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Weak Singularity in Graded Rings

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Abstract: In this paper we introduce the notion of graded weak singular ideals of a graded ring R. It is shown that every graded weak singular ideal of R is graded singular. A graded weakly nil ring is graded weak singular. If R is a graded weak non-singular ring then R is graded semiprime. Every graded strongly prime ring R is graded weak non-singular and the same holds for every reduced graded ring.

Keywords: Graded Ring, Graded Ideals, Graded Singular Ideals, Graded Semiprime Ring.

1 Introduction

The notion of singularity plays a very important role in the study of algebraic structures. It was remarked by Miguel Ferrero and Edmund R. Puczylowski in [3] that studying properties of rings one can usually say more assuming that the considered rings are either singular or nonsingular. Therefore, in the studies of rings, the importance of the concept of singularity is remarkable. In [3], the authors have established some properties of singular ideals of a ring. T. K. Dutta et al have introduced the notion of singular ideals in ternary semirings and investigated various properties of such ideals [2].

In this paper, we attempt to study some properties of graded weak singular ideals. It is shown that every graded weak singular ideal is graded singular, but the converse is not true. It is established that a graded weak nonsingular ring is graded semi prime and hence every graded strongly prime ring R is graded semi prime. Finally we prove that if I is a graded ideal of R and I is graded semi prime then $Z_W(I) = I \cap Z_W(R)$, where $Z_W(R)$ denotes the graded weak singular ideal of R. This implies that the class of semiprime graded weak nonsingular rings is hereditary.

2 Preliminaries

Throughout our discussion, we consider R to be a commutative graded ring. The basic definitions used in this paper on rings, modules and graded rings are available in [4,5].

We now present the following definitions that are needed in the sequel.

Definition 2.1: The graded singular ideal of R, denoted by $Z(R)$ is defined as
$Z(R) = \{x \in R | ann_R(x) \cap H \neq 0$ for every nonzero graded ideal H of $R\}$ $= \{x \in h(R) | xK = 0$ for some graded essential ideal $K\}$

Definition 2.2: A graded ring R is called graded singular provided $Z(R) = R$. On the other hand, R is called graded nonsingular provided $Z(R) = 0$.

Definition 2.3: Graded weak singular ideal denoted by $Zw(R)$ is defined as
$Z_w(R) = \{r \in R | r = r_1 + r_2 + ...r_k, r_i \in h(R)\ and\ r_i K = 0\ \forall i,\ for\ some\ graded\ essential\ ideal\ K\ of\ R\}$

Definition 2.4: R is a graded weak nonsingular ring if $Z_W(R) = 0$ and a graded weak singular ring if $Z_W(R) = R$.

Definition 2.5: R is called a graded weakly nil ring if there is such a multiplicatively closed subset S of nilpotent homogeneous elements from R that each homogeneous element of R is a finite sum of elements from S.

Definition 2.6: A graded ring R is said to be graded (right) strongly prime if every nonzero graded ideal I of R contains a finite subset $F \subseteq h(I)$ such that $ann_R(F) = 0$.

Definition 2.7: A graded ring R is called reduced if it has no nonzero homogeneous nilpotent elements.

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3 Main Results

In this section we present our main results.

Proposition 3.1: Every graded weak singular ideal of R is graded singular.

Proof: We need to prove $Zw(R) \subseteq Z(R)$.
Let $r \in Z_w(R)$ such that $r = r_1 + r_2 + ... + r_n$. Then $r_iK = 0$ i,and $K \leq_e R$
$\Rightarrow r_1K + r_2K + ... + r_nK = 0,\ K \leq_e R$
$\Rightarrow (r_1 + r_2 + ... + r_n)K = 0,\ K \leq_e R$
$\Rightarrow rK = 0, K \leq_e R$, where $r = r_1 + r_2 + ... + r_n$.
Hence $r \in Z(R)$. As a consequence we get $Z_w(R) \subseteq Z(R)$.
Remark: The converse of the above is not true.
For if $r \in Z(R)$ such that $r = r_1 + r_2 + ... + r_k$, then $rK = 0, K \leq_e R$.
But it does not imply $r_iK = 0$ where $K \leq_e R$.
For example: Let $R = Z_{12} \bigoplus Z_{12}$. Here $G = Z_2$, $R_0 = Z_{12},\ R_1 = Z_{12}$
$Z(R) = \{x \in R | xK = 0, K \leq_e R\}$
$Z_w(R) = \{x = x_1 + x_2 + ... + x_k \in R | x_iK = 0\ \forall i, K \leq_e R\}$
$I = \{(2x, y) | x, y \in Z_{12}\} \leq_e R$.
Then $Z(R) = \{(0,0), (6,0)\}$.
Now $6 \in R_0,\ 0 \in R_1$ and $6I \neq 0$.
Thus we have $(6,0) \notin Z(R)$ such that $(6,0) \notin Z_w(R)$. Hence $Z(R) \not\subset Z_w(R)$.

It is known that every nil ring is singular [1]. The following result establishes the same for weaker conditions.

Proposition 3.2: Every graded weakly nil ring R is graded weak singular.

Proof: Let $x \in R$ be such that $x = x_1 + x_2 + ... + x_k$,where $x_i \in h(R), degx_i > 0$, for all i. And $0 \neq y \in R$ such that $y = y_1 + y_2 + ... + y_s,\ 0 \neq y_j \in h(R), degy_j > 0$, for all i.
Since R is graded weakly nil, we have a smallest natural number n such that $x_i^n y_j R = 0$, So $x_i^{n-1} y_j R \neq 0$. Thus we have $0 \neq x_i^{n-1} y_j \subseteq r(x_i)$.
Also $0 \neq x_i^{n-1} y_j = y_j x_i^{n-1} \subseteq y_j R$.
Thus $y_j R \cap r(x_i) \neq 0$ which implies $r(x_i) \leq_e R$, for all i.
Hence $x \in Z_w(R)$ giving thereby $R \subseteq Z_w(R)$.
It is obvious that $Z_w(R) \subseteq R$. So $Z_w(R) = R$.

In [1], it is shown that if the singular ideal $Z(R)$ of a ring $= 0$, then R is a semiprime ring. It is interesting to note that a similar results holds in the graded case.

Proposition 3.3: If graded weak singular ideal $Z_W(R) = 0$ then R is a graded semiprime ring.

Proof: We assume $Z_W(R) = 0$.
Let $a \in R$ such that $a = a_1 + a_2 + ... + a_s,\ a_i \in h(R)$ with $deg\ a_j > 0$ for all i and $a^2 = 0$.
We need to show $a = 0$.
Let $0 \neq x \in R$ such that $x = x_1 + x_2 + ... + x_t$ with $deg\ x > 0$ and $x_k \neq 0$. Then either $ax = 0$ or $ax \neq 0$.
Thus $\sum a_j x_k = 0$ or $\sum a_j x_k \neq 0$ which implies $a_j x_k = 0$ or $a_j x_k \neq 0$.
Thus $x_k \in ann(a_j)$ or $a_j x_k \neq 0$.
We show that $ann(a_j) \leq_e R$.
Now $a^2 = 0$ implies $(a_1 + a_2 + ... + a_s)^2 = 0$.
Therefore

$$\sum_{j=1}^{s} a_j^2 + 2\sum_{j \neq i} a_j a_i = 0$$

.
This gives

$$(\sum_{j=1}^{s} a_j^2 + 2\sum_{j \neq i} a_j a_i) x_k = 0$$

.
Hence

$$\sum_{j=1}^{s} a_j^2 x_k + 2\sum_{j \neq i} a_j a_i x_k = 0$$

which implies $a_j^2 x_k = 0$, $a_j a_i x_k = 0$. Thus $a_j x_k \in ann(a_j)$, $a_i x_k \in ann(a_j)$ for each $j = 1, ..., s;\ k = 1, ..., t$. Hence $x_k R \cap ann(aj) \neq 0$, so that $a = a_1 + a_2 + ... + a_s \in Z_w(R)$. Thus we have $a = 0$.

Cor 3.4: If R is graded weak nonsingular ring, then R is graded semiprime.

Proposition 3.5: Every graded strongly prime ring R is graded weak non-singular.

Proof: Let $Z_W(R) \neq 0$.
Since R is graded strongly prime, so there exists a finite subset $S = x, y, z, ..., t \subseteq Z_W(R)$ such that $ann(S) = 0$. This implies $ann(x) \cap ann(y) \cap\ ...\ ann(t) = 0$.
We take $x = x_1 + x_2 + ... + x_k$.
Since $x \in Z_W(R)$, so we have $x_iK = 0$ for all $i,\ K \leq_e R$.
Thus $ann(x_i) \leq_e R$, for all i. This implies $ann(x) \leq_e R$. So we have $Z_W(R) \cap ann(S) \neq 0$, a contradiction.
So $Z_w(R) = 0$. Hence R is graded weak nonsingular.

Proposition 3.6: If I is a graded right ideal of R, then $Z_w(I)I \subseteq Z_w(R)$.

Proof: Let $z \in Z_w(I)$. Then $z \in I$ such that $z = z_1 + z_2 + ... + z_t$ where each $z_i \in h(I)$.
And $z_tk = 0,\ K \leq_e I$.
Let $i \in I$ such that $i = i_1 + i_2 + ... + i_j$. Then $z_i \in Z_w(I)I$ is such that

$$z_i = \sum_{s,k} z_s i_k,\ \ z = 1, ..., t; k = 1, ..., j$$

.
It is clear that $z_i \in R$ such that $z_s i_k \in h(R)$.
Let $0 \neq H$ be a graded ideal of R.
If $i_kH = 0$, then $z_s i_k H = 0$ implies $H \subseteq ann_R(z_s i_k)$.
Therefore $0 \neq H \subseteq ann_R(z_s i_k)H \cap ann_R(z_s i_k) \neq 0$.
This gives $ann_R(z_s i_k) \leq_e$R.
If $i_kH \neq 0$, then i_kH is a non-zero graded right ideal of I.
Since $z \in Z_w(I)$,so we have $ann_R(z_s) \cap i_kH \neq 0$, So there exists a nonzero homogeneous element h in H such that $z_s i_k h = 0$.
Hence we have $ann_R(z_s i_k) \cap H \neq 0$. So $ann_R(z_s i_k) \leq_e R$ giving thereby $z_i \in Z_w(R)$.
Hence the result follows.

Proposition 3.7: Every reduced graded ring R(with an identity) is graded weak nonsingular.

Proof: Let $r \in R$ such that $r = r_1 + r_2 + ... + r_k,\ r_j \neq 0$ and $deg\ r_1 < deg\ r_2 < ... < deg\ r_k$.
Let $0 \neq H$ be a nonzero homogeneous ideal of R.
We consider a homogeneous element x such that $x \in ann_R(r_j) \cap rH,\ \ j = 1,2,...,k$. Then $x = rh$ and $r_j r_h = 0$,for each j. So $(rhRr_j)^2 = 0$ gives $rhRr_j = 0$.
Now $xRx = rhRrh = \sum rhRr_j h = 0$. Thus $x = 0$. This implies $ann_R(r_j) \cap rH = 0$.
Hence $Z_w(R) = 0$.

Proposition 3.8: Let I be a graded ideal of R such that I is graded semiprime then

$$Z_w(I) = I \cap Z_w(R)$$

.

Proof: Let $i \in Z_w(I)$.Then $i \in I$ such that $i = i_1 + i_2 + ... + i_t$, where each $i_k \in h(I)$.
And $i_k ann(i_k) = 0,\ ann_I(i_k) \leq_e I$.
Let $0 \neq H$ be a right graded ideal of I. Then either $HI = 0$ or $HI \neq 0$.
Suppose $HI = 0$. Then $(IH)^2 = 0$.
This implies $IH = 0$. Thus $0 \neq H \subseteq ann_R(I) \subseteq ann_R(i) \subseteq ann_R(i_k)$. This gives $H \cap ann_R(i_k) \neq 0$.
Hence $ann_R(i_k) \leq_e R$.
Next we consider $HI \neq 0$. Then HI is a nonzero graded right ideal of I. So we have $ann_I(i_k) \cap HI \neq 0$. This implies $ann_R(i_k) \cap H \neq 0$ which proves that $ann_R(i_k) \leq_e R$. Hence $i \in Z_w(R)$. Also since $i \in I$, so $i \in I \cap Z_w(R)$. Conversely, let us consider a homogeneous element i such that $i \in I \cap Z_w(R)$ where $i = i_1 + i_2 + ... + i_t$ and $deg\ i_1 < deg i_2 < ... < deg\ i_t$.
Let $0 \neq H$ be a graded ideal of I. Then we have $HI \neq 0$, since I is semiprime.
Now HI is a nonzero graded right ideal of R, So we have $ann_R(i_k) \cap HI \neq 0$. Thus $ann_I(i_k) \cap H \neq 0$.
This implies $ann_I(i_k) \leq_e I$. Hence $i \in Z_w(I)$.
Thus we have $Z_w(I) = I \cap Z_w(R)$.

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