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New Fractional Inequalities Involving Saigo Fractional Integral Operator

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Abstract: The aim of this paper is to obtain some new results related to Minkowski fractional integral inequality and other integral inequalities using Saigo fractional integral .

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1 Introduction

In recent years, many researcher have worked on fractional differential and integral inequalities using Riemann-Liouville, Caputo and q-fractional integrals see [1-10, 19]. Some author have studied on Siago fractional integral operator, for example, we refer the reader to [11-18] and references cited therein. In [4], Dahmani established reverse Minkowski fractional integral inequality. Also, in [1] Ahmed Anber and et al., have studied the fractional integral inequalities using Riemann-Liouville fractional integral as follows.
Let $\alpha \geq 0$, $p \geq 1$, $\frac{1}{p}+\frac{1}{q}=1$ and let f, g be two positive function on $[0,\infty[$, such that for all $t>0$, $J^{\alpha}f^{p}(t)<\infty$, $J^{\alpha}g^{q}(t)<\infty$. If $0<m\leq\frac{f(\tau)}{g(\tau)}\leq M<\infty$, $\tau\in[0,t]$ then the inequality

$$[J^{\alpha}f(t)]^{\frac{1}{p}}+[J^{\alpha}g(t)]^{\frac{1}{q}}\leq(\frac{M}{m})^{\frac{1}{pq}}J^{\alpha}\left[(f(t)^{\frac{1}{p}}g(t)^{\frac{1}{q}}\right], \quad (1.1)$$

hold. And
Let $\alpha\geq 0, f$ and g be two positive function on $[0,\infty[$, such that $J^{\alpha}f^{p}(t)<\infty$, $J^{\alpha}g^{q}(t)<\infty$, $t>0$. If $0<m\leq\frac{f(\tau)^{p}}{g(\tau)^{q}}\leq M<\infty$, $\tau\in[0,t]$ then we have

$$[J^{\alpha}f^{p}(t)]^{\frac{1}{p}}+[J^{\alpha}g^{q}(t)]^{\frac{1}{q}}\leq(\frac{M}{m})^{\frac{1}{pq}}J^{\alpha}\left[(f(t)g(t)\right], \quad (1.2)$$

where $p\geq 1$ and $\frac{1}{p}+\frac{1}{q}=1$.

And author have established the following theorem.
Let $\alpha>0$, and f,g be two positive function on $[0,\infty[$, such that f is nondecreasing and g is non-increasing. Then

$$J^{\alpha}f^{\gamma}(t)g^{\delta}(t)\leq\frac{\Gamma(\alpha+1)}{t^{\alpha}}J^{\alpha}f^{\gamma}(t)J^{\alpha}g^{\delta}(t), \quad (1.3)$$

for any $t>0$, $\gamma>0$, $\delta>0$.
For $\beta>0$,

$$\frac{t^{\beta}}{\Gamma(\beta+1)}J^{\alpha}(f^{\gamma}(t)g^{\delta}(t))+\frac{t^{\alpha}}{\Gamma(\alpha+1)}J^{\beta}(f^{\gamma}(t)g^{\delta}(t)) \leq(J^{\alpha}f^{\gamma}(t))(J^{\beta}g^{\delta}(t))+(J^{\alpha}g^{\delta}(t))(J^{\beta}f^{\gamma}(t)). \quad (1.4)$$

In literature few results have been obtained on some fractional integral inequalities using Saigo fractional integral operator in [15, 16]. Motivated from [1, 3, 4], our purpose in this paper is to establish some new results using Saigo fractional integral ([15]). The paper has been organized as follows, in Section 2, we define basic definitions and proposition related to Saigo fractional derivatives and integrals. In Section 3, we give the results about reverse Minkowski fractional integral inequality using fractional Saigo integral, In Section 4, we give some other inequalities using fractional Saigo integral.

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2 Preliminariese

We give some necessary definitions and properties which will be used latter. For more details, see [15, 17].

Definition 1. *A real-valued function $f(x)$, ($x > 1$), is said to be in space C_μ ($\mu \in R$), if there exists a real number $p > \mu$ such that $f(x) = x^p\phi(x)$; where $\phi(x) \in C(0,\infty)$.*

Definition 2. *Let $\alpha > 0$,$\beta,\eta \in R$,then the Saigo fractional integral $I_{0,x}^{\alpha,\beta,\eta}[f(x)]$ of order α for a real-valued continuous function $f(x)$ is defined by([17]), see also ([12, p 19], [14])*

$$I_{0,x}^{\alpha,\beta,\eta}[f(x)] = \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}\,{}_2F_1(\alpha+\beta,-\eta;\alpha;1-\frac{t}{x})f(t)dt. \tag{2.1}$$

where, the function ${}_2F_1(-)$ in the right-hand side of (2.1) is the Gaussian hypergeometric function defined by

$${}_2F_1(a,b;c;x) = \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{x^n}{n!}. \tag{2.2}$$

and $(a)_n$ is the Pochhammer symbol

$$(a)_n = a(a+1)...(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)},\ (a)_0 = 1.$$

For $f(x) = x^\mu$ in (2.1) we have the known result [17] as:

$$I_{0,x}^{\alpha,\beta,\eta}[x^\mu] = \frac{\Gamma(\mu+1)\Gamma(\mu+1-\beta+\eta)}{\Gamma(\mu+1-\beta)\Gamma(\mu+1+\alpha+\eta)}x^{\mu-\beta}. \tag{2.3}$$

$(\alpha > 0, min(\mu,\mu-\beta+\eta) > -1, x > 0)$,

3 Reverse Minkowski fractional integral inequality

In this section, we establish reverse Minkowski fractional integral inequality involving Siago fractional integral operator (2.1).

Theorem 1. *Let $p \geq 1$ and let f, g be two positive function on $[0,\infty)$, such that for all $x > 0$, $I_{0,x}^{\alpha,\beta,\eta}[f^p(x)] < \infty$, $I_{0,x}^{\alpha,\beta,\eta}[g^q(x)] < \infty$. If $0 < m \leq \frac{f(\tau)}{g(\tau)} \leq M$, $\tau \in (0,x)$ we have*

$$\left[I_{0,x}^{\alpha,\beta,\eta}[f^p(x)]\right]^{\frac{1}{p}} + \left[I_{0,x}^{\alpha,\beta,\eta}[g^q(x)]\right]^{\frac{1}{p}} \leq \frac{1+M(m+2)}{(m+1)(M+1)}\left[I_{0,x}^{\alpha,\beta,\eta}[(f+g)^p(x)]\right]^{\frac{1}{p}}, \tag{3.1}$$

for any $\alpha > max\{0,-\beta\}$, $\beta < 1$, $\beta-1 < \eta < 0$.

Proof: Using the condition $\frac{f(\tau)}{g(\tau)} \leq M$, $\tau \in (0,x)$, $x > 0$, we can write

$$(M+1)^p f(\tau) \leq M^p(f+g)^p(\tau). \tag{3.2}$$

Consider

$$\begin{aligned} G(x,\tau) &= \frac{x^{-\alpha-\beta}(x-\tau)^{\alpha-1}}{\Gamma(\alpha)}\,{}_2F_1(\alpha+\beta,-\eta;\alpha;1-\frac{\tau}{x}),\ (\tau\in(0,x);x>0)\\ &= \frac{1}{\Gamma(\alpha)}\frac{(x-\tau)^{\alpha-1}}{x^{\alpha+\beta}} + \frac{(\alpha+\beta)(-\eta)}{\Gamma(\alpha+1)}\frac{(x-\tau)^{\alpha}}{x^{\alpha+\beta+1}}\\ &+ \frac{(\alpha+\beta)(\alpha+\beta+1)(-\eta)(-\eta+1)}{\Gamma(\alpha+2)}\frac{(x-\tau)^{\alpha+1}}{x^{\alpha+\beta+2}} + ... \end{aligned} \tag{3.3}$$

Clearly, we can say that the function $G(x,\tau)$ remain positive because for all $\tau \in (0,x)$, $(x > 0)$ since each term of the (3.3) is positive. Multiplying both side of (3.2) by $G(x,\tau)$, then integrating resulting identity with respect to τ from 0 to x, we get

$$\begin{aligned} &(M+1)^p\frac{x^{-\alpha-\beta}}{\Gamma(\alpha)}\int_0^x(x-\tau)^{\alpha-1}\,{}_2F_1(\alpha+\beta,-\eta;\alpha;1-\frac{\tau}{x})f^p(\tau)d\tau\\ &\leq \frac{M^p}{\Gamma(\alpha)}\int_0^x(x-\tau)^{\alpha-1}\,{}_2F_1(\alpha+\beta,-\eta;\alpha;1-\frac{\tau}{x})(f+g)^p(\tau)d\tau. \end{aligned} \tag{3.4}$$

which is equivalent to

$$I_{0,x}^{\alpha,\beta,\eta}[f^p(x)] \leq \frac{M^p}{(M+1)^p}\left[I_{0,x}^{\alpha,\beta,\eta}[(f+g)^p(x)]\right], \tag{3.5}$$

hence, we can write

$$\left[I_{0,x}^{\alpha,\beta,\eta}[f^p(x)]\right]^{\frac{1}{p}} \leq \frac{M}{(M+1)}\left[I_{0,x}^{\alpha,\beta,\eta}[(f+g)^p(x)]\right]^{\frac{1}{p}}. \tag{3.6}$$

On other hand, using condition $m \leq \frac{f(\tau)}{g(\tau)}$, we obtain

$$(1+\frac{1}{m})g(\tau) \leq \frac{1}{m}(f(\tau)+g(\tau)), \tag{3.7}$$

therefore,

$$(1+\frac{1}{m})^p g^p(\tau) \leq (\frac{1}{m})^p(f(\tau)+g(\tau))^p. \tag{3.8}$$

Now, multiplying both side of (3.8) by $G(x,\tau)$, ($\tau \in (0,x)$, $x > 0$), where $G(x,\tau)$ is defined by (3.3). Then integrating resulting identity with respect to τ from 0 to x, we have

$$\left[I_{0,x}^{\alpha,\beta,\eta}[g^p(x)]\right]^{\frac{1}{p}} \leq \frac{1}{(m+1)}\left[I_{0,x}^{\alpha,\beta,\eta}[(f+g)^p(x)]\right]^{\frac{1}{p}}. \tag{3.9}$$

The inequalities (3.1) follows on adding the inequalities (3.6) and (3.9).
Our second result is as follows.

Theorem 2. *Let $p \geq 1$ and f, g be two positive function on $[0,\infty)$, such that for all $x > 0$, $I_{0,x}^{\alpha,\beta,\eta}[f^p(x)] < \infty$, $I_{0,x}^{\alpha,\beta,\eta}[g^q(x)] < \infty$. If $0 < m \leq \frac{f(\tau)}{g(\tau)} \leq M$, $\tau \in (0,t)$ then we have*

$$\left[I_{0,x}^{\alpha,\beta,\eta}[f^p(x)]\right]^{\frac{2}{p}} + \left[I_{0,x}^{\alpha,\beta,\eta}[g^q(x)]\right]^{\frac{2}{p}} \geq (\frac{(M+1)(m+1)}{M}-2)\left[I_{0,x}^{\alpha,\beta,\eta}[f^p(x)]\right]^{\frac{1}{p}} + \left[I_{0,x}^{\alpha,\beta,\eta}[g^q(x)]\right]^{\frac{1}{p}}. \tag{3.10}$$

Proof: Multiplying the inequalities (3.6) and (3.9), we obtain

$$\frac{(M+1)(m+1)}{M}\left[I_{0,x}^{\alpha,\beta,\eta}[f^p(x)]\right]^{\frac{1}{p}}+\left[I_{0,x}^{\alpha,\beta,\eta}[g^q(x)]\right]^{\frac{1}{p}}\leq\left([I_{0,x}^{\alpha,\beta,\eta}[(f(x)+g(x))^p]]^{\frac{1}{p}}\right)^2. \tag{3.11}$$

Applying Minkowski inequalities to the right hand side of (3.11), we have

$$(\left[I_{0,x}^{\alpha,\beta,\eta}[(f(x)+g(x))^p]\right]^{\frac{1}{p}})^2\leq(\left[I_{0,x}^{\alpha,\beta,\eta}[f^p(x)]\right]^{\frac{1}{p}}+\left[I_{0,x}^{\alpha,\beta,\eta}[g^q(x)]\right]^{\frac{1}{p}})^2, \tag{3.12}$$

which implies that

$$\left[I_{0,x}^{\alpha,\beta,\eta}[(f(x)+g(x))^p]\right]^{\frac{2}{p}}\leq\left[I_{0,x}^{\alpha,\beta,\eta}[f^p(x)]\right]^{\frac{2}{p}}+\left[I_{0,x}^{\alpha,\beta,\eta}[g^q(x)]\right]^{\frac{2}{p}}+2\left[I_{0,x}^{\alpha,\beta,\eta}[f^p(x)]\right]^{\frac{1}{p}}\left[I_{0,x}^{\alpha,\beta,\eta}[g^q(x)]\right]^{\frac{1}{p}}. \tag{3.13}$$

using (3.11) and (3.13) we obtain (3.10). Theorem 2 is thus proved.

4 Fractional integral inequalities involving Saigo fractional integral operator

In this section, we establish some new integral inequalities involving the Siago fractional integral operator (2.1).

Theorem 3. *Let $p>1$, $\frac{1}{p}+\frac{1}{q}=1$ and f,g be two positive function on $[0,\infty)$, such that $I_{0,x}^{\alpha,\beta,\eta}[f(x)]<\infty$, $I_{0,x}^{\alpha,\beta,\eta}[g(x)]<\infty$. If $0<m\leq\frac{f(\tau)}{g(\tau)}\leq M<\infty$, $\tau\in[0,x]$ we have*

$$\left[I_{0,x}^{\alpha,\beta,\eta}[f(x)]\right]^{\frac{1}{p}}\left[I_{0,x}^{\alpha,\beta,\eta}[g(x)]\right]^{\frac{1}{q}}\leq(\frac{M}{m})^{\frac{1}{pq}}\left[I_{0,x}^{\alpha,\beta,\eta}[[f(x)]^{\frac{1}{p}}[g(t)]^{\frac{1}{q}}]\right], \tag{4.1}$$

hold. For all $x>0$, $\alpha>max\{0,-\beta\}$, $\beta<1$, $\beta-1<\eta<0$.

Proof:- Since $\frac{f(\tau)}{g(\tau)}\leq M$, $\tau\in[0,x]$ $x>0$, therefore

$$[g(\tau)]^{\frac{1}{p}}\geq M^{\frac{-1}{q}}[f(\tau)]^{\frac{1}{q}}, \tag{4.2}$$

and also,

$$\begin{aligned}[f(\tau)]^{\frac{1}{p}}[g(\tau)]^{\frac{1}{q}}&\geq M^{\frac{-1}{q}}[f(\tau)]^{\frac{1}{q}}[f(\tau)]^{\frac{1}{p}}\\&\geq M^{\frac{-1}{q}}[f(\tau)]^{\frac{1}{q}+\frac{1}{q}}\\&\geq M^{\frac{-1}{q}}[f(\tau)].\end{aligned} \tag{4.3}$$

Multiplying both side of (4.3) by $G(x,\tau)$, ($\tau\in(0,x)$, $x>0$), where $G(x,\tau)$ is defined by (3.3). Then integrating resulting identity with respect to τ from 0 to x, we have

$$\begin{aligned}&\frac{x^{-\alpha-\beta}}{\Gamma(\alpha)}\int_0^x(x-\tau)^{\alpha-1}{}_2F_1(\alpha+\beta,-\eta;\alpha;1-\frac{\tau}{x})f(\tau)^{\frac{1}{p}}g(\tau)^{\frac{1}{q}}d\tau\\&\leq\frac{M^{\frac{-1}{q}}x^{-\alpha-\beta}}{\Gamma(\alpha)}\int_0^x(x-\tau)^{\alpha-1}{}_2F_1(\alpha+\beta,-\eta;\alpha;1-\frac{\tau}{x})f(\tau)d\tau.\end{aligned} \tag{4.4}$$

which implies that

$$I_{0,x}^{\alpha,\beta,\eta}\left[[f(x)]^{\frac{1}{p}}[g(x)]^{\frac{1}{q}}\right]\leq M^{\frac{-1}{q}}\left[I_{0,x}^{\alpha,\beta,\eta}f(x)\right]. \tag{4.5}$$

Consequently,

$$\left(I_{0,x}^{\alpha,\beta,\eta}\left[[f(x)]^{\frac{1}{p}}[g(x)]^{\frac{1}{q}}\right]\right)^{\frac{1}{p}}\leq M^{\frac{-1}{pq}}\left[I_{0,x}^{\alpha,\beta,\eta}f(x)\right]^{\frac{1}{p}}. \tag{4.6}$$

on other hand, since $mg(\tau)\leq f(\tau)$, $\tau\in[0,x)$, $x>0$, then we have

$$[f(\tau)]^{\frac{1}{p}}\geq m^{\frac{1}{p}}[g(\tau)]^{\frac{1}{p}}, \tag{4.7}$$

multiplying equation (4.7) by $[g(\tau)]^{\frac{1}{q}}$, we have,

$$[f(\tau)]^{\frac{1}{p}}[g(\tau)]^{\frac{1}{q}}\geq m^{\frac{1}{p}}[g(\tau)]^{\frac{1}{q}}[g(\tau)]^{\frac{1}{p}}=m^{\frac{1}{p}}[g(\tau)]. \tag{4.8}$$

Multiplying both side of (4.8) by $G(x,\tau)$, ($\tau\in(0,x)$, $x>0$), where $G(x,\tau)$ is defined by (3.3). Then integrating resulting identity with respect to τ from 0 to x, we have

$$\begin{aligned}&\frac{x^{-\alpha-\beta}}{\Gamma(\alpha)}\int_0^x(x-\tau)^{\alpha-1}{}_2F_1(\alpha+\beta,-\eta;\alpha;1-\frac{\tau}{x})[f(\tau)^{\frac{1}{p}}g(\tau)^{\frac{1}{q}}]d\tau\\&\leq\frac{M^{\frac{1}{p}}x^{-\alpha-\beta}}{\Gamma(\alpha)}\int_0^x(x-\tau)^{\alpha-1}{}_2F_1(\alpha+\beta,-\eta;\alpha;1-\frac{\tau}{x})g(\tau)d\tau.\end{aligned} \tag{4.9}$$

that is,

$$I_{0,x}^{\alpha,\beta,\eta}\left[[f(x)]^{\frac{1}{p}}[g(x)]^{\frac{1}{q}}\right]\leq m^{\frac{1}{p}}\left[I_{0,x}^{\alpha,\beta,\eta}g(x)\right]. \tag{4.10}$$

Hence we can write,

$$\left(I_{0,x}^{\alpha,\beta,\eta}\left[[f(x)]^{\frac{1}{p}}[g(x)]^{\frac{1}{q}}\right]\right)^{\frac{1}{q}}\leq m^{\frac{1}{pq}}\left[I_{0,x}^{\alpha,\beta,\eta}f(x)\right]^{\frac{1}{q}}. \tag{4.11}$$

multiplying equation (4.6) and (4.11) we get the result (4.1).

Theorem 4. *Let f and g be two positive function on $[0,\infty[$, such that $I_{0,x}^{\alpha,\beta,\eta}[f^p(x)]<\infty$, $I_{0,x}^{\alpha,\beta,\eta}[g^q(x)]<\infty$. $x>0$, If $0<m\leq\frac{f(\tau)^p}{g(\tau)^q}\leq M<\infty$, $\tau\in[0,x]$. Then we have*

$$\left[I_{0,x}^{\alpha,\beta,\eta}f^p(x)\right]^{\frac{1}{p}}\left[I_{0,x}^{\alpha,\beta,\eta}g^q(x)\right]^{\frac{1}{q}}\leq(\frac{M}{m})^{\frac{1}{pq}}\left[I_{0,x}^{\alpha,\beta,\eta}(f(x)g(x))\right]\ hold. \tag{4.12}$$

Where $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, for all $x>0$, $\alpha>max\{0,-\beta\}$, $\beta<1$, $\beta-1<\eta<0$.

Proof:- Replacing $f(\tau)$ and $g(\tau)$ by $f(\tau)^p$ and $g(\tau)^q$, $\tau\in[0,x]$, $x>0$ in theorem 3, we obtain (4.12).

Theorem 5. *Let f, g be two positive function on $[0,\infty)$, such that f is non-decreasing and g is non-increasing. Then*

$$I_{0,x}^{\alpha,\beta,\eta}f^\gamma(x)g^\delta(x)\leq\frac{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)}{\Gamma(1-\beta+\eta)}x^\beta I_{0,x}^{\alpha,\beta,\eta}[f^\gamma(x)]I_{0,x}^{\alpha,\beta,\eta}[g^\delta(x)]. \tag{4.13}$$

For all $x>0$, $\alpha>max\{0,-\beta\}$, $\beta<1$, $\beta-1<\eta<0.\gamma>0$ $\delta>0$.

Proof:- let $\tau,\rho \in [0,x]$, $x>0$, for any $\delta>0$, $\gamma>0$, we have

$$\left(f^{\gamma}(\tau)-f^{\gamma}(\rho)\right)\left(g^{\delta}(\rho)-g^{\delta}(\tau)\right)\geq 0. \tag{4.14}$$

$$f^{\gamma}(\tau)g^{\delta}(\rho)-f^{\gamma}(\tau)g^{\delta}(\tau)-f^{\gamma}(\rho)(g^{\delta}(\rho)+f^{\gamma}(\rho)g^{\delta}(\tau)\geq 0. \tag{4.15}$$

Therfore

$$f^{\gamma}(\tau)g^{\delta}(\tau)+f^{\gamma}(\rho)(g^{\delta}(\rho)\leq f^{\gamma}(\tau)g^{\delta}(\rho)+f^{\gamma}(\rho)g^{\delta}(\tau), \tag{4.16}$$

Now, multiplying both side of (4.16) by $G(x,\tau)$, ($\tau\in(0,x)$, $x>0$), where $G(x,\tau)$ is defined by (3.3). Then integrating resulting identity with respect to τ from 0 to x, we have

$$\begin{aligned}&\frac{x^{-\alpha-\beta}}{\Gamma(\alpha)}\int_0^x(x-\tau)^{\alpha-1}{}_2F_1(\alpha+\beta,-\eta;\alpha;1-\frac{\tau}{x})[f^{\gamma}(\tau)g^{\delta}(\tau)]d\tau\\&+f^{\gamma}(\rho)g^{\delta}(\rho)\frac{x^{-\alpha-\beta}}{\Gamma(\alpha)}\int_0^x(x-\tau)^{\alpha-1}{}_2F_1(\alpha+\beta,-\eta;\alpha;1-\frac{\tau}{x})[1]d\tau\\&\leq g^{\delta}(\rho)\frac{x^{-\alpha-\beta}}{\Gamma(\alpha)}\int_0^x(x-\tau)^{\alpha-1}{}_2F_1(\alpha+\beta,-\eta;\alpha;1-\frac{\tau}{x})f^{\gamma}(x)d\tau\\&+f^{\gamma}(x)\frac{x^{-\alpha-\beta}}{\Gamma(\alpha)}\int_0^x(x-\tau)^{\alpha-1}{}_2F_1(\alpha+\beta,-\eta;\alpha;1-\frac{\tau}{x})g^{\delta}(x).\end{aligned} \tag{4.17}$$

$$I_{0,x}^{\alpha,\beta,\eta}[f^{\gamma}(x)g^{\delta}(x)]+f^{\gamma}(\rho)(g^{\delta}(\rho)I_{0,x}^{\alpha,\beta,\eta}[1]\leq g^{\delta}(\rho)I_{0,x}^{\alpha,\beta,\eta}[f^{\gamma}(x)]+f^{\gamma}(\rho)I_{0,x}^{\alpha,\beta,\eta}[g^{\delta}(x)]. \tag{4.18}$$

Again, multiplying both side of (4.18) by $G(x,\rho)$, ($\rho\in(0,x)$, $x>0$), where $G(x,\rho)$ is defined by (3.3). Then integrating resulting identity with respect to ρ from 0 to x, we have

$$\begin{aligned}&I_{0,x}^{\alpha,\beta,\eta}[f^{\gamma}(t)g^{\delta}(x)]I_{0,x}^{\alpha,\beta,\eta}[1]+I_{0,x}^{\alpha,\beta,\eta}[f^{\gamma}(x)g^{\delta}(x)]I_{0,x}^{\alpha,\beta,\eta}[1]\\&\leq I_{0,x}^{\alpha,\beta,\eta}[g^{\delta}(x)]I_{0,x}^{\alpha,\beta,\eta}[f^{\gamma}(x)]+I_{0,x}^{\alpha,\beta,\eta}[f^{\gamma}(x)]I_{0,x}^{\alpha,\beta,\eta}[g^{\delta}(x)],\end{aligned}$$

then we can write

$$2I_{0,x}^{\alpha,\beta,\eta}[f^{\gamma}(x)g^{\delta}(x)]\leq\frac{1}{[I_{0,x}^{\alpha,\beta,\eta}[1]]^{-1}}2I_{0,x}^{\alpha,\beta,\eta}[f^{\gamma}(x)]I_{0,x}^{\alpha,\beta,\eta}[g^{\delta}(x)].$$

This proves the result (4.13).

Theorem 6. *Let f, g be two positive function on $[0,\infty)$, such that f is non-decreasing and g is non-increasing. Then we have*

$$\begin{aligned}&\frac{\Gamma(1-\beta+\eta)}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)x^{\beta}}I_{0,x}^{\psi,\phi,\zeta}[f^{\gamma}(x)g^{\delta}(x)]\\&+\frac{\Gamma(1-\phi+\zeta)}{\Gamma(1-\phi)\Gamma(1+\psi+\zeta)x^{\phi}}I_{0,x}^{\alpha,\beta,\eta}[f^{\gamma}(x)g^{\delta}(x)]\\&\leq\left(I_{0,x}^{\alpha,\beta,\eta}[f^{\gamma}(x)]\right)\left(I_{0,x}^{\psi,\phi,\zeta}[g^{\delta}(x)]\right)+\left(I_{0,x}^{\psi,\phi,\zeta}[g^{\delta}(x)]\right)\left(I_{0,x}^{\alpha,\beta,\eta}[f^{\gamma}(x)]\right).\end{aligned} \tag{4.19}$$

For all $x>0$, $\alpha>max\{0,-\beta\}$, $\psi>max\{0,-\phi\}\beta<1$, $\beta-1<\eta<0$, $\phi<1$, $\phi-1<\zeta<0$,$\gamma>0$ $\delta>0$.

Proof:- Multiplying both side of equation (4.18) by $\frac{x^{-\psi-\phi}}{\Gamma(\psi)}(x-\rho)^{\psi-1}{}_2F_1(\psi+\phi,-\zeta;\psi;1-\frac{\rho}{x})$ ($\rho\in(0,x)$, $x>0$), which (in view of the argument mentioned above in proof of theorem 5) remain positive. Then integrating resulting identity with respect to ρ from 0 to x, we have

$$\begin{aligned}&I_{0,x}^{\alpha,\beta,\eta}[f^{\gamma}(x)g^{\delta}(x)]\frac{x^{-\psi-\phi}}{\Gamma(\psi)}\int_0^x(x-\rho)^{\psi-1}{}_2F_1(\psi+\phi,-\zeta;\psi;1-\frac{\rho}{x})[1]d\rho\\&+I_{0,x}^{\alpha,\beta,\eta}[1]\frac{x^{-\psi-\phi}}{\Gamma(\psi)}\int_0^x(x-\rho)^{\psi-1}{}_2F_1(\psi+\phi,-\zeta;\psi;1-\frac{\rho}{x})f^{\gamma}(\rho)g^{\delta}(\rho)d\rho\\&\leq I_{0,x}^{\alpha,\beta,\eta}[f^{\gamma}(x)]\frac{x^{-\psi-\phi}}{\Gamma(\psi)}\int_0^x(x-\rho)^{\psi-1}{}_2F_1(\psi+\phi,-\zeta;\psi;1-\frac{\rho}{x})g^{\delta}(\rho)d\rho\\&+I_{0,x}^{\alpha,\beta,\eta}[g^{\delta}(x)]\frac{x^{-\psi-\phi}}{\Gamma(\psi)}\int_0^x(x-\rho)^{\psi-1}{}_2F_1(\psi+\phi,-\zeta;\psi;1-\frac{\rho}{x})f^{\gamma}(\rho)d\rho.\end{aligned} \tag{4.20}$$

which implies (4.19). This completes proof.

*Remark.*It may be noted that the inequality (4.13) and (4.19) are reversed if the functions are

$$\left(f^{\gamma}(\tau)-f^{\gamma}(\rho)\right)\left(g^{\delta}(\rho)-g^{\delta}(\tau)\right)\leq 0.$$

*Remark.*For $\alpha=\psi$, $\beta=\phi$ and $\eta=\zeta$, in theorem 6 directly we reduces to the theorem 5.

Theorem 7. *Let $f\geq 0$, $g\geq 0$ be two functions defined on $[0,\infty)$, such that g is non-decreasing. If*

$$I_{0,x}^{\alpha,\beta,\eta}f(x)\geq I_{0,x}^{\alpha,\beta,\eta}g(x),x>o. \tag{4.21}$$

then for all $\alpha>max\{0,-\beta\}$, $\beta<1$, $\beta-1<\eta<0$, $\gamma>0$, $\delta>0$ and $\gamma-\delta>0$, we have

$$I_{0,x}^{\alpha,\beta,\eta}f^{\gamma-\delta}(x)\leq I_{0,x}^{\alpha,\beta,\eta}f^{\gamma}(x)g^{-\delta}(x), \tag{4.22}$$

Proof:-We use arithmetic-geometric inequality, for $\gamma>0$, $\delta>0$, we have:

$$\tfrac{\gamma}{\gamma-\delta}f^{\gamma-\delta}(\tau)-\tfrac{\delta}{\gamma-\delta}g^{\gamma-\delta}(\tau)\leq f^{\gamma}(\tau)g^{-\delta}(\tau),\ \tau\in(0,x),x>0. \tag{4.23}$$

Now, multiplying both side of (4.23) by $G(x,\tau)$, ($\tau\in(0,x)$, $x>0$), where $G(x,\tau)$ is defined by (3.3). Then integrating resulting identity with respect to τ from 0 to x, we have

$$\begin{aligned}&\frac{\gamma}{\gamma-\delta}\frac{x^{-\alpha-\beta}}{\Gamma(\alpha)}\int_0^x(x-\tau)^{\alpha-1}{}_2F_1(\alpha+\beta,-\eta;\alpha;1-\frac{\tau}{x})[f^{\gamma-\delta}(\tau)]d\tau\\&-\frac{\delta}{\gamma-\delta}\frac{x^{-\alpha-\beta}}{\Gamma(\alpha)}\int_0^x(x-\tau)^{\alpha-1}{}_2F_1(\alpha+\beta,-\eta;\alpha;1-\frac{\tau}{x})[g^{\gamma-\delta}(\tau)]d\tau\\&\leq\frac{x^{-\alpha-\beta}}{\Gamma(\alpha)}\int_0^x(x-\tau)^{\alpha-1}{}_2F_1(\alpha+\beta,-\eta;\alpha;1-\frac{\tau}{x})[f^{\gamma}(\tau)g^{-\delta}(\tau)]d\tau.\end{aligned} \tag{4.24}$$

consequently

$$\tfrac{\gamma}{\gamma-\delta}I_{0,x}^{\alpha,\beta,\eta}[f^{\gamma-\delta}(x)]-\tfrac{\delta}{\gamma-\delta}I_{0,x}^{\alpha,\beta,\eta}[g^{\gamma-\delta}(x)]\leq I_{0,x}^{\alpha,\beta,\eta}[f^{\gamma}(x)g^{-\delta}(x)]. \tag{4.25}$$

which implies that

$$\tfrac{\gamma}{\gamma-\delta}I_{0,x}^{\alpha,\beta,\eta}[f^{\gamma-\delta}(x)]\leq I_{0,x}^{\alpha,\beta,\eta}[f^{\gamma}(x)g^{-\delta}(x)]+\tfrac{\delta}{\gamma-\delta}I_{0,x}^{\alpha,\beta,\eta}[g^{\gamma-\delta}(x)]. \tag{4.26}$$

that is

$$I_{0,x}^{\alpha,\beta,\eta}[f^{\gamma-\delta}(x)] \leq \frac{\gamma-\delta}{\gamma} I_{0,x}^{\alpha,\beta,\eta}[f^{\gamma}(t)g^{-\delta}(x)] + \frac{\delta}{\gamma} I_{0,x}^{\alpha,\beta,\eta}[f^{\gamma-\delta}(x)]. \quad (4.27)$$

thus we get the result (4.22).

Theorem 8. *Suppose that f, g and h be positive and continuous functions on $[0,\infty)$, such that*

$$(g(\tau)-g(\rho))\left(\frac{f(\rho)}{h(\rho)}-\frac{f(\tau)}{h(\tau)}\right) \geq 0;\ \tau,\rho \in [0,x)\ \ x>0, \quad (4.28)$$

then for all $x>0$, $\alpha > max\{0,-\beta\}$, $\beta<1$, $\beta-1<\eta<0$, we have

$$\frac{I_{0,x}^{\alpha,\beta,\eta}[f(x)]}{I_{0,x}^{\alpha,\beta,\eta}[h(x)]} \geq \frac{I_{0,x}^{\alpha,\beta,\eta}[(gf)(x)]}{I_{0,x}^{\alpha,\beta,\eta}[(gh)(x)]}. \quad (4.29)$$

Proof: Since f, g and h be three positive and continuous functions on $[0,\infty[$ by (4.28), we can write

$$g(\tau)\frac{f(\rho)}{h(\rho)} + g(\rho)\frac{f(\tau)}{h(\tau)} - g(\rho)\frac{f(\rho)}{h(\rho)} - g(\tau)\frac{f(\tau)}{h(\tau)} \geq 0;\ \ \tau,\rho \in (0,x),\ \ x>0. \quad (4.30)$$

Now, multiplying equation (4.30) by $h(\rho)h(\tau)$, on both side, we have,

$$g(\tau)f(\rho)h(\tau) - g(\tau)f(\tau)h(\rho) - g(\rho)f(\rho)h(\tau) + g(\rho)f(\tau)h(\rho) \geq 0, \quad (4.31)$$

Now multiplying equation (4.31) by $G(x,\tau)$, ($\tau \in (0,x)$, $x>0$), where $G(x,\tau)$ is defined by (3.3). Then integrating resulting identity with respect to τ from 0 to x, we have

$$\begin{aligned}
& f(\rho)\frac{x^{-\alpha-\beta}}{\Gamma(\alpha)}\int_0^x (x-\tau)^{\alpha-1}{}_2F_1(\alpha+\beta,-\eta;\alpha;1-\frac{\tau}{x})[g(\tau)h(\tau)]d\tau \\
& -h(\rho)\frac{x^{-\alpha-\beta}}{\Gamma(\alpha)}\int_0^x (x-\tau)^{\alpha-1}{}_2F_1(\alpha+\beta,-\eta;\alpha;1-\frac{\tau}{x})[f(\tau)g(\tau)]d\tau \\
& +f(\rho)g(\rho)\frac{x^{-\alpha-\beta}}{\Gamma(\alpha)}\int_0^x (x-\tau)^{\alpha-1}{}_2F_1(\alpha+\beta,-\eta;\alpha;1-\frac{\tau}{x})[h(\tau)]d\tau \\
& g(\rho)h(\rho)\frac{x^{-\alpha-\beta}}{\Gamma(\alpha)}\int_0^x (x-\tau)^{\alpha-1}{}_2F_1(\alpha+\beta,-\eta;\alpha;1-\frac{\tau}{x})[f(\tau)]d\tau \geq 0,
\end{aligned} \quad (4.32)$$

we get

$$\begin{aligned}
& f(\rho)I_{0,x}^{\alpha,\beta,\eta}[(gh)(x)] + g(\rho)h(\rho)I_{0,x}^{\alpha,\beta,\eta}[f(x)] \\
& -g(\rho)f(\rho)I_{0,x}^{\alpha,\beta,\eta}[h(x)] - h(\rho)I_{0,x}^{\alpha,\beta,\eta}[(gf)(x)] \geq 0.
\end{aligned} \quad (4.33)$$

Again multiplying (4.33) by $G(x,\rho)$, ($\rho \in (0,x)$, $x>0$), where $G(x,\rho)$ is defined by (3.3). Then integrating resulting identity with respect to ρ from 0 to x, we have

$$\begin{aligned}
I_{0,x}^{\alpha,\beta,\eta}[f(x)]I_{0,x}^{\alpha,\beta,\eta}[(gh)(x)] - I_{0,x}^{\alpha,\beta,\eta}[h(x)]I_{0,x}^{\alpha,\beta,\eta}[(gf)(x)] \\
- I_{0,x}^{\alpha,\beta,\eta}[(gf)(x)]I_{0,x}^{\alpha,\beta,\eta}[h(x)] \\
+ I_{0,x}^{\alpha,\beta,\eta}[(gh)(x)]I_{0,x}^{\alpha,\beta,\eta}[f(x)] \geq 0,
\end{aligned} \quad (4.34)$$

which implies that

$$I_{0,x}^{\alpha,\beta,\eta}[f(x)]I_{0,x}^{\alpha,\beta,\eta}[(gh)(x)] \geq I_{0,x}^{\alpha,\beta,\eta}[h(x)]I_{0,x}^{\alpha,\beta,\eta}[(gf)(x)], \quad (4.35)$$

we get

$$\frac{I_{0,x}^{\alpha,\beta,\eta}[f(x)]}{I_{0,x}^{\alpha,\beta,\eta}[h(x)]} \geq \frac{I_{0,x}^{\alpha,\beta,\eta}[(gf)(x)]}{I_{0,x}^{\alpha,\beta,\eta}[(gh)(x)]}. \quad (4.36)$$

This completes the proof.

Theorem 9. *Suppose that f, g and h be positive and continuous functions on $[0,\infty)$, such that*

$$(g(\tau)-g(\rho))\left(\frac{f(\rho)}{h(\rho)}-\frac{f(\tau)}{h(\tau)}\right) \geq 0, \tau,\rho \in (0,t)\ \ t>0, \quad (4.37)$$

then for all $x>0$, $\alpha > max\{0,-\beta\}$, $\psi > max\{0,-\phi\}$, $\beta<1$, $\beta-1<\eta<0$, $\phi<1$, $\phi-1<\zeta<0$,

$$\frac{I_{0,x}^{\alpha,\beta,\eta}[f(x)]I_{0,x}^{\psi,\phi,\zeta}[(gh)(x)] + I_{0,x}^{\psi,\phi,\zeta}[f(x)]I_{0,x}^{\alpha,\beta,\eta}[(gh)(x)]}{I_{0,x}^{\alpha,\beta,\eta}[h(x)]I_{0,x}^{\psi,\phi,\zeta}[(gf)(x)] + I_{0,x}^{\psi,\phi,\zeta}[h(x)]I_{0,x}^{\alpha,\beta,\eta}[(gf)(x)]} \geq 1, \quad (4.38)$$

hold.

Proof: Multiplying equation (4.33) by $\frac{x^{-\psi-\phi}}{\Gamma(\psi)}(x-\rho)^{\psi-1}{}_2F_1(\psi+\phi,-\zeta;\psi;1-\frac{\rho}{x})$ ($\rho \in (0,x)$, $x>0$), which (in view of the argument mentioned above in proof of theorem 5) remain positive. Then integrating resulting identity with respect to ρ from 0 to x, we have,

$$\begin{aligned}
I_{0,x}^{\psi,\phi,\zeta}[f(x)]I_{0,x}^{\alpha,\beta,\eta}[(gh)(x)] - I_{0,x}^{\psi,\phi,\zeta}[h(x)]I_{0,x}^{\alpha,\beta,\eta}[(gf)(x)] \\
- I_{0,x}^{\psi,\phi,\zeta}[(gf)(x)]I_{0,x}^{\alpha,\beta,\eta}[h(x)] \\
+ I_{0,x}^{\psi,\phi,\zeta}[(gh)(x)]I_{0,x}^{\alpha,\beta,\eta}[f(x)] \geq 0,
\end{aligned} \quad (4.39)$$

we get,

$$\begin{aligned}
& I_{0,x}^{\psi,\phi,\zeta}[f(x)]I_{0,x}^{\alpha,\beta,\eta}[(gh)(x)] + I_{0,x}^{\psi,\phi,\zeta}[(gh)(x)]I_{0,x}^{\alpha,\beta,\eta}[f(x)] \\
& \geq I_{0,x}^{\psi,\phi,\zeta}[h(x)]I_{0,x}^{\alpha,\beta,\eta}[(gf)(x)] + I_{0,x}^{\psi,\phi,\zeta}[(gf)(x)]I_{0,x}^{\alpha,\beta,\eta}[h(x)],
\end{aligned} \quad (4.40)$$

this gives the required inequality (4.38).

*Remark.*For $\alpha=\psi$, $\beta=\phi$ and $\eta=\zeta$, in theorem 9 directly we reduces to the theorem 8.

Theorem 10. *Suppose that f and h are two positive continuous function such that $f \leq h$ on $[0,\infty)$. If $\frac{f}{h}$ is decreasing and f is increasing on $[0,\infty)$, then for any $p \geq 0$, For all $x>0$, $\alpha > max\{0,-\beta\}$, $\beta<1$, $\beta-1<\eta<0$ then*

$$\frac{h^p(\tau)[f(x)]}{I_{0,x}^{\alpha,\beta,\eta}[h(x)]} \geq \frac{I_{0,x}^{\alpha,\beta,\eta}[f^p(x)]}{I_{0,x}^{\alpha,\beta,\eta}[h^p(x)]}. \quad (4.41)$$

Proof: We take $g=f^{p-1}$ in theorem 8

$$\frac{I_{0,x}^{\alpha,\beta,\eta}[f(x)]}{I_{0,x}^{\alpha,\beta,\eta}[h(x)]} \geq \frac{I_{0,x}^{\alpha,\beta,\eta}[(ff^{p-1})(x)]}{I_{0,x}^{\alpha,\beta,\eta}[(hf^{p-1})(x)]}. \quad (4.42)$$

Since $f \leq h$ on $[0,\infty)$, then we can write,

$$hf^{p-1}(x) \leq h^p. \tag{4.43}$$

Multiplying equation (4.43) by $G(x,\tau)$, ($\tau \in (0,x)$, $x>0$), where $G(x,\tau)$ is defined by (3.3). Then integrating resulting identity with respect to τ from 0 to x, we have,

$$\frac{x^{-\alpha-\beta}}{\Gamma(\alpha)}\int_0^x (x-\tau)^{\alpha-1}{}_2F_1(\alpha+\beta,-\eta;\alpha;1-\frac{\tau}{x})[f^{p-1}h(\tau)]d\tau \leq x^{-\alpha-\beta}\Gamma(\alpha)\int_0^x (x-\tau)^{\alpha-1}{}_2F_1(\alpha+\beta,-\eta;\alpha;1-\frac{\tau}{x})[h^p(\tau)]d\tau. \tag{4.44}$$

implies that

$$I_{0,x}^{\alpha,\beta,\eta}[hf^{p-1}(x)] \leq I_{0,x}^{\alpha,\beta,\eta}[h^p(x)], \tag{4.45}$$

and so we have,

$$\frac{h^p(\tau)[(ff^{p-1})(x)]}{h^p(\tau)[(hf^{p-1})(x)]} \geq \frac{h^p(\tau)[f^p(x)]}{h^p(\tau)[h^p(x)]}, \tag{4.46}$$

then from equation (4.42) and (4.46), we obtain (4.41).

Theorem 11. *Suppose that f and h are two positive continuous function such that $f \leq h$ on $[0,\infty)$. If $\frac{f}{h}$ is decreasing and f is increasing on $[0,\infty)[$, then for any $p \geq 1$, for all $x>0$, $\alpha > max\{0,-\beta\}$, $\psi > max\{0,-\phi\}\beta < 1$, $\beta-1<\eta<0$, $\phi<1$, $\phi-1<\zeta<0$,*

$$\frac{I_{0,x}^{\alpha,\beta,\eta}[f(x)]I_{0,x}^{\psi,\phi,\zeta}h^p[(x)]+I_{0,x}^{\psi,\phi,\zeta}[f(x)]I_{0,x}^{\alpha,\beta,\eta}[h^p(x)]}{I_{0,x}^{\alpha,\beta,\eta}[h(x)]I_{0,x}^{\psi,\phi,\zeta}f^p(x)]+I_{0,x}^{\psi,\phi,\zeta}[h(x)]I_{0,x}^{\alpha,\beta,\eta}[f^p(x)]} \geq 1. \tag{4.47}$$

Proof: We take $g = f^{p-1}$ in theorem 9, then we obtain

$$\frac{I_{0,x}^{\alpha,\beta,\eta}[f(x)]I_{0,x}^{\psi,\phi,\zeta}[hf^{p-1}(x)]+I_{0,x}^{\psi,\phi,\zeta}[f(x)]I_{0,x}^{\alpha,\beta,\eta}[hf^{p-1}(x)]}{I_{0,x}^{\alpha,\beta,\eta}[h(x)]I_{0,x}^{\psi,\phi,\zeta}[f^p(x)]+I_{0,x}^{\psi,\phi,\zeta}[h(x)]I_{0,x}^{\alpha,\beta,\eta}[f^p(x)]} \geq 1, \tag{4.48}$$

then by hypothesis, $f \leq h$ on $[0,\infty)$, which implies that

$$hf^{p-1} \leq h^p. \tag{4.49}$$

Now, multiplying both side of (4.49) by $\frac{x^{-\psi-\phi}}{\Gamma(\psi)}(x-\rho)^{\psi-1}{}_2F_1(\psi+\phi,-\zeta;\psi;1-\frac{\rho}{x})$ $(\rho \in (0,x)$, $x>0)$, which (in view of the argument mentioned above in proof of theorem 5) remain positive. Then integrating resulting identity with respect to ρ from 0 to x, we have,

$$I_{0,x}^{\psi,\phi,\zeta}[hf^{p-1}(x)] \leq I_{0,x}^{\psi,\phi,\zeta}[h^p(x)], \tag{4.50}$$

multiplying on both side of (4.50) by $I_{0,x}^{\alpha,\beta,\eta}[f(x)]$, we obtain

$$I_{0,x}^{\alpha,\beta,\eta}[f(x)]I_{0,x}^{\psi,\phi,\zeta}[hf^{p-1}(x)] \leq I_{0,x}^{\alpha,\beta,\eta}[f(x)]I_{0,x}^{\psi,\phi,\zeta}[h^p(x)], \tag{4.51}$$

hence by (4.50) and (4.51), we obtain

$$I_{0,x}^{\alpha,\beta,\eta}[f(x)]I_{0,x}^{\psi,\phi,\zeta}[hf^{p-1}(x)]+I_{0,x}^{\psi,\phi,\zeta}[f(x)]I_{0,x}^{\alpha,\beta,\eta}[hf^{p-1}(x)] \leq I_{0,x}^{\alpha,\beta,\eta}[f(x)]I_{0,x}^{\psi,\phi,\zeta}[h^p(x)]+I_{0,x}^{\psi,\phi,\zeta}[f(x)]I_{0,x}^{\alpha,\beta,\eta}[h^p(x)]. \tag{4.52}$$

By (4.48) and (4.52), we complete the proof of this theorem.

Theorem 12. *Suppose that f, g and h be positive and continuous functions on $[0,\infty)$, such that*

$$(f(\tau)-f(\rho))(g(\tau)-g(\rho))(h(\tau)+h(\rho)) \geq 0;\ \tau,\rho \in (0,x)\ \ x>0, \tag{4.53}$$

then for all $x>0$, $\alpha > max\{0,-\beta\}$, $\beta<1$, $\beta-1<\eta<0$, we have

$$I_{0,x}^{\alpha,\beta,\eta}[(fgh)(x)]I_{0,x}^{\alpha,\beta,\eta}[1]+I_{0,x}^{\alpha,\beta,\eta}[(fg)(x)]I_{0,x}^{\alpha,\beta,\eta}[h(x)] \geq I_{0,x}^{\alpha,\beta,\eta}[g(x)]I_{0,x}^{\alpha,\beta,\eta}[(fh)(x)]+I_{0,x}^{\alpha,\beta,\eta}[f(x)]I_{0,x}^{\alpha,\beta,\eta}[(gh)(x)]. \tag{4.54}$$

Proof: By the assumption for any τ,ρ, we have

$$f(\tau)g(\tau)h(\tau)+f(\tau)g(\rho)h(\rho)-f(\tau)g(\rho)h(\tau)-f(\tau)g(\rho)h(\rho) - f(\rho)g(\tau)h(\tau)-f(\rho)g(\tau)h(\rho)+f(\rho)g(\rho)h(\tau)+f(\rho)g(\rho)h(\rho) \geq 0, \tag{4.55}$$

multiplying equation (4.55) by $G(x,\tau)$, ($\tau \in (0,x)$, $x>0$), where $G(x,\tau)$ is defined by (3.3). Then integrating resulting identity with respect to τ from 0 to x, we have

$$I_{0,x}^{\alpha,\beta,\eta}[(fgh)(x)]+h(\rho)I_{0,x}^{\alpha,\beta,\eta}[(fg)(x)]+f(\rho)g(\rho)I_{0,x}^{\alpha,\beta,\eta}[h(x)] + f(\rho)g(\rho)h(\rho)I_{0,x}^{\alpha,\beta,\eta}[1] \geq g(\rho)I_{0,x}^{\alpha,\beta,\eta}[(fh)(x)]+g(\rho)h(\rho)I_{0,x}^{\alpha,\beta,\eta}[f(x)] + f(\rho)I_{0,x}^{\alpha,\beta,\eta}[(gh)(x)]+f(\rho)h(\rho)I_{0,x}^{\alpha,\beta,\eta}[g(x)]. \tag{4.56}$$

Again multiplying (4.56) by $G(x,\rho)$, ($\rho \in (0,x)$, $x>0$), where $G(x,\rho)$ is defined by (3.3). Then integrating resulting identity with respect to ρ from 0 to x, we have

$$I_{0,x}^{\alpha,\beta,\eta}[1]I_{0,x}^{\alpha,\beta,\eta}[(fgh)(x)]+I_{0,x}^{\alpha,\beta,\eta}[h(x)]I_{0,x}^{\alpha,\beta,\eta}[(fg)(x)] + I_{0,x}^{\alpha,\beta,\eta}[h(x)]I_{0,x}^{\alpha,\beta,\eta}[(fg)(x)]+I_{0,x}^{\alpha,\beta,\eta}[(fgh)(x)]I_{0,x}^{\alpha,\beta,\eta}[1] \geq I_{0,x}^{\alpha,\beta,\eta}[g(x)]I_{0,x}^{\alpha,\beta,\eta}[(fh)(x)]+I_{0,x}^{\alpha,\beta,\eta}[(gh)(x)]I_{0,x}^{\alpha,\beta,\eta}[f(x)] + I_{0,x}^{\alpha,\beta,\eta}[(gh)(x)]I_{0,x}^{\alpha,\beta,\eta}[f(x)]+I_{0,x}^{\alpha,\beta,\eta}[(fh)(x)]I_{0,x}^{\alpha,\beta,\eta}[g(x)]. \tag{4.57}$$

which gives the equation (4.54). This proves the Theorem.

Theorem 13. *Suppose that f, g and h be positive and continuous functions on $[0,\infty)$, such that*

$$(f(\tau)-f(\rho))(g(\tau)+g(\rho))(h(\tau)+h(\rho)) \geq 0;\ \tau,\rho \in (0,x)\ \ x>0, \tag{4.58}$$

then for all $x>0$, $\alpha > max\{0,-\beta\}$, $\beta<1$, $\beta-1<\eta<0$, we have

$$I_{0,x}^{\alpha,\beta,\eta}[f(x)]I_{0,x}^{\alpha,\beta,\eta}[(gh)(x)]+I_{0,x}^{\alpha,\beta,\eta}[(fh)(x)]I_{0,x}^{\alpha,\beta,\eta}[g(x)] \geq I_{0,x}^{\alpha,\beta,\eta}[(gh)(x)]I_{0,x}^{\alpha,\beta,\eta}[f(x)]+I_{0,x}^{\alpha,\beta,\eta}[h(x)]I_{0,x}^{\alpha,\beta,\eta}[(fg)(x)]. \tag{4.59}$$

Proof: For any τ,ρ, we have

$$f(\tau)g(\tau)h(\tau)+f(\tau)g(\rho)h(\rho)+f(\tau)g(\rho)h(\tau)+f(\tau)g(\rho)h(\rho)$$
$$\geq f(\rho)g(\tau)h(\tau)+f(\rho)g(\tau)h(\rho)+f(\rho)g(\rho)h(\tau)+f(\rho)g(\rho)h(\rho); \quad (4.60)$$

Similar to the proof theorem 12, we have,

$$I_{0,x}^{\alpha,\beta,\eta}[(fgh)(x)]+h(\rho)I_{0,x}^{\alpha,\beta,\eta}[(fg)(x)]+g(\rho)I_{0,x}^{\alpha,\beta,\eta}[(fh)(x)]$$
$$+g(\rho)h(\rho)I_{0,x}^{\alpha,\beta,\eta}[f(x)]\geq f(\rho)g(\rho)I_{0,x}^{\alpha,\beta,\eta}[h(x)]+f(\rho)h(\rho)I_{0,x}^{\alpha,\beta,\eta}[g(x)]$$
$$+f(\rho)I_{0,x}^{\alpha,\beta,\eta}[(gh)(x)]+f(\rho)g(\rho)h(\rho)I_{0,x}^{\alpha,\beta,\eta}[1]. \quad (4.61)$$

Again similar to proof of theorem 12, we obtain,

$$I_{0,x}^{\alpha,\beta,\eta}[1]I_{0,x}^{\alpha,\beta,\eta}[(fgh)(x)]+I_{0,x}^{\alpha,\beta,\eta}[h(x)]I_{0,x}^{\alpha,\beta,\eta}[(fg)(x)]$$
$$+I_{0,x}^{\alpha,\beta,\eta}[g(x)]I_{0,x}^{\alpha,\beta,\eta}[(fh)(x)]+I_{0,x}^{\alpha,\beta,\eta}[(gh)(x)]I_{0,x}^{\alpha,\beta,\eta}[f(x)]$$
$$\geq I_{0,x}^{\alpha,\beta,\eta}[(fg)(x)]I_{0,x}^{\alpha,\beta,\eta}[h(x)]+I_{0,x}^{\alpha,\beta,\eta}[(fh)(x)]I_{0,x}^{\alpha,\beta,\eta}[g(x)]$$
$$+I_{0,x}^{\alpha,\beta,\eta}[(fg)(x)]I_{0,x}^{\alpha,\beta,\eta}[h(x)]+I_{0,x}^{\alpha,\beta,\eta}[(fgh)(x)]I_{0,x}^{\alpha,\beta,\eta}[1]. \quad (4.62)$$

we get equation (4.59), this complete the proof of theorem 13.

References

[1] A. Anber, Z. Dahmani and B. Bendoukha, *New integral inequalities of Feng Qi type via Riemann-Liouville fractional integration*, Facta Universitatis (NIS) Ser. Math. Imform. 27(2)(2012), 13-22.

[2] S. Belarbi and Z. Dahmani, *On some new fractional integral inequality*, J. Inequal. Pure and Appl. Math., 10(3)(2009), Art.86, 5 pp.

[3] Z. Dahmani and A. Benzidane, *New inequalities using Q-fractional theory*, Bull. Math. Anal. Appl. 4(1)(2012), 190-196.

[4] Z. Dahmani, *On Minkowski and Hermit-Hadamard integral inequalities via fractional integration*, Ann. Funct. Anal. 1(2010), no.1, 51-58.

[5] Z. Dahmani, *New inequalities in fractional integrals*, Int. J. Nonlinear Sci. 9(4)(2010), 493-497.

[6] Z. Dahmani, *The Riemann-Liouville operator to generate some new inequalities*, Int. J. Nonlinear Sci. 12(4)(2011), 452-455.

[7] Z. Dahmani, *Some results associate with fractional integrals involving the extended Chebyshev*, Acta Univ. Apulensis Math. Inform. 27(2011), 217-224.

[8] Z. Dahmani, Louiza Tabharit and Sabrina Taf, *New generalisations of Gruss inequality using Riemann-Liouville fractional integrals*, Bull. Math. Anal. Appl. 2(3)(2010), 93-99.

[9] Z. Denton and A. S. Vatsala, *Fractional integral inequalities and application*, Comput. Math. Appl. 59(2010), 1087-1094

[10] G. A. Anastassiou, *Fractional Differentiation Inequalities*, Springer Publishing Company, New York, 2009.

[11] V. Kiryakova, *On two Saigo's fractional integral operator in the class of univalent functions*, Fract. Calc. Appl. Anal 9(2)(2006).

[12] V. S. Kiryakova, *Generalized Fractional Calculus and Applications*, Pitman Res. Notes Math. Ser. 301, Longman Scientific and Technical, Harlow, 1994.

[13] A. R. Prabhakaran and K. Srinivasa Rao, *Saigo operator of fractional integration of Hypergeometric functions*, International Journal of Pure and Applied Mathematics 81(5) 2012, 755-763.

[14] R. K. Raina, *Solution of Abel-type integral equation involving the Appell hypergeometric function*, Integral Transforms Spec. Funct. 21 (7) (2010), 515-522.

[15] S. D. Purohit and R. K. Raina, *Chebyshev type inequalities for the Saigo fractional integral and their q- analogues*, J. Math. Inequal.7(2)(2013), 239-249.

[16] S. D. Purohit, R. K. Yadav, *On generalized fractional q-integral operator involving the q-Gauss Hypergeometric functions*, Bull. Math. Anal. 2(4)(2010), 35-44.

[17] M. Saigo, A remark on integral operators involving the Gauss hypergeometric functions, Math. Rep.Kyushu Univ., 11 (1978), 135-143.

[18] N. Virchenko and O. Lisetska, *On some fractional integral operators involving generalized Gauss hypergeometric functions*, Appl. Appl. Math. 5(10) (2010), 1418-1427.

[19] W. Yang, *Some new fractional quantum integral inequalities*, Appl. Math. Latt. 25(6), 2012, 963-969.

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