family, then T_i and S have a common fixed point for every $i \in I$ and the set $\{x\in\Omega : T_i(x)=S(x)=x, \forall i \in I\}$ is compact.

Proof. To prove the first part of theorem we consider the sequence Ω_n defined as $\Omega_0=\Omega$ and $\Omega_n=\overline{Conv}(S(\Omega_{n-1}))$ for $n=1,2,3,\ldots$. Then, we show that

$$\Omega_n \subset \Omega_{n-1} \quad , \quad T_i(\Omega_n) \subset \Omega_n \quad , \quad \mu(\Omega_n) \le \varphi^n(\mu(\Omega_n)) \quad (4)$$

for every $n=1,2,3,\ldots$ and $i \in I$.

It is clear that $\Omega_1 \subset \Omega_0$ and

$$\begin{aligned} T_i(\Omega_1) &\subset \overline{Conv}(S(T_i(\Omega_0))) \\ &\subset \overline{Conv}(S(\Omega_0)) \\ &= \Omega_1. \end{aligned}$$

There for, we have

$$\begin{aligned} \mu(\Omega_1) &= \mu(\overline{Conv}S(\Omega_0)) \\ &= \mu(S(\Omega_0)) \\ &\le \varphi(\sup_{i\in I}\mu(T_i(\Omega_0)) \\ &\le \varphi(\mu(\Omega_0)) \end{aligned}$$

So (4) holds for $n=1$. Assuming now that (4) is true for some $n \ge 1$ and $i \in I$. Then

$$\begin{aligned} \Omega_{n+1} &= \overline{Conv}(S(\Omega_n)) \\ &\subset \overline{Conv}(S(\Omega_{n-1})) \\ &= \Omega_n \end{aligned}$$

and

$$\begin{aligned} T_i(\Omega_{n+1}) &= T_i(\overline{Conv}(S(\Omega_n))) \\ &\subset \overline{Conv}(S(T_i\Omega_n)) \\ &\subset \overline{Conv}(S(\Omega_n)) \\ &= \Omega_{n+1} \end{aligned}$$

for any $i \in I$. Hence, the assertion (4) is true by the induction.

Next, since $\lim_{n\to\infty}\varphi^n(t)=0$ for each $t \ge 0$ and for any $A \subset \Omega$ and $\mu(\Omega_n) \le \varphi^n(\mu(\Omega_n))$, we have $\mu(\Omega_n) \to 0$ as $n \to \infty$. Since the sequence (Ω_n) is nested, in view of axiom (6) of Definition (1), $\Omega_\infty = \bigcap_{n=1}^{\infty}\Omega_n$ is nonempty, closed and convex subset of Ω. Hence Ω_∞ is the member of kerμ. So, Ω_∞ is compact. Next, keeping in mind that S maps Ω_∞ into itself and taking into account the Schauder fixed point principle as Theorem (1) we infer that the operator S has a fixed point x in the set Ω_∞. Obviously $x \in \Omega$. Thus the set $F = \{x \in \Omega : Sx = x\}$ is closed by the continuity of S. On the other hand, T_i commutes with S for any $i \in I$, we see that $T_i x$ is a fixed point of S for any $x \in F$.

Thus $T_i(F) \subset F$, and using lemma (1)

$$\begin{aligned}\mu(F) &= \mu(S(F)) \\ &\le \varphi(\sup_{i\in I}\mu(T_i(F)) \\ &\le \varphi(\mu(F)),\end{aligned}$$

we conclude that $\mu(F) = 0$ and according to the closedness of F, F is compact.

(2) The second part of the theorem has been proved in [10].

(3) For every $i \in I$, F_i is convex since T_i is affine mapping. Also, we have $S(F_i) \subset F_i$ and $T_j(F_i) \subset F_i$ for every $j \in I$ with F_i is convex, closed and bounded, and for any $A \subset F_i$, we get

$$\mu(S(A)) \le \varphi\big(\sup_{j\in I}\mu(T_j(A))\big).$$

Then by using part (1) S has a fixed point in F_i, therefore S and T_i have a common fixed point. Since S is continuous and by the hypothesis (3), we see that the set of common fixed point of S and T_i is a compact.

(4) The fourth part of the theorem has been proved in [10].

*Remark.*In the theorem (3) replacing hypothesis (iii) by the following condition implies that theorem (3) is still correct.

(3*) Suppose that μ is an arbitrary measure of noncompactness and $\psi,\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ are given functions such that φ is lower semicontinuous and ψ is increasing and continuous on $\mathbb{R}_+$. Moreover, $\varphi(0) = 0$ and $\varphi(t) > 0$ for $t > 0$ and

$$\psi(\mu(SA)) \le \psi(\mu(T_iA)) - \varphi(\mu(T_iA)) \qquad (5)$$

for any nonempty subset A of Ω.

*Proof.*To prove this fact, we argue similar to the proof of remark 2.1 in [2]. Let us first observe that from inequality (5) we infer that $\psi(t) - \varphi(t) \ge 0$ for $t \ge 0$. Thus, since the function ψ is invertible and the inverse function ψ^{-1} is defined and continuous on an subinterval of $\mathbb{R}_+$, we can equivalently write inequality (5) in the form

$$\mu(SA) \le \psi^{-1}(\psi(\mu(T_iA)) - \varphi(\mu(T_iA))) \qquad (6)$$

for any $A \in \mathfrak{M}_E$. Further, let us consider the function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ defined by the formula

$$\phi(t) = \psi^{-1}(\psi(t) - \varphi(t)).$$

Observe that ϕ is continuous on $\mathbb{R}_+$. Moreover, inequality (6) can be written in the form

$$\mu(S(A)) \le \varphi(\sup_{i\in I}\mu((T_i(A)))$$

for any $A \in \mathfrak{M}_E$, which has the same form as inequality (3) from Theorem (3). Notice that in view of the fact that the function ψ^{-1} is increasing on $\mathbb{R}_+$ we deduce that for $t > 0$ the following inequality holds

$$\phi(t) = \psi^{-1}(\psi(t) - \varphi(t)) < \psi^{-1}(\psi(t)) = t.$$

Thus, in view of Lemma (1), the function f satisfies the condition $\lim_{n\to\infty}\varphi^n(t) = 0$ for each $t \ge 0$ from Theorem (3). This shows that we can apply Theorem (3) which justifies our above stated assertion.

Theorem 4.*Let E be a Banach space and Ω be a nonempty convex, closed and bounded subset of E. Let T_1, T_2 and S be continuous functions from Ω into Ω such that*

(1) $T_1T_2 = T_2T_1$ and $T_iS = ST_i$ for any $i \in \{1,2\}$.

(2) T_1, T_2 are affine.

(3) There exists an upper semicontinuous and nondecreasing function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\lim_{n\to\infty}\varphi^n(t) = 0$ for each $t \ge 0$ and for any $A \subset \Omega$ we have

$$\mu(S(A)) \le \varphi(\mu(A)).$$

Then the set $\{x \in \Omega : Sx = T_1x = T_2x = x\}$ is nonempty and compact.

*Proof.*To prove this fact, we argue similar to the proof of Theorem 3.2 in [9]. We consider the operator $H(x) = S(T_1(X))$. It is clear that H maps Ω into Ω, commutes with T_1, and is continuous. Moreover, we have

$$\mu(H(A)) = \mu(S(T_1(A))) \le \varphi(\mu(T_1(A)))$$

for any $A \subset \Omega$. Hence, by Theorem (3), H and T_1 have a common fixed point which is a fixed point with S. Thus, the nonempty set $F = \{x \in \Omega : T_1x = x\}$ is closed, convex and bounded subset of Ω, for T_1 being continuous and affine. Moreover, by (1) we have $S(F) \subset F$ and $T_2(F) \subset F$. Therefore, we have

$$\mu(S(T_2(A))) \le \varphi(\mu(T_2(A)))$$

for any $A \subset F$. By the same argument as before, we consider $H_1(x) = ST_2(x)$ for $x \in F$. It follows that the set $\{x \in \Omega : Sx = T_1x = T_2x = x\}$ is nonempty and compact.

3 Application

In this section as an application, we study the existence of common solutions for the following equations:

$$x(t) = f(t, T_1x(t)), \quad (7)$$
$$x(t) = f(t, x(t)), \quad (8)$$
$$x(t) = T_2x(t), \quad (9)$$
$$x(t) = \lambda T_2x(t) + (1-\lambda)f(t, T_1x(t)), \qquad \lambda \in [0,1], \quad (10)$$

under some appropriate assumptions on the functions f, T_1 and T_2 weaker than those in [9]. Let $(E, \|.\|)$ be a Banach space and B be a convex, closed and bounded subset of E. Denote by $C([0,b],B)$ the space of all continuous functions from $[0,b]$; $b > 0$, into B endowed with the norm

$$\|x\|_\infty = \sup_{t\in[0,b)} \|x(t)\|.$$

Assume that
(a) for given fixed $f : [0,b] \times B \to B$, there exists $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ where φ is an upper semicontinuous and nondecreasing function such that $\lim_{n\to\infty} \varphi^n(t) = 0$ for each $t \geq 0$ and

$$\|f(t,x) - f(t,y)\| \leq \varphi(\|x-y\|)$$

for all $x, y \in B$, $t \in [0,b]$;
(b) $T_i : B \to B$ are linear continuous operators, satisfying $T_i(f(t,x)) = f(t, T_i(x))$ for any $(t,x) \in [0,b] \times B$ and $i \in \{1,2\}$.

Theorem 5. *Under hypotheses (a) and (b), equations (7), (8), (9), and (10) have at least one common solution in C([0, b], B).*

Proof. We argue similar to the proof of theorem 3.2 in ([9]). First, it is clear that $C([0,b],B)$ is a closed, bounded and convex subset of $C([0,b],E)$. On the other hand, setting $Sx(t) := f(t,x(t))$, for $x \in C([0,b],B)$, we obtain that

$$\|Sx(t) - Sy(t)\| \leq \varphi(\|x(t) - y(t)\|) \leq \varphi(\|x-y\|_\infty).$$

This implies that

$$\|Sx - Sy\|_\infty \leq \varphi(\|x-y\|_\infty)$$

for any $x, y \in C([0,b],B)$.
Let $\mu : \mathfrak{M}_E \to \mathbb{R}_+$ be defined by the formula

$$\mu(X) = diamX,$$

where $diamX = \sup\{\|x-y\| : x,y \in X\}$ stands for the diameter of X. It is easily seen that μ is a measure of noncompactness in $C([0,b],B)$ (see [6]) in the sense of Definition (1) and

$$\mu(S(A)) \leq \varphi(\mu(A))$$

for any $A \in C([0,b],B)$. Finally, since S and T_i commute, we conclude from Theorem (4) that T_1, T_2, and S have a common fixed point. Therefore, equations (7), (8), (9), and (10) have at least one common solution in $C([0,b],B)$, and the proof is complete.

References

[1] R. P. Agarwal, M. Meehan and D. O'Regan, Fixed Point Theory and Applications, Cambridge University Press, Cambridge, 2004.
[2] A. Aghajani, J. Banaś, N, Sabzali, Some generalizations of Darbo fixed point theorem and applictions, Bull. Belg. Math. Soc. Simon Stevin 20 (2013), 345-358.
[3] A. Aghajani, M. Aliaskari, Generalization of Darbo's fixed point theorem and application. International Journal of Nonlinear Analysis and Applications, 5(2), (86-95), 2014.
[4] A. Aghajani, M. Aliaskari, Measure of noncompactness in Banach algebra and application to the solvability of integral equations in $BC(R_+)$, Inf. Sci. Lett. In press.
[5] J. Banaś, K. Goebel, Measures of Noncompactness in Banach spaces, Lect. Notes Pure Appl. Math., vol. 60, Dekker, New York, 1980.
[6] J. Banaś, On measures of noncompactness in Banach spaces, Comment. Math. Univ. Carolinae 21 (1980) 131-143.
[7] J. Banaś, J. Rocha Martin and K. Sadarangani, On of a quadratic integral equation of Hammerstein type, Math. Comput. Model 43 (2006), 97-104.
[8] G. Darbo, Punti uniti in transformazioni a condominio non compatto, Rend. Sem. Math. Univ. Padova, 24(1955), 84-92.
[9] A. Hajji, A generalization of Darbo's fixed point and common solutions of equations in Banach spaces. Fixed Point Theory and Applications, 2013(1), 1-9.
[10] A.Hajji, E. Hanebaly, Commuting mappings and α-compact type fixed point theorems in locally convex spaces. Int. J. Math. Anal, 1(14), 661-680.

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