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On Some New Entire Sequence Spaces

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Abstract: In this paper we introduce entire sequence spaces and analytic sequence spaces on seminormed spaces defined by a Musielak-Orlicz function and study some toplological properties and inclusion relations between these spaces. We also make an effort to study these sequence spaces over n-normed spaces.

Keywords: paranorm space, Orlicz function, Musielak-Orlicz function, solid, monotone, entire sequence space, analytic sequence space.
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1 Introduction

An Orlicz function $M : [0,\infty) \to [0,\infty)$ is a continuous, non-decreasing and convex function such that $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \longrightarrow \infty$ as $x \longrightarrow \infty$. Lindenstrauss and Tzafriri [17] used the idea of Orlicz function to define the following sequence space. Let w be the space of all real or complex sequences $x = (x_k)$, then

$$l_M = \left\{ x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty \right\}$$

which is called a Orlicz sequence space. Also l_M is a Banach space with the norm

$$||x|| = \inf\left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\}.$$

Also, it was shown in [17] that every Orlicz sequence space l_M contains a subspace isomorphic to $l_p (p \geq 1)$. The Δ_2- condition is equivalent to $M(Lx) \leq LM(x)$, for all L with $0 < L < 1$. An Orlicz function M can always be represented in the following integral form

$$M(x) = \int_0^x \eta(t)dt$$

where η is known as the kernel of M, is right differentiable for $t \geq 0, \eta(0) = 0, \eta(t) > 0$, η is non-decreasing and $\eta(t) \to \infty$ as $t \to \infty$.
A sequence $\mathcal{M} = (M_k)$ of Orlicz functions is called a Musielak-Orlicz function see ([18],[20]). A sequence $\mathcal{N} = (N_k)$ of Orlicz functions defined by

$$N_k(v) = \sup\{|v|u - M_k(u) : u \geq 0\},\ k = 1,2,\ldots$$

is called the complementary function of the Musielak-Orlicz function $\mathcal{M}$. For a given Musielak-Orlicz function $\mathcal{M}$, the Musielak-Orlicz sequence space $t_{\mathcal{M}}$ and its subspace $h_{\mathcal{M}}$ are defined as follows

$$t_{\mathcal{M}} = \left\{ x \in w : I_{\mathcal{M}}(cx) < \infty,\ \text{for some}\ c > 0 \right\},$$

$$h_{\mathcal{M}} = \left\{ x \in w : I_{\mathcal{M}}(cx) < \infty,\ \text{for all}\ c > 0 \right\},$$

where $I_{\mathcal{M}}$ is a convex modular defined by

$$I_{\mathcal{M}}(x) = \sum_{k=1}^{\infty} M_k(x_k), x = (x_k) \in t_{\mathcal{M}}.$$

We consider $t_{\mathcal{M}}$ equipped with the Luxemburg norm

$$||x|| = \inf\left\{ k > 0 : I_{\mathcal{M}}\left(\frac{x}{k}\right) \leq 1 \right\}$$

or equipped with the Orlicz norm

$$||x||^0 = \inf\left\{ \frac{1}{k}\Big(1 + I_{\mathcal{M}}(kx)\Big) : k > 0 \right\}.$$

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Let X be a linear metric space. A function $p: X \to \mathbb{R}$ is called paranorm, if

1. $p(x) \geq 0$, for all $x \in X$,
2. $p(-x) = p(x)$, for all $x \in X$,
3. $p(x+y) \leq p(x) + p(y)$, for all $x, y \in X$,
4. if (λ_n) is a sequence of scalars with $\lambda_n \to \lambda$ as $n \to \infty$ and (x_n) is a sequence of vectors with $p(x_n - x) \to 0$ as $n \to \infty$, then $p(\lambda_n x_n - \lambda x) \to 0$ as $n \to \infty$.

A paranorm p for which $p(x) = 0$ implies $x = 0$ is called total paranorm and the pair (X, p) is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm (see [28], Theorem 10.4.2, P-183). For more details about sequence spaces see([1], [3], [5], [15], [16], [21], [22], [23], [24], [25], [26], [27]).
A complex sequence, whose k^{th} term is x_k is denoted by (x_k). Let φ be the set of all finite sequences. A sequence $x = (x_k)$ is said to be analytic if $\sup_k |x_k|^{\frac{1}{k}} < \infty$. The vector space of all analytic sequences will be denoted by Λ. A sequence x is called entire sequence if $\lim_{k\to\infty} |x_k|^{\frac{1}{k}} = 0$. The vector space of all entire sequences will be denoted by Γ.
Let σ be a one-one mapping of the set of positive integers into itself such that $\sigma^m(n) = \sigma(\sigma^{m-1}(n)), m = 1, 2, 3, \cdots$.
A continuous linear functional ϕ on Λ is said to be an invariant mean or a $\sigma-$mean if and only if

1. $\phi(x) \geq 0$ when the sequence $x = (x_n)$ has $x_n \geq 0$ for all n,
2. $\phi(e) = 1$ where $e = (1, 1, 1, \cdots)$ and
3. $\phi(\{x_{\sigma(n)}\}) = \phi(\{x_n\})$ for all $x \in \Lambda$.

For certain kinds of mappings σ, every invariant mean ϕ extends the limit functional on the space $\mathscr{C}$ of all convergent sequences in the sense that $\phi(x) = \lim x$ for all $x \in \mathscr{C}$. Consequently $\mathscr{C} \subset V_\sigma$, where V_σ is the set of analytic sequences all of those $\sigma-$means are equal. If $x = (x_n)$, set $Tx = (Tx)^{\frac{1}{n}} = (x_{\sigma(n)})$. It can be shown that

$$V_\sigma = \Big\{x = (x_n) : \lim_{m\to\infty} t_{mn}(x_n)^{\frac{1}{n}} = L \text{ uniformly in n}, \ L = \sigma - \lim_{n\to\infty}(x_n)^{\frac{1}{n}}\Big\},$$

where

$$t_{mn}(x) = \frac{(x_n + Tx_n + \cdots + T^m x_n)^{\frac{1}{n}}}{m+1}.$$

Given a sequence $x = \{x_k\}$ its n^{th} section is the sequence $x^{(n)} = \{x_1, x_2, \cdots x_n, 0, 0, \cdots\}$, $\delta^{(n)} = (0, 0, \cdots, 1, 0, 0, \cdots)$, in the n^{th} place and zeros elsewhere.
The space consisting of all those sequences x in w such that $M_k\Big(\frac{|x_k|^{1/k}}{\rho}\Big) \to 0$ as $k \to \infty$ for some arbitrary fixed $\rho > 0$ is denoted by $\Gamma_{\mathscr{M}}$ and is known as Musielak-Orlicz space of entire sequences. The space $\Gamma_{\mathscr{M}}$ is a metric space with the metric $d(x, y) = \sup_k M_k\Big(\frac{|x_k - y_k|^{1/k}}{\rho}\Big)$ for all $x = \{x_k\}$ and $y = \{y_k\}$ in $\Gamma_{\mathscr{M}}$.
The space consisting of all those sequences x in w such that $\Big(\sup_k\Big(M_k\Big(\frac{|x_k|^{1/k}}{\rho}\Big)\Big)\Big) < \infty$ for some arbitrarily fixed $\rho > 0$ is denoted by $\Lambda_{\mathscr{M}}$ and is known as Musielak-Orlicz space of analytic sequences.
A sequence space E is said to be solid or normal if $(\alpha_k x_k) \in E$ whenever $(x_k) \in E$ and for all sequences of scalars (α_k) with $|\alpha_k| \leq 1$ (see [20]). The following inequality will be used throughout the paper. Let $p = (p_k)$ be a sequence of positive real numbers with $0 \leq p_k \leq \sup p_k = G, K = \max(1, 2^{G-1})$ then

$$|a_k + b_k|^{p_k} \leq K\{|a_k|^{p_k} + |b_k|^{p_k}\} \text{ for all } k \text{ and } a_k, b_k \in \mathbb{C}. \quad (1.1)$$

Also $|a|^{p_k} \leq \max(1, |a|^G)$ for all $a \in \mathbb{C}$.
Let $\mathscr{M} = (M_k)$ be a Musielak-Orlicz function, X be locally convex Hausdorff topological linear space whose topology is determined by a set of continuous seminorms q. The symbol $\Lambda(X)$, $\Gamma(X)$ denotes the space of all analytic and entire sequences recpectively defined over X. In this paper we define the following classes of sequences:

$$\Lambda_{\mathscr{M}}(p,\sigma,q,s) = \Big\{x \in \Lambda(x) : \sup_{n,k} k^{-s}\Big[M_k\Big(q\Big(\frac{|x_{\sigma^k(n)}|^{\frac{1}{k}}}{\rho}\Big)\Big)\Big]^{p_k} < \infty \text{ uniformly in } n \geq 0, \ s \geq 0 \text{ and for some } \rho > 0\Big\},$$

$$\Gamma_{\mathscr{M}}(p,\sigma,q,s) = \Big\{x \in \Gamma(x) : \sum_{k=1}^{n} k^{-s}\Big[M_k\Big(q\Big(\frac{|x_{\sigma^k(n)}|^{\frac{1}{k}}}{\rho}\Big)\Big)\Big]^{p_k} \to 0 \text{ as } k \to \infty \text{ uniformly in } n \geq 0, \ s \geq 0 \text{ and for some } \rho > 0\Big\}.$$

If we take $p = (p_k) = 1$, we get

$$\Lambda_{\mathscr{M}}(\sigma,q,s) = \Big\{x \in \Lambda(x) : \sup_{n,k} k^{-s}\Big[M_k\Big(q\Big(\frac{|x_{\sigma^k(n)}|^{\frac{1}{k}}}{\rho}\Big)\Big)\Big] < \infty \text{ uniformly in } n \geq 0, \ s \geq 0 \text{ and for some } \rho > 0\Big\},$$

$$\Gamma_{\mathscr{M}}(\sigma,q,s) = \Big\{x \in \Gamma(x) : \sum_{k=1}^{n} k^{-s}\Big[M_k\Big(q\Big(\frac{|x_{\sigma^k(n)}|^{\frac{1}{k}}}{\rho}\Big)\Big)\Big] \to 0 \text{ as } k \to \infty \text{ uniformly in } n \geq 0, \ s \geq 0 \text{ and for some } \rho > 0\Big\}.$$

The main purpose of this paper is to study some entire and analytic sequence spaces on seminormed spaces defined by a Musielak-Orlicz function $\mathscr{M} = (M_k)$. We study some topological properties and inclusion relations between the spaces $\Lambda_{\mathscr{M}}(p,\sigma,q,s)$ and $\Gamma_{\mathscr{M}}(p,\sigma,q,s)$ in the second section of this paper. In the third section we make an effort to study some properties of these sequence spaces over n-normed spaces.

2 Some topological properties of spaces $\Lambda_{\mathscr{M}}(p,\sigma,q,s)$ and $\Gamma_{\mathscr{M}}(p,\sigma,q,s)$

Theorem 2.1 *Let $\mathscr{M} = (M_k)$ be a Musielak-Orlicz function and $p = (p_k)$ be a sequence of strictly positive* *real numbers. Then the spaces $\Gamma_{\mathscr{M}}(p,\sigma,q,s)$ and $\Lambda_{\mathscr{M}}(p,\sigma,q,s)$ are linear spaces over the field of complex numbers $\mathbb{C}$.*

Proof. Let $x=(x_k),\ y=(y_k)\in\Gamma_{\mathscr{M}}(p,\sigma,q,s)$. Then there exist positive numbers ρ_1 and ρ_2 such that

$$\sum_{k=1}^{n} k^{-s}\Big[M_k\Big(q\Big(\frac{|x_{\sigma^k(n)}|^{\frac{1}{k}}}{\rho_1}\Big)\Big)\Big]^{p_k}\to 0 \text{ as } k\to\infty$$

and

$$\sum_{k=1}^{n} k^{-s}\Big[M_k\Big(q\Big(\frac{|y_{\sigma^k(n)}|^{\frac{1}{k}}}{\rho_2}\Big)\Big)\Big]^{p_k}\to 0 \text{ as } k\to\infty.$$

Let $\rho_3=\max(2|\alpha|\rho_1,2|\beta|\rho_2)$. Since $\mathscr{M}=(M_k)$ is non decreasing, convex and q is a seminorm so by using inequality (1.1), we have

$$\begin{aligned}
&\sum_{k=1}^{n} k^{-s}\Big[M_k(q(\frac{|\alpha x_{\sigma^k(n)}+\beta y_{\sigma^k(n)}|^{\frac{1}{k}}}{\rho_3}))\Big]^{p_k}\\
&\le \sum_{k=1}^{n} k^{-s}\Big[M_k\Big(q\Big(\frac{|\alpha x_{\sigma^k(n)}|}{\rho_3}+\frac{|\beta y_{\sigma^k(n)}|}{\rho_3}\Big)^{\frac{1}{k}}\Big)\Big]^{p_k}\\
&\le \sum_{k=1}^{n}\frac{1}{2^{p_k}}k^{-s}\Big[M_k\Big(q\Big(\frac{|x_{\sigma^k(n)}|^{\frac{1}{k}}}{\rho_1}\Big)\Big)+M_k\Big(q\Big(\frac{|y_{\sigma^k(n)}|^{\frac{1}{k}}}{\rho_2}\Big)\Big)\Big]^{p_k}\\
&\le \sum_{k=1}^{n} k^{-s}\Big[M_k\Big(q\Big(\frac{|x_{\sigma^k(n)}|^{\frac{1}{k}}}{\rho_1}\Big)\Big)+M_k\Big(q\Big(\frac{|y_{\sigma^k(n)}|^{\frac{1}{k}}}{\rho_2}\Big)\Big)\Big]^{p_k}\\
&\le K\sum_{k=1}^{n} k^{-s}\Big[M_k\Big(q\Big(\frac{|x_{\sigma^k(n)}|^{\frac{1}{k}}}{\rho_1}\Big)\Big)\Big]^{p_k}\\
&+K\sum_{k=1}^{n} k^{-s}\Big[M_k\Big(q\Big(\frac{|y_{\sigma^k(n)}|^{\frac{1}{k}}}{\rho_2}\Big)\Big)\Big]^{p_k}\\
&\to 0 \text{ as } k\to\infty.
\end{aligned}$$

Thus $\alpha x+\beta y\in\Gamma_{\mathscr{M}}(p,\sigma,q,s)$. Hence $\Gamma_{\mathscr{M}}(p,\sigma,q,s)$ is a linear space. Similarly, we can show that $\Lambda_{\mathscr{M}}(p,\sigma,q,s)$ is a linear space.

Theorem 2.2*Suppose $\mathscr{M}=(M_k)$ is Musielak-Orlicz function and $p=(p_k)$ be a sequence of strictly positive real numbers. Then the space $\Gamma_{\mathscr{M}}(p,\sigma,q,s)$ is a paranormed space with the paranorm defined by*

$$g(x)=\inf\Big\{\rho^{\frac{p_m}{M}}:\sup_{k\ge1}k^{-s}\Big[M_k(q(\frac{|x_{\sigma^k}(n)|^{\frac{1}{k}}}{\rho}))\Big]^{\frac{p_k}{M}}\le 1, \text{uniformly in } n>0,\ \rho>0\Big\},$$

where $M=\max(1,\sup_k p_k)$.

Proof. Clearly $g(x)\ge0, g(x)=g(-x)$ and $g(\theta)=0$, where θ is the zero sequence of X. Let $(x_k),\ (y_k)\in\Gamma_{\mathscr{M}}(p,\sigma,q,s)$. Let $\rho_1,\ \rho_2>0$ be such that

$$\sup_{k\ge1}k^{-s}\Big[M_k\Big(q\Big(\frac{|x_{\sigma^k(n)}|^{\frac{1}{k}}}{\rho}\Big)\Big)\Big]^{\frac{p_k}{M}}\le 1$$

and

$$\sup_{k\ge1}k^{-s}\Big[M_k\Big(q\Big(\frac{|y_{\sigma^k(n)}|^{\frac{1}{k}}}{\rho}\Big)\Big)\Big]^{\frac{p_k}{M}}\le 1.$$

Let $\rho=\rho_1+\rho_2$ and by using Minkowski's inequality, we have

$$\begin{aligned}
\sup_{k\ge1}k^{-s}\Big[M_k\Big(q\Big(\frac{|x_{\sigma^k(n)}+y_{\sigma^k(n)}|^{\frac{1}{k}}}{\rho}\Big)\Big)\Big]^{\frac{p_k}{M}} &\le \frac{\rho_1}{\rho_1+\rho_2}\sup_{k\ge1}k^{-s}\Big[M_k\Big(q\Big(\frac{|x_{\sigma^k(n)}|^{\frac{1}{k}}}{\rho_1}\Big)\Big)\Big]^{\frac{p_k}{M}}\\
&+\frac{\rho_2}{\rho_1+\rho_2}\sup_{k\ge1}k^{-s}\Big[M_k\Big(q\Big(\frac{|y_{\sigma^k(n)}|^{\frac{1}{k}}}{\rho_2}\Big)\Big)\Big]^{\frac{p_k}{M}}\\
&\le 1.
\end{aligned}$$

Hence
$g(x+y)$

$$\begin{aligned}
&\le \inf\Big\{(\rho_1+\rho_2)^{\frac{p_m}{M}}:\sup_{k\ge1}k^{-s}\Big[M_k\Big(q\Big(\frac{|x_{\sigma^k(n)}+y_{\sigma^k(n)}|^{\frac{1}{k}}}{\rho_1+\rho_2}\Big)\Big)\Big]^{\frac{p_k}{M}}\le 1,\ \rho_1,\ \rho_2>0, m\in N\Big\}\\
&\le \inf\Big\{(\rho_1)^{\frac{p_m}{M}}:\sup_{k\ge1}k^{-s}\Big[M_k\Big(q\Big(\frac{|x_{\sigma^k(n)}|^{\frac{1}{k}}}{\rho_1}\Big)\Big)\Big]^{\frac{p_k}{M}}\le 1,\ \rho_1>0,\ m\in N\Big\}\\
&+\inf\Big\{(\rho_2)^{\frac{p_m}{M}}:\sup_{k\ge1}k^{-s}\Big[M_k\Big(q\Big(\frac{|y_{\sigma^k(n)}|^{\frac{1}{k}}}{\rho_2}\Big)\Big)\Big]^{\frac{p_k}{M}}\le 1,\ \rho_2>0, m\in N\Big\}.
\end{aligned}$$

Thus we have $g(x+y)\le g(x)+g(y)$. Hence g satisfies the triangle inequality. Now

$$\begin{aligned}
g(\lambda x) &= \inf\Big\{(\rho)^{\frac{p_m}{M}}:\sup_{k\ge1}k^{-s}\Big[M_k\Big(q\Big(\frac{|x_{\sigma^k(n)}|^{\frac{1}{k}}}{\rho}\Big)\Big)\Big]^{\frac{p_k}{M}}\le 1,\ \rho>0,\ m\in N\Big\}\\
&= \inf\Big\{(r|\lambda|)^{\frac{p_m}{M}}:\sup_{k\ge1}k^{-s}\Big[M_k\Big(q\Big(\frac{|x_{\sigma^k(n)}|^{\frac{1}{k}}}{\rho}\Big)\Big)\Big]^{\frac{p_k}{M}}\le 1,\ r>0,\ m\in N\Big\},
\end{aligned}$$

where $r=\frac{\rho}{|\lambda|}$. Hence $\Gamma_{\mathscr{M}}(p,\sigma,q,s)$ is a paranormed space.

Theorem 2.3*Let $\mathscr{M}=(M_k)$ be a Musielak-Orlicz function. Then*

$$\Gamma_{\mathscr{M}}(p,\sigma,q,s)\cap\Lambda_{\mathscr{M}}(p,\sigma,q,s)\subseteq\Gamma_{\mathscr{M}}(p,\sigma,q,s).$$

Proof. The proof is trivial so we omit.

Theorem 2.4$\Gamma_{\mathscr{M}}(p,\sigma,q,s)\subseteq\Lambda_{\mathscr{M}}(p,\sigma,q,s)$.

Proof. The proof is trivial so we omit.

Theorem 2.5*Let $0\le p_k\le r_k$ and let $\{\frac{r_k}{p_k}\}$ be bounded. Then $\Gamma_{\mathscr{M}}(r,\sigma,q,s)\subset\Gamma_{\mathscr{M}}(p,\sigma,q,s)$.*

Proof. Let $x\in\Gamma_{\mathscr{M}}(r,\sigma,q,s)$. Then

$$\sum_{k=1}^{n}k^{-s}\Big[M_k\Big(q\Big(\frac{|x_{\sigma^k(n)}|^{\frac{1}{k}}}{\rho}\Big)\Big)\Big]^{r_k}\to 0 \text{ as } n\to\infty. \quad (2.1)$$

Let $t_k=\sum_{k=1}^{n}k^{-s}\Big[M_k\Big(q\Big(\frac{|x_{\sigma^k(n)}|^{\frac{1}{k}}}{\rho}\Big)\Big)\Big]^{q_k}$ and $\lambda_k=\frac{p_k}{r_k}$. Since $p_k\le r_k$, we have $0\le\lambda_k\le1$. Take $0<\lambda<\lambda_k$. Define

$$u_k=\begin{cases} t_k, & \text{if } t_k\ge1\\ 0, & \text{if } t_k<1\end{cases}$$

and

$$v_k = \begin{cases} 0, & \text{if } t_k \geq 1 \\ t_k, & \text{if } t_k < 1 \end{cases}$$

$t_k = u_k + v_k$, $t_k^{\lambda_k} = u_k^{\lambda_k} + v_k^{\lambda_k}$. It follows that $u_k^{\lambda_k} \leq u_k \leq t_k$, $v_k^{\lambda_k} \leq v_k^{\lambda}$. Since $t_k^{\lambda_k} = u_k^{\lambda_k} + v_k^{\lambda_k}$, then $t_k^{\lambda_k} \leq t_k + v_k^{\lambda}$. Now

$$\sum_{k=1}^{n} k^{-s}\Big[\Big[M_k\Big(q\Big(\frac{|x_{\sigma^k(n)}|^{\frac{1}{k}}}{\rho}\Big)\Big)\Big]^{r_k}\Big]^{\lambda_k} \leq \sum_{k=1}^{n} k^{-s}\Big[M_k\Big(q\Big(\frac{|x_{\sigma^k(n)}|^{\frac{1}{k}}}{\rho}\Big)\Big)\Big]^{r_k}$$

$$\Longrightarrow \sum_{k=1}^{n} k^{-s}\Big[M_k\Big(q\Big(\frac{|x_{\sigma^k(n)}|^{\frac{1}{k}}}{\rho}\Big)\Big)^{r_k}\Big]^{p_k/r_k} \leq \sum_{k=1}^{n} k^{-s}\Big[M_k\Big(q\Big(\frac{|x_{\sigma^k(n)}|^{\frac{1}{k}}}{\rho}\Big)\Big)\Big]^{r_k}$$

$$\Longrightarrow \sum_{k=1}^{n} k^{-s}\Big[M_k\Big(q\Big(\frac{|x_{\sigma^k(n)}|^{\frac{1}{k}}}{\rho}\Big)\Big)\Big]^{p_k} \leq \sum_{k=1}^{n} k^{-s}\Big[M_k\Big(q\Big(\frac{|x_{\sigma^k(n)}|^{\frac{1}{k}}}{\rho}\Big)\Big)\Big]^{r_k}.$$

But

$$\sum_{k=1}^{n} k^{-s}\Big[M_k\Big(q\Big(\frac{|x_{\sigma^k(n)}|^{\frac{1}{k}}}{\rho}\Big)\Big)\Big]^{r_k} \to 0 \text{ as } n \to \infty \text{ (by(2.1))}.$$

Therefore

$$\sum_{k=1}^{n} k^{-s}\Big[M_k\Big(q\Big(\frac{|x_{\sigma^k(n)}|^{\frac{1}{k}}}{\rho}\Big)\Big)\Big]^{p_k} \to 0 \text{ as } n \to \infty.$$

Hence $x \in \Gamma_{\mathscr{M}}(p,\sigma,q,s)$. From (2.1), we get $\Gamma_{\mathscr{M}}(r,\sigma,q,s) \subset \Gamma_{\mathscr{M}}(p,\sigma,q,s)$.

Theorem 2.6*(i) Let* $0 < \inf p_k \leq p_k \leq 1$. *Then* $\Gamma_{\mathscr{M}}(p,\sigma,q,s) \subset \Gamma_{\mathscr{M}}(\sigma,q,s)$,
(ii) let $1 \leq p_k \leq \sup p_k < \infty$. *Then* $\Gamma_{\mathscr{M}}(\sigma,q,s) \subset \Gamma_{\mathscr{M}}(p,\sigma,q,s)$.

Proof. (i) Let $x \in \Gamma_{\mathscr{M}}(p,\sigma,q,s)$. Then

$$\sum_{k=1}^{n} k^{-s}\Big[M_k\Big(q\Big(\frac{|x_{\sigma^k(n)}|^{\frac{1}{k}}}{\rho}\Big)\Big)\Big]^{p_k} \to 0 \text{ as } n \to \infty. \quad (2.2)$$

Since $0 < \inf p_k \leq p_k \leq 1$,

$$\sum_{k=1}^{n} k^{-s}\Big[M_k\Big(q\Big(\frac{|x_{\sigma^k(n)}|^{\frac{1}{k}}}{\rho}\Big)\Big)\Big] \leq \sum_{k=1}^{n} k^{-s}\Big[M_k\Big(q\Big(\frac{|x_{\sigma^k(n)}|^{\frac{1}{k}}}{\rho}\Big)\Big)\Big]^{p_k} \to 0 \text{ as } n \to \infty. \quad (2.3)$$

From (2.2) and (2.3) it follows that, $x \in \Gamma_{\mathscr{M}}(\sigma,q,s)$. Thus $\Gamma_{\mathscr{M}}(p,\sigma,q,s) \subset \Gamma_{\mathscr{M}}(\sigma,q,s)$.
(ii) Let $p_k \geq 1$ for each k and $\sup p_k < \infty$ and let $x \in \Gamma_{\mathscr{M}}(\sigma,q,s)$. Then

$$\sum_{k=1}^{n} k^{-s}\Big[M_k\Big(q\Big(\frac{|x_{\sigma^k(n)}|^{\frac{1}{k}}}{\rho}\Big)\Big)\Big] \to 0 \text{ as } n \to \infty. \quad (2.4)$$

Since $1 \leq p_k \leq \sup p_k < \infty$, we have

$$\sum_{k=1}^{n} k^{-s}\Big[M_k\Big(q\Big(\frac{|x_{\sigma^k(n)}|^{\frac{1}{k}}}{\rho}\Big)\Big)\Big]^{p_k} \leq \sum_{k=1}^{n} k^{-s}\Big[M_k\Big(q\Big(\frac{|x_{\sigma^k(n)}|^{\frac{1}{k}}}{\rho}\Big)\Big)\Big]$$

$$\sum_{k=1}^{n} k^{-s}\Big[M_k\Big(q\Big(\frac{|x_{\sigma^k(n)}|^{\frac{1}{k}}}{\rho}\Big)\Big)\Big]^{p_k} \to 0 \text{ as } n \to \infty.$$

This implies that $x \in \Gamma_{\mathscr{M}}(p,\sigma,q,s)$. Therefore $\Gamma_{\mathscr{M}}(\sigma,q,s) \subset \Gamma_{\mathscr{M}}(p,\sigma,q,s)$.

Theorem 2.7*Suppose* $\sum_{k=1}^{n} k^{-s}\Big[M_k\Big(q\Big(\frac{|x_{\sigma^k(n)}|^{\frac{1}{k}}}{\rho}\Big)\Big)\Big]^{p_k} \leq |x_k|^{1/k}$, *then* $\Gamma \subset \Gamma_{\mathscr{M}}(p,\sigma,q,s)$.

Proof. Let $x \in \Gamma$. Then we have,

$$|x_k|^{1/k} \to 0 \text{ as } k \to \infty. \quad (2.5)$$

But $\sum_{k=1}^{n} k^{-s}\Big[M_k\Big(q\Big(\frac{|x_{\sigma^k(n)}|^{\frac{1}{k}}}{\rho}\Big)\Big)\Big]^{p_k} \leq |x_k|^{1/k}$, by our assumption, implies that

$$\sum_{k=1}^{n} k^{-s}\Big[M_k\Big(q\Big(\frac{|x_{\sigma^k(n)}|^{\frac{1}{k}}}{\rho}\Big)\Big)\Big]^{p_k} \to 0 \text{ as } n \to \infty. \text{ by(2.5)}$$

Then $x \in \Gamma_{\mathscr{M}}(p,\sigma,q,s)$ and $\Gamma \subset \Gamma_{\mathscr{M}}(p,\sigma,q,s)$.

Theorem 2.8$\Gamma_{\mathscr{M}}(p,\sigma,q,s)$ *is solid.*

Proof. Let $|x_k| \leq |y_k|$ and let $y = (y_k) \in \Gamma_{\mathscr{M}}(p,\sigma,q,s)$, because $\mathscr{M} = (M_k)$ is non-decreasing

$$\sum_{k=1}^{n} k^{-s}\Big[M_k\Big(q\Big(\frac{|x_{\sigma^k(n)}|^{\frac{1}{k}}}{\rho}\Big)\Big)\Big]^{p_k} \leq \sum_{k=1}^{n} k^{-s}\Big[M_k\Big(q\Big(\frac{|y_{\sigma^k(n)}|^{\frac{1}{k}}}{\rho}\Big)\Big)\Big]^{p_k}.$$

Since $y \in \Gamma_{\mathscr{M}}(p,\sigma,q,s)$. Therefore,

$$\sum_{k=1}^{n} k^{-s}\Big[M_k\Big(q\Big(\frac{(|y_{\sigma^k(n)}|^{\frac{1}{k}}}{\rho}\Big)\Big)\Big]^{p_k} \to 0 \text{ as } n \to \infty$$

and hence

$$\sum_{k=1}^{n} k^{-s}\Big[M_k\Big(q\Big(\frac{|x_{\sigma^k(n)}|^{\frac{1}{k}}}{\rho}\Big)\Big)\Big]^{p_k} \to 0 \text{ as } n \to \infty.$$

Therefore $x = \{x_k\} \in \Gamma_{\mathscr{M}}(p,\sigma,q,s)$.

Theorem 2.9$\Gamma_{\mathscr{M}}(p,\sigma,q,s)$ *is monotone.*

Proof. The proof is trivial.

3 Sequence spaces over n- normed spaces

The concept of 2-normed spaces was initially developed by Gähler[11] in the mid of 1960's, while that of n-normed spaces one can see in Misiak[19]. Since then, many others have studied this concept and obtained various results, see Gunawan ([12,[13]) and Gunawan and Mashadi [14]. Let $n \in \mathbb{N}$ and X be a linear space over the field $\mathbb{R}$, where $\mathbb{R}$ is field of reals of dimension d, where $d \geq n \geq 2$. A real valued function $||\cdot,\cdots,\cdot||$ on X^n satisfying the following four conditions:

1. $||x_1,x_2,\cdots,x_n|| = 0$ if and only if $x_1,x_2,\cdots,x_n$ are linearly dependent in X;

2. $||x_1, x_2, \cdots, x_n||$ is invariant under permutation;
3. $||\alpha x_1, x_2, \cdots, x_n|| = |\alpha| \ ||x_1, x_2, \cdots, x_n||$ for any $\alpha \in \mathbb{R}$, and
4. $||x + x', x_2, \cdots, x_n|| \leq ||x, x_2, \cdots, x_n|| + ||x', x_2, \cdots, x_n||$

is called an n-norm on X, and the pair $(X, ||\cdot, \cdots, \cdot||)$ is called a n-normed space over the field $\mathbb{R}$.
For example, we may take $X = \mathbb{R}^n$ being equipped with the n-norm $||x_1, x_2, \cdots, x_n||_E$ = the volume of the n-dimensional parallelopiped spanned by the vectors $x_1, x_2, \cdots, x_n$ which may be given explicitly by the formula

$$||x_1, x_2, \cdots, x_n||_E = |\det(x_{ij})|,$$

where $x_i = (x_{i1}, x_{i2}, \cdots, x_{in}) \in \mathbb{R}^n$ for each $i = 1, 2, \cdots, n$. Let $(X, ||\cdot, \cdots, \cdot||)$ be an n-normed space of dimension $d \geq n \geq 2$ and $\{a_1, a_2, \cdots, a_n\}$ be linearly independent set in X. Then the function $||\cdot, \cdots, \cdot||_\infty$ on X^{n-1} defined by

$$||x_1, x_2, \cdots, x_{n-1}||_\infty = \max\{||x_1, x_2, \cdots, x_{n-1}, a_i|| : i = 1, 2, \cdots, n\}$$

is known as an $(n-1)$-norm on X with respect to $\{a_1, a_2, \cdots, a_n\}$.
Let $n \in \mathbb{N}$ and X be a real vector space of dimension d, where $2 \leq n \leq d$. Let β_{n-1} be the collection of linearly independent sets B with $n-1$ elements. For $B \in \beta_{n-1}$, let us define

$$q_B(x_1) = \|x_1, x_2, \cdots x_n\|, \ \ x_1 \in X.$$

Then q_B is a seminorm on X and the family $q = \{q_B : B \in \beta_{n-1}\}$ of seminorms generates a locally convex topology on X. The seminorms q_B have the following properties:

1. $\ker(q_B)$ = the linear span of B.
2. For $B \in \beta_{n-1}$, $y \in B$ and $x \in X \backslash$the linear span of B we have

$$q_{B \cup \{x\} \backslash y}(y) = q_B(x). \quad \text{See ([10])}$$

A sequence (x_k) in a n-normed space $(X, ||\cdot, \cdots, \cdot||)$ is said to converge to some $L \in X$ if

$$\lim_{k\to\infty} ||x_k - L, z_1, \cdots, z_{n-1}|| = 0 \ \text{ for every } \ z_1, \cdots, z_{n-1} \in X.$$

A sequence (x_k) in a n-normed space $(X, ||\cdot, \cdots, \cdot||)$ is said to be Cauchy if

$$\lim_{k,p\to\infty} ||x_k - x_p, z_1, \cdots, z_{n-1}|| = 0 \text{ for every } z_1, \cdots, z_{n-1} \in X.$$

If every Cauchy sequence in X converges to some $L \in X$, then X is said to be complete with respect to the n-norm. Any complete n-normed space is said to be n-Banach space. For more details about $n-$normed spaces one can see ([2], [4], [6], [7], [8], [9]) and references therein.
Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function, X be locally convex Hausdorff topological real linear $n-$normed space whose topology is determined by a set of continuous seminorms q. The symbol $\Lambda(X)$, $\Gamma(X)$ denotes the space of all analytic and entire sequences respectively defined over X. In this section, for each $z_1, \cdots, z_{n-1} \in X$ we define the following classes of sequences:

$$\Lambda_{\mathcal{M}}(p,\sigma,q,s,||.,\cdots,.||) = \Big\{x \in \Lambda(x) : \sup_{n,k} k^{-s}\Big[M_k\Big(q\Big(||\frac{(x_{\sigma^k(n)})^{\frac{1}{k}}}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big)\Big]^{p_k} < \infty \text{ uniformly in } n \geq 0, \ s \geq 0 \text{ for some } \rho > 0\Big\},$$

$$\Gamma_{\mathcal{M}}(p,\sigma,q,s,||.,\cdots,.||) = \Big\{x \in \Gamma(x) : \sum_{k=1}^{n} k^{-s}\Big[M_k\Big(q\Big(||\frac{(x_{\sigma^k(n)})^{\frac{1}{k}}}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big)\Big]^{p_k} \to 0 \text{ as } k \to \infty \text{ uniformly in } n \geq 0, \ s \geq 0 \text{ for some } \rho > 0\Big\}.$$

If we take $p = (p_k) = 1$, we get

$$\Lambda_{\mathcal{M}}(\sigma,q,s,||.,\cdots,.||) = \Big\{x \in \Lambda(x) : \sup_{n,k} k^{-s}\Big[M_k\Big(q\Big(||\frac{(x_{\sigma^k(n)})^{\frac{1}{k}}}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big)\Big] < \infty \text{ uniformly in } n \geq 0, \ s \geq 0 \text{ for some } \rho > 0\Big\},$$

$$\Gamma_{\mathcal{M}}(\sigma,q,s,||.,\cdots,.||) = \Big\{x \in \Gamma(x) : \sum_{k=1}^{n} k^{-s}\Big[M_k\Big(q\Big(||\frac{(x_{\sigma^k(n)})^{\frac{1}{k}}}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big)\Big] \to 0 \text{ as } k \to \infty \text{ uniformly in } n \geq 0, \ s \geq 0 \text{ for some } \rho > 0\Big\}.$$

In the present section we study some topological properties of the spaces $\Lambda_{\mathcal{M}}(p,\sigma,q,s,||.,\cdots,.||)$ and $\Gamma_{\mathcal{M}}(p,\sigma,q,s,||.,\cdots,.||)$ and also examine some inclusion relation between these spaces.

Theorem 3.1 *Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function and $p = (p_k)$ be a sequence of strictly positive real numbers. Then the spaces $\Gamma_{\mathcal{M}}(p,\sigma,q,s,||.,\cdots,.||)$ and $\Lambda_{\mathcal{M}}(p,\sigma,q,s,||.,\cdots,.||)$ are linear space over the field of real numbers $\mathbb{R}$.*

Proof. Let $x = (x_k)$, $y = (y_k) \in \Gamma_{\mathcal{M}}(p,\sigma,q,s,||.,\cdots,.||)$. Then there exist positive numbers ρ_1 and ρ_2 such that

$$\sum_{k=1}^{n} k^{-s}\Big[M_k\Big(q\Big(||\frac{(x_{\sigma^k(n)})^{\frac{1}{k}}}{\rho_1}, z_1, \cdots, z_{n-1}||\Big)\Big)\Big]^{p_k} \to 0 \text{ as } k \to \infty$$

and

$$\sum_{k=1}^{n} k^{-s}\Big[M_k\Big(q\Big(||\frac{(y_{\sigma^k(n)})^{\frac{1}{k}}}{\rho_2}, z_1, \cdots, z_{n-1}\Big)\Big)\Big]^{p_k} \to 0 \text{ as } k \to \infty.$$

Let $\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$. Since $\mathcal{M} = (M_k)$ is non decreasing, convex and q is a seminorm and by using inequality (1.1), we have

$$\sum_{k=1}^{n} k^{-s}\Big[M_k(q||\Big(\frac{(\alpha x_{\sigma^k(n)}+\beta y_{\sigma^k(n)})^{\frac{1}{k}}}{\rho_3}\Big),z_1,\cdots,z_{n-1}||\Big]^{p_k}$$

$$\leq \sum_{k=1}^{n} k^{-s}\Big[M_k\Big(q||\Big(\frac{\alpha(x_{\sigma^k(n)})}{\rho_3}+\frac{(y_{\sigma^k(n)})}{\rho_3}\Big)^{\frac{1}{k}},z_1,\cdots,z_{n-1}||\Big)\Big]^{p_k}$$

$$\leq \sum_{k=1}^{n} \frac{1}{2^{p_k}}k^{-s}\Big[M_k\Big(q\Big(||\frac{(x_{\sigma^k(n)})^{\frac{1}{k}}}{\rho_1},z_1,\cdots,z_{n-1}||\Big)\Big)$$

$$+M_k\Big(q\Big(||\frac{(y_{\sigma^k(n)})^{\frac{1}{k}}}{\rho_2},z_1,\cdots,z_{n-1}||\Big)\Big)\Big]^{p_k}$$

$$\leq \sum_{k=1}^{n} k^{-s}\Big[M_k\Big(q\Big(||\frac{(x_{\sigma^k(n)})^{\frac{1}{k}}}{\rho_1},z_1,\cdots,z_{n-1}||\Big)\Big)$$

$$+M_k\Big(q\Big(||\frac{(y_{\sigma^k(n)})^{\frac{1}{k}}}{\rho_2},z_1,\cdots,z_{n-1}||\Big)\Big)\Big]^{p_k}$$

$$\leq K\sum_{k=1}^{n} k^{-s}\Big[M_k\Big(q\Big(||\frac{(x_{\sigma^k(n)})^{\frac{1}{k}}}{\rho_1},z_1,\cdots,z_{n-1}||\Big)\Big)\Big]^{p_k}$$

$$+K\sum_{k=1}^{n} k^{-s}\Big[M_k\Big(q\Big(||\frac{(y_{\sigma^k(n)})^{\frac{1}{k}}}{\rho_2},z_1,\cdots,z_{n-1}||\Big)\Big)\Big]^{p_k}$$

$$\to 0 \text{ as } k\to\infty.$$

Thus $\alpha x+\beta y \in \Gamma_{\mathscr{M}}(p,\sigma,q,s,||.,\cdots,.||)$. Hence $\Gamma_{\mathscr{M}}(p,\sigma,q,s,||.,\cdots,.||)$ is a linear space. Similarly, we can prove $\Lambda_{\mathscr{M}}(p,\sigma,q,s,||.,\cdots,.||)$ is a linear space.

Theorem 3.2*Suppose $\mathscr{M}=(M_k)$ is Musielak-Orlicz function and $p=(p_k)$ be a sequence of strictly positive real numbers. Then the space $\Gamma_{\mathscr{M}}(p,\sigma,q,s,||.,\cdots,.||)$ is a paranormed space with the paranorm defined by*

$$g(x)=\inf\Big\{\rho^{\frac{p_m}{M}}:\sup_{k\geq 1}k^{-s}\Big[M_k(q||\Big(\frac{(x_{\sigma^k}(n))^{\frac{1}{k}}}{\rho},z_1,\cdots,z_{n-1})||\Big)\Big]^{\frac{p_k}{M}}\leq 1,$$

$$\text{uniformly in } n>0,\ \rho>0\Big\},\ \textit{where}$$

$$M=\max(1,\sup_k p_k).$$

Proof. Clearly $g(x)\geq 0, g(x)=g(-x)$ and $g(\theta)=0$, where θ is the zero sequence of X. Let $(x_k),\ (y_k)\in\Gamma_{\mathscr{M}}(p,\sigma,q,s,||.,\cdots,.||)$. Let $\rho_1,\ \rho_2>0$ be such that

$$\sup_{k\geq 1}k^{-s}\Big[M_k\Big(q\Big(||\frac{(x_{\sigma^k(n)})^{\frac{1}{k}}}{\rho},z_1,\cdots,z_{n-1}||\Big)\Big)\Big]^{\frac{p_k}{M}}\leq 1$$

and

$$\sup_{k\geq 1}k^{-s}\Big[M_k\Big(q\Big(||\frac{(y_{\sigma^k(n)})^{\frac{1}{k}}}{\rho},z_1,\cdots,z_{n-1}||\Big)\Big)\Big]^{\frac{p_k}{M}}\leq 1.$$

Then

$$\sup_{k\geq 1}k^{-s}\Big[M_k\Big(q\Big(||\frac{(x_{\sigma^k(n)}+y_{\sigma^k(n)})^{\frac{1}{k}}}{\rho},z_1,\cdots,z_{n-1}||\Big)\Big)\Big]^{\frac{p_k}{M}}$$

$$\leq \frac{\rho_1}{\rho_1+\rho_2}\sup_{k\geq 1}k^{-s}\Big[M_k\Big(q\Big(||\frac{(x_{\sigma^k(n)})^{\frac{1}{k}}}{\rho_1},z_1,\cdots,z_{n-1}||\Big)\Big)\Big]^{\frac{p_k}{M}}$$

$$+\frac{\rho_2}{\rho_1+\rho_2}\sup_{k\geq 1}k^{-s}\Big[M_k\Big(q\Big(||\frac{(y_{\sigma^k(n)})^{\frac{1}{k}}}{\rho_2},z_1,\cdots,z_{n-1}||\Big)\Big)\Big]^{\frac{p_k}{M}}$$

$$\leq 1.$$

Hence
$g(x+y)$

$$\leq \inf\Big\{(\rho_1+\rho_2)^{\frac{p_m}{M}}:\sup_{k\geq 1}k^{-s}\Big[M_k\Big(q\Big(||\frac{(x_{\sigma^k(n)}+y_{\sigma^k(n)})^{\frac{1}{k}}}{\rho_1+\rho_2},z_1,\cdots,z_{n-1}||\Big)\Big)\Big]^{\frac{p_k}{M}}\leq 1,$$
$$\rho_1,\ \rho_2>0,\ m\in N\Big\}$$

$$\leq \inf\Big\{(\rho_1)^{\frac{p_m}{M}}:\sup_{k\geq 1}k^{-s}\Big[M_k\Big(q\Big(||\frac{(x_{\sigma^k(n)})^{\frac{1}{k}}}{\rho},z_1,\cdots,z_{n-1}||\Big)\Big)\Big]^{\frac{p_k}{M}}\leq 1,$$
$$\rho_1>0,\ m\in N\Big\}$$

$$+\inf\Big\{(\rho_2)^{\frac{p_m}{M}}:\sup_{k\geq 1}k^{-s}\Big[M_k\Big(q\Big(||\frac{(y_{\sigma^k(n)})^{\frac{1}{k}}}{\rho},z_1,\cdots,z_{n-1}||\Big)\Big)\Big]^{\frac{p_k}{M}}\leq 1,$$
$$\rho_2>0,\ m\in N\Big\}.$$

Thus we have $g(x+y)\leq g(x)+g(y)$. Hence g satisfies the triangle inequality. Now
$g(\lambda x)$

$$=\inf\Big\{(\rho)^{\frac{p_m}{M}}:\sup_{k\geq 1}k^{-s}\Big[M_k\Big(q\Big(||\frac{(x_{\sigma^k(n)})^{\frac{1}{k}}}{\rho},z_1,\cdots,z_{n-1}||\Big)\Big)\Big]^{\frac{p_k}{M}}\leq 1,\ \rho>0,\ m\in N\Big\}$$

$$=\inf\Big\{(r|\lambda|)^{\frac{p_m}{M}}:\sup_{k\geq 1}k^{-s}\Big[M_k\Big(q\Big(||\frac{(x_{\sigma^k(n)})^{\frac{1}{k}}}{\rho},z_1,\cdots,z_{n-1}||\Big)\Big)\Big]^{\frac{p_k}{M}}\leq 1, r>0, m\in N\Big\},$$

where $r=\frac{\rho}{|\lambda|}$. Hence $\Gamma_{\mathscr{M}}(p,\sigma,q,s,||.,\cdots,.||)$ is a paranormed space.

Theorem 3.3*Let $\mathscr{M}=(M_k)$ be a Musielak-Orlicz function. Then*

$$\Gamma_{\mathscr{M}}(p,\sigma,q,s,||.,\cdots,.||)\cap\Lambda_{\mathscr{M}}(p,\sigma,q,s,||.,\cdots,.||)\subseteq\Gamma_{\mathscr{M}}(p,\sigma,q,s,||.,\cdots,.||).$$

Proof. It is easy to prove so we omit the proof.

Theorem 3.4
$\Gamma_{\mathscr{M}}(p,\sigma,q,s,||.,\cdots,.||)\subseteq\Lambda_{\mathscr{M}}(p,\sigma,q,s,||.,\cdots,.||)$.

Proof. It is easy to prove so we omit the proof.

Theorem 3.5$\Gamma_{\mathscr{M}}(p,\sigma,q,s,||.,\cdots,.||)$ *is solid.*

Proof. Let $|x_k|\leq|y_k|$ and let $y=(y_k)\in\Gamma_{\mathscr{M}}(p,\sigma,q,s,||.,\cdots,.||)$, since $\mathscr{M}=(M_k)$ is non-decreasing, so

$$\sum_{k=1}^{n}k^{-s}\Big[M_k\Big(q\Big(||\frac{(x_{\sigma^k}(n))^{\frac{1}{k}}}{\rho},z_1,\cdots,z_{n-1}||\Big)\Big)\Big]^{p_k}$$

$$\leq\sum_{k=1}^{n}k^{-s}\Big[M_k\Big(q\Big(||\frac{(y_{\sigma^k}(n))^{\frac{1}{k}}}{\rho},z_1,\cdots,z_{n-1}||\Big)\Big)\Big]^{p_k}.$$

Since $y \in \Gamma_{\mathscr{M}}(p,\sigma,q,s,||.,\cdots,.||)$. Therefore,

$$\sum_{k=1}^{n} k^{-s}\Big[M_k\Big(q\Big(||\frac{(y_{\sigma^k(n)})^{\frac{1}{k}}}{\rho},z_1,\cdots,z_{n-1}||\Big)\Big)\Big]^{p_k} \to 0 \text{ as } n\to\infty.$$

So that

$$\sum_{k=1}^{n} k^{-s}\Big[M_k\Big(q\Big(||\frac{(x_{\sigma^k(n)})^{\frac{1}{k}}}{\rho},z_1,\cdots,z_{n-1}||\Big)\Big)\Big]^{p_k} \to 0 \text{ as } n\to\infty.$$

Therefore $x=(x_k)\in \Gamma_{\mathscr{M}}(p,\sigma,q,s,||.,\cdots,.||)$. Hence $\Gamma_{\mathscr{M}}(p,\sigma,q,s,||.,\cdots,.||)$ is solid.

Theorem 3.6$\Gamma_{\mathscr{M}}(p,\sigma,q,s,||.,\cdots,.||)$ *is monotone.*

Proof. The proof is trivial so we omit it.

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