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On Homotopy of Volterrian Quadratic Stochastic Operators

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In the present paper we introduce a notion of homotopy of two Volterra operators which is related to fixed points of such operators. We establish a criterion for determining when two Volterra operators are homotopic, and as a consequence we obtain that the corresponding tournaments of that operators are the same. This gives us a possibility to know some information about the trajectory of homotopic Volterra operators. Moreover, it is shown that any Volterra q.s.o. given on a face has at least two homotopic extensions to the whole simplex.

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1 Introduction

Since Lotka and Volterra's seminal and pioneering works [29, 30] many decades ago, modeling of interacting, competing species has received considerable attention in the fields of biology, ecology, mathematics [7, 9, 17, 25, 26] and, more recently, in the physics literature as well [2, 12–14, 21, 23]. In their remarkably simple deterministic model, Lotka and Volterra considered two coupled nonlinear differential equations that mimic the temporal evolution of a two-species system of competing predator and prey populations. They demonstrated that coexistence of both species was not only possible but inevitable in their model. Moreover, similar to observations in real populations, both predator and prey densities in this deterministic system display regular oscillations in time, with both the amplitude and the period determined by the prescribed initial conditions.

To investigate the computational aspects of such dynamical systems, we need to consider discretization of such systems. This leads to the study of the trajectory of discrete time Volterra operators. Therefore, in [3–5, 18, 19] discrete time Volterra operators were considered and investigated. (Note that the more general *quadratic operators* were studied by many authors, see for example, [1, 10, 16]). A connection between such dynamical systems and the theory of tournaments were established. This allows us to to get some information about the trajectory of Volterra operators by looking at the corresponding tournaments, which are related to fixed points of Volterra operators. Moreover, some ergodic properties of such operators, in small dimensions, were studied in [6,28,31]. However, still much information is unknown about behavior of Volterra operators.

In the present paper we introduce a notion of homotopy of two Volterra operators which is related to fixed points of such operators. Further, we will establish a criterion for determining when two Volterra operators are homotopic, and as a consequence we obtain that the corresponding tournaments of that operators are the same. This, due to [3], gives us a possibility to know some information about the trajectory of homotopic Volterra operators. Moreover, it is shown that any Volterra q.s.o. given on a face has at least two homotopic extensions to the whole simplex.

2 Preliminaries

We denote by

$$S^{m-1} = \Big\{x = (x_1, x_2, \dots, x_m) \in R^m : \sum_{k=1}^{m} x_k = 1,\ x_k \geq 0\Big\}$$

the $(m-1)-$dimensional simplex. The vertices of the simplex S^{m-1} are described by the elements $e_k = (\delta_{1k}, \delta_{2k}, \dots, \delta_{mk})$, where δ_{ik} is the Kronecker's symbol. Let $I = \{1, 2, \dots m\}$ and $\alpha \subset I$ be an arbitrary subset. By Γ_α we denote the convex hull of the vertices $\{e_i\}_{i\in\alpha}$. The set Γ_α is usually called $(|\alpha|-1)-$*dimensional face* of the simplex, where $|\alpha|$ stands for the cardinality of α. An interior of Γ_α in the induced topology of $\mathbb{R}^m$ to affine hull Γ_α is called *relative interior* and is denoted by $ri\Gamma_\alpha$. One can see that

$$ri\Gamma_\alpha = \Big\{x \in S^{m-1} : x_k > 0\ \forall k \in \alpha;\ x_k = 0\ \forall k \notin \alpha\Big\}.$$

Similarly, one can define *relative boundary* $\partial\Gamma_\alpha$ of the face Γ_α. In particular, we have

$$riS^{m-1} = \{x \in S^{m-1} : x_k > 0\ \forall k \in I\},$$

$$\partial S^{m-1} = \{x \in S^{m-1} : \exists k \in I; x_k = 0\}.$$

A *Volterra quadratic stochastic operator (q.s.o.)* $V : S^{m-1} \to S^{m-1}$ is defined by

$$(V(x))_k = x_k\left(1 + \sum_{i=1}^{m} a_{ki}x_i\right),\ k = \overline{1, m}, \tag{2.1}$$

where $a_{ki} = -a_{ik}$, $|a_{ki}| \le 1$, i.e. $A_m = (a_{ki})_{k,i=1}^m$ is a skew-symmetrical matrix.

Note that in what follows sometimes for the sake for shortness we will use a terminology *Volterra operator* instead of Volterra q.s.o.

Let

$$Fix(V) = \{x \in S^{m-1} : Vx = x\}$$

be the set of all fixed points of the Volterra operator V.

One can see that for any Volterra operator V the set $Fix(V)$ contains all the vertices of the simplex S^{m-1}. Therefore, it is nonempty.

For given $x \in S^{m-1}$ we consider a sequence $\{x, Vx, \dots, V^n x, \dots\}$ called *the trajectory* of a Volterra operator V. Limit points of such a sequence is denoted by $\omega_V(x)$.

For an arbitrary $x \in \mathbb{R}^m$,

$$supp(x) = \{i \in I : x_i \neq 0\}$$

is the *support* of x. The following statements can be easily proved.

Theorem 2.1 ([3]). *Let $V : S^{m-1} \to S^{m-1}$ be a Volterra operator. Let V_α be the restriction of V to the face Γ_α. Then the following assertions are true:*

(i) *For any $\alpha \subset I$ one has $V(\Gamma_\alpha) \subset \Gamma_\alpha$.*
(ii) *For any $\alpha \subset I$ one has $V(ri\Gamma_\alpha) \subset ri\Gamma_\alpha$ and $V(\partial\Gamma_\alpha) \subset \partial\Gamma_\alpha$.*
(iii) *The restriction $V_\alpha : \Gamma_\alpha \to \Gamma_\alpha$ is also a Volterra q.s.o.*
(iv) *If $x \in Fix(V)$, then $Supp(x) \cap Supp(A_m x) = \emptyset$. In particular, if $x \in Fix(V) \cap riS^{m-1}$, then $x \in KerA_m$, where $KerA_m$ is the kernel of the matrix A_m.*
(v) *For any $x \in S^{m-1}$, the set $\omega_V(x)$ either consists of a single point or is infinite.*
(vi) *The set of all volterra q.s.o.s geometrically can be considered as a $(m(m-1)/2)$ –dimensional cub on $\mathbb{R}^{m(m-1)/2}$.*

Let A_m be a skew-symmetrical matrix corresponding to a Volterra operator given by (2.1). It is known [24] that if the order of a skew-symmetrical matrix is odd, then the determinant of this matrix is 0, otherwise the determinant is the square of some polynomial of its entries, which situated above the main diagonal. Such a polynomial is called *pffaffian* and can by calculated by the following rule:

Lemma 2.1 ([24]). *Let p_m be a pffaffian of an even order skew-symmerical matrix A_m, $m > 2$. By p_{im} we denote a pffaffian of the skew-symmerical matrix A_{im}, which is obtained from A_m by deleting of the m-th and i-th rows and columns, where $i = \overline{1, m-1}$. Then one has*

$$p_m = \sum_{i=1}^{m-1} (-1)^{i-1} p_{im} a_{im}; \quad p_2 = a_{12},$$

where p_{im} is obtained via p_{m-2} by adding 1 to all indexes greater or equal to i.

A pffaffian of the main minor of an even order skew-symmetric matrix A_m with rows and columns $\{i_1, i_2, \ldots, i_{2k}\}$ is called *main subpffafian* of order $2k$, and is denoted by $gp_{i_1 i_2 \ldots i_{2k}}$.

For example,

$$gp_{i_1 i_2} = a_{i_1 i_2}; \;\; gp_{i_1 i_2 i_3 i_4} = a_{i_1 i_2} a_{i_3 i_4} + a_{i_1 i_4} a_{i_2 i_3} - a_{i_1 i_3} a_{i_2 i_4}.$$

Definition 2.1. A skew-symmetrical matrix A_m is called transversal if all even order main minors are nonzero.

It is clear that if a skew-symmetrical matrix A_m is transversal, then all even order main subpffaffians are nonzero, that is

$$gp_{i_1 i_2 \ldots i_{2k}} \neq 0, \;\; \forall i_1, i_2, \ldots, i_{2k} \in I,$$

in particular, $a_{ki} \neq 0$ for $k \neq i$.

Definition 2.2. We say that a Volterra q.s.o. V is *transversal* if the corresponding skew-symmetrical matrix A_m is transversal.

We denote $\mathcal{V}_t^{m-1}$ the set of all transversal Volterra q.s.o.s defined in the simplex S^{m-1}. Henceforth, we will consider only transversal Volterra q.s.o. without using the word "transversal".

Theorem 2.2 ([3])**.** *Let $V \in \mathcal{V}_t^{m-1}$, then*

(i) *$Fix(V)$ is a finite set;*
(ii) *if $x \in Fix(V)$, then the cardinality of $Supp(x)$ is odd;*
(iii) *for any face Γ_α of the simplex S^{m-1} one has $|Fix(V) \cap ri\Gamma_\alpha| \leq 1$.*

Remark 2.1. Note that there is no fixed points of any (transversal) Volterra q.s.o. in the interior of odd-dimensional faces.

3 Homotopy of Volterra Operators

In this section we are going to define a notion of homotopy for Volterra operator. Further, we will show that two homotopic Volterra operators have 'similar' trajectories under some conditions.

Definition 3.1. Two Volterra operators $V_0, V_1 \in \mathcal{V}_t^{m-1}$ are called homotopic, if there exists a family of Volterra operators $\{V_\lambda\}_{\lambda \in [0,1]} \subset \mathcal{V}_t^{m-1}$ such that it is continuous with respect to λ with $V_\lambda \mid_{\lambda=0} = V_0$, $V_\lambda \mid_{\lambda=1} = V_1$ and one has $|Fix(V_\lambda)| = |Fix(V_0)| = |Fix(V_1)|$ for any $\lambda \in [0, 1]$.

Remark 3.1. Note that if a family $\{V_\lambda\}_{\lambda\in[0,1]} \subset \mathcal{V}_t^{m-1}$ is continuous then one can see that the main subpffaffians $gp^{(\lambda)}_{i_1 i_2 \dots i_{2k}}$ of the corresponding skew-symmetric matrices $A_m^{(\lambda)}$ are also continuous with respect to λ.

One can see that the introduced homotopy defines an equivalency relation in the set $\mathcal{V}_t^{m-1}$. Therefore, two operators $V_0, V_1 \in \mathcal{V}_t^{m-1}$ are called *equivalent* and denoted by $V_0 \sim V_1$, if they are homotopic. Hence, one can consider a factor set $\mathcal{V}_t^{m-1}/\sim$.

Example 3.1. Let $m = 2$. Then Volterra operators corresponding to the matrices

$$A_a = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}, \quad 0 < a \le 1$$

are always homopotic.

Let $m = 3$. Then Volterra operators corresponding to the matrices

$$A_{abc} = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}, \quad 0 < a, b, c \le 1$$

are always homotopic.

Now we are interested in determining when two Volterra q.s.o. are equivalent.

Theorem 3.1. *Let $V_0, V_1 \in \mathcal{V}_t^{m-1}$ with $V_0 \sim V_1$ and $A_m^{(0)}, A_m^{(1)}$ be their corresponding skew-symmetric matrices. Then all corresponding even order main subpffaffians of the matrices $A_m^{(0)}$ and $A_m^{(1)}$ have the same sign i.e.*

$$Sign\Big(gp^{(0)}_{i_1 i_2 \dots i_{2k}}\Big) = Sign\Big(gp^{(1)}_{i_1 i_2 \dots i_{2k}}\Big) \quad \forall i_1, i_2, \dots, i_{2k} \in I.$$

Proof. Due to $V_0 \sim V_1$ there exists a continuous family $\{V_\lambda\}_{\lambda\in[0,1]} \in \mathcal{V}_t^{m-1}$ such that $V_\lambda \mid_{\lambda=0} = V_0$ and $V_\lambda \mid_{\lambda=1} = V_1$. Let us consider a skew-symmetrical matrix $A_m^{(\lambda)}$ corresponding to V_λ. Then $A_m^{(\lambda)} \mid_{\lambda=0} = A_m^{(0)}$ and $A_m^{(\lambda)} \mid_{\lambda=1} = A_m^{(1)}$.

Assume that the assertion of the theorem is not true, that is there are $2k_0$ order main subpffaffians of the matrices $A_m^{(0)}$ and $A_m^{(1)}$ such that

$$Sign\Big(gp^{(0)}_{i_1 i_2 \dots i_{2k_0}}\Big) \ne Sign\Big(gp^{(1)}_{i_1 i_2 \dots i_{2k_0}}\Big),$$

which implies

$$gp^{(0)}_{i_1 i_2 \dots i_{2k_0}} \cdot gp^{(1)}_{i_1 i_2 \dots i_{2k_0}} < 0.$$

Continuity of $gp^{(\lambda)}_{i_1 i_2 \dots i_{2k_0}}$ with respect to λ (see Remark 3.1) yields the existence of $\lambda_0 \in [0, 1]$ such that $gp^{(\lambda_0)}_{i_1 i_2 \dots i_{2k_0}} = 0$. But the last contradicts to $V_{\lambda_0} \in \mathcal{V}_t^{m-1}$. □

Let us recall some definitions relating to tournaments associated with a skew-symmetrical matrix $A_m = (a_{ki})_{k,i=1}^m$. Put

$$Sign(A_m) = (Sign\ a_{ki})_{k,i=1}^m .$$

Define a *tournament* T_m, as a complete (full) graph consisting of m vertices labeled with $\{1, 2, \ldots, m\}$, corresponding to a skew-symmetrical matrix A_m by the following rule: there is an arrow from i to k if $a_{ki} < 0$, a reverse arrow otherwise. Note that if signs of two skew-symmetric matrices are the same, then the corresponding tournaments are the same as well.

Recall that a tournament is said to be *strong* if it is possible to go from any vertex to any other vertex with directions taken into account. A *strong component* of a tournament is a maximal strong subtournament of the tournament. The tournament with the strong components of T_m as vertices and with the edge directions induced from T_m is called *the factor tournament* of the tournament T_m and denoted by $\widetilde{T}_m$. *Transitivity* of the tournament means that there is no strong subtournament consisting of three vertices of the given tournament. A tournament containing fewer than three vertices is regarded as *transitive* by definition. As is known [8], the factor tournament $\widetilde{T}_m$ of any tournament T_m is transitive. Further, after a suitable renumbering of the vertices of T_m we can assume that the subtournament T_r contains the vertices of T_m as its vertices, i.e., $\{1\}, \{2\}, \ldots, \{r\}$. Obviously, $r \geq m$, and $r = m$ if and only if T_m is a strong tournament.

Corollary 3.1. *If $V_0 \sim V_1$, then the corresponding tournaments $T_m^{(0)}$ and $T_m^{(1)}$ are the same.*

Proof. Since $gp_{ki} = a_{ki}$, Theorem 3.1 implies that

$$Sign\Big(A_m^{(0)}\Big) = Sign\Big(A_m^{(1)}\Big).$$

Hence, the corresponding tournaments $T_m^{(0)}$ and $T_m^{(1)}$ are the same. □

This corollary gives some information about the trajectory of equivalent Volttera operators. Namely, due to results of [3] and Corollary 3.1 we get the following:

Corollary 3.2. *Let $V_0 \sim V_1$. The following assertions hold true:*

(i) *Assume that the tournament $T_m^{(0)}$ corresponding to V_0 is not strong. Then for any $x^0 \in riS^{m-1}$ one has $\omega_{V_0}(x^0) \subset \Gamma_\alpha$, $\omega_{V_1}(x^0) \subset \Gamma_\alpha$, here $\alpha = \{1, 2, \ldots, r\}$.*

(ii) *Assume that $T_m^{(0)}$ is transitive, then for any $x^0 \in riS^{n-1}$ one has $\omega_{V_0}(x^0) = \omega_{V_1}(x^0) = (1, 0, \ldots, 0)$.*

In Theorem 3.1 we have formulated a necessary condition of equivalence of two Volterra operators. Now in small dimensions, we are going to provide certain criterions for the equivalence.

Theorem 3.2. *Let $m \leq 3$. Then $V_0 \sim V_1$ if and only if*

$$Sign\Big(A_m^{(0)}\Big) = Sign\Big(A_m^{(1)}\Big).$$

Proof. Necessity immediately follows from Theorem 3.1. Therefore, we shall prove sufficiency. Let us separately consider two distinct cases with respect to m.

Let $m = 2$. Then

$$A_2^{(0)} = \begin{pmatrix} 0 & a_{12}^{(0)} \\ -a_{12}^{(0)} & 0 \end{pmatrix} \quad \text{and} \quad A_2^{(1)} = \begin{pmatrix} 0 & a_{12}^{(1)} \\ -a_{12}^{(1)} & 0 \end{pmatrix},$$

where $Sign\, a_{12}^{(0)} = Sign\, a_{12}^{(1)}$.

Consider $A_2^{(\lambda)} = (1-\lambda)A_2^{(0)} + \lambda A_2^{(1)}$. It is clear that $A_2^{(\lambda)}$ is transversal for any $\lambda \in [0,1]$. Let V_λ be the corresponding Volterra q.s.o. (see (2.1)). Then one has $V_\lambda = (1-\lambda)V_0 + \lambda V_1$ and $\{V_\lambda\}_{\lambda\in[0,1]} \subset \mathcal{V}_t^1$. According to Theorem 2.2 and Remark 2.1 the set of all fixed points of $V_\lambda \in \mathcal{V}_t^1$ consists of the vertices of S^1. Therefore, $|Fix(V_\lambda)| = |Fix(V_0)| = |Fix(V_1)| = 2$ for any $\lambda \in [0,1]$, which means $V_0 \sim V_1$.

Let $m = 3$. Then

$$A_3^{(0)} = \begin{pmatrix} 0 & a_{12}^{(0)} & a_{13}^{(0)} \\ -a_{12}^{(0)} & 0 & a_{23}^{(0)} \\ -a_{13}^{(0)} & -a_{23}^{(0)} & 0 \end{pmatrix} \quad \text{and} \quad A_3^{(1)} = \begin{pmatrix} 0 & a_{12}^{(1)} & a_{13}^{(1)} \\ -a_{12}^{(1)} & 0 & a_{23}^{(1)} \\ -a_{13}^{(1)} & -a_{23}^{(1)} & 0 \end{pmatrix},$$

where $Sign\, a_{ij}^{(0)} = Sign\, a_{ij}^{(1)}$ for $i < j$.

One can check that the skew-symmetrical matrix $A_3^{(\lambda)} = (1-\lambda)A_3^{(0)} + \lambda A_3^{(1)}$ is transversal for any $\lambda \in [0,1]$. Therefore, the corresponding Volterra q.s.o. V_λ belongs to $\mathcal{V}_t^2$ and one has $V_\lambda = (1-\lambda)V_0 + \lambda V_1$ for any $\lambda \in [0,1]$.

Assume that $V \in \mathcal{V}_t^2$, and its corresponding matrix be

$$A_3 = \begin{pmatrix} 0 & a_{12} & a_{13} \\ -a_{12} & 0 & a_{23} \\ -a_{13} & -a_{23} & 0 \end{pmatrix}.$$

Then one can find that if

$$Sign(a_{12}) = Sign(a_{13}) = Sign(a_{23}),$$

then $|Fix(V)| = 4$, otherwise $|Fix(V)| = 3$.

Hence, if the condition of the theorem is satisfied, i.e.,

$$Sign\Big(A_3^{(0)}\Big) = Sign\Big(A_3^{(1)}\Big),$$

then we conclude that either $|Fix(V_0)| = |Fix(V_1)| = 4$ or $|Fix(V_0)| = |Fix(V_1)| = 3$.

Due to

$$Sign\Big(A_3^{(\lambda)}\Big) = Sign\Big(A_3^{(0)}\Big) = Sign\Big(A_3^{(1)}\Big),$$

one gets

$$|Fix(V_\lambda)| = |Fix(V_0)| = |Fix(V_1)|$$

for any $\lambda \in [0,1]$. These imply that $V_0 \sim V_1$. □

Corollary 3.3. *If $m=2$, then $|\mathcal{V}_t^2 / \sim| = 2$. If $m=3$, then $|\mathcal{V}_t^3 / \sim| = 8$.*

Theorem 3.3. *Let $m=4$. Then $V_0 \sim V_1$ if and only if*

$$Sign\Big(A_4^{(0)}) = Sign(A_4^{(1)}\Big), \ \ Sign\Big(gp_{1234}^{(0)}) = Sign(gp_{1234}^{(1)}\Big),$$

where

$$gp_{1234}^{(i)} = a_{12}^{(i)}a_{34}^{(i)} + a_{14}^{(i)}a_{23}^{(i)} - a_{13}^{(i)}a_{24}^{(i)}, \ \ i=0,1.$$

Proof. As before, the necessity immediately follows from Theorem 3.1. Let us prove the sufficiency.

Let

$$A_4^{(0)} = \begin{pmatrix} 0 & a_{12}^{(0)} & a_{13}^{(0)} & a_{14}^{(0)} \\ -a_{12}^{(0)} & 0 & a_{23}^{(0)} & a_{24}^{(0)} \\ -a_{13}^{(0)} & -a_{23}^{(0)} & 0 & a_{34}^{(0)} \\ -a_{14}^{(0)} & -a_{24}^{(0)} & -a_{34}^{(0)} & 0 \end{pmatrix}, \ A_4^{(1)} = \begin{pmatrix} 0 & a_{12}^{(1)} & a_{13}^{(1)} & a_{14}^{(1)} \\ -a_{12}^{(1)} & 0 & a_{23}^{(1)} & a_{24}^{(1)} \\ -a_{13}^{(1)} & -a_{23}^{(1)} & 0 & a_{34}^{(1)} \\ -a_{14}^{(1)} & -a_{24}^{(1)} & -a_{34}^{(1)} & 0 \end{pmatrix},$$

where $Sign\ a_{ij}^{(0)} = Sign\ a_{ij}^{(1)}$ for $i<j$ and

$$Sign\left(a_{12}^{(0)}a_{34}^{(0)} + a_{14}^{(0)}a_{23}^{(0)} - a_{13}^{(0)}a_{24}^{(0)}\right) = Sign\left(a_{12}^{(1)}a_{34}^{(1)} + a_{14}^{(1)}a_{23}^{(1)} - a_{13}^{(1)}a_{24}^{(1)}\right).$$

Let us consider the following skew-symmetric matrix $A_4^{(\lambda)}$ defined by

$$\begin{pmatrix} 0 & A_{12} & A_{13} & A_{14} \\ -A_{12} & 0 & A_{23} & A_{24} \\ -A_{13} & -A_{23} & 0 & A_{34} \\ -A_{14} & -A_{24} & -A_{34} & 0 \end{pmatrix},$$

where

$$A_{12} = (1-\lambda)a_{12}^{(0)} + \lambda a_{12}^{(1)}, \qquad A_{13} = (1-\lambda)a_{13}^{(0)} + \lambda a_{13}^{(1)},$$

$$A_{14} = (1-\lambda)a_{14}^{(0)} + \lambda a_{14}^{(1)}, \qquad A_{23} = \frac{(1-\lambda)a_{14}^{(0)}a_{23}^{(0)} + \lambda a_{14}^{(1)}a_{23}^{(1)}}{(1-\lambda)a_{14}^{(0)} + \lambda a_{14}^{(1)}},$$

$$A_{24} = \frac{(1-\lambda)a_{13}^{(0)}a_{24}^{(0)} + \lambda a_{13}^{(1)}a_{24}^{(1)}}{(1-\lambda)a_{13}^{(0)} + \lambda a_{13}^{(1)}}, \qquad A_{34} = \frac{(1-\lambda)a_{12}^{(0)}a_{34}^{(0)} + \lambda a_{12}^{(1)}a_{34}^{(1)}}{(1-\lambda)a_{12}^{(0)} + \lambda a_{12}^{(1)}}.$$

It is then clear that

$$Sign\Big(a_{ij}^{(\lambda)}\Big) = Sign\Big(a_{ij}^{(0)}\Big) = Sign\Big(a_{ij}^{(1)}\Big) \quad \text{for } i < j$$

and

$$\begin{aligned} & Sign\Big(a_{12}^{(\lambda)}a_{34}^{(\lambda)} + a_{14}^{(\lambda)}a_{23}^{(\lambda)} - a_{13}^{(\lambda)}a_{24}^{(\lambda)}\Big) \\ = & Sign\Big((1-\lambda)a_{12}^{(0)}a_{34}^{(0)} + \lambda a_{12}^{(1)}a_{34}^{(1)} + (1-\lambda)a_{14}^{(0)}a_{23}^{(0)} + \lambda a_{14}^{(1)}a_{23}^{(1)} \\ & \quad - (1-\lambda)a_{13}^{(0)}a_{24}^{(0)} - \lambda a_{13}^{(1)}a_{24}^{(1)}\Big) \\ = & Sign\left(a_{12}^{(0)}a_{34}^{(0)} + a_{14}^{(0)}a_{23}^{(0)} - a_{13}^{(0)}a_{24}^{(0)}\right) \\ = & Sign\left(a_{12}^{(1)}a_{34}^{(1)} + a_{14}^{(1)}a_{23}^{(1)} - a_{13}^{(1)}a_{24}^{(1)}\right) \end{aligned}$$

for any $\lambda \in [0,1]$. This implies that the corresponding Volterrian operator V_λ belongs to $\mathcal{V}_t^4$ for any $\lambda \in [0,1]$.

Since $m = 4$ is even, according to Remark 2.1 there is no fixed point in the interior of the simplex S^3. Thanks to Theorem 3.2 one gets

$$|Fix(V_\lambda) \cap \partial S^3| = |Fix(V_0) \cap \partial S^3| = |Fix(V_1) \cap \partial S^3|$$

for any $\lambda \in [0,1]$. Therefore, $|Fix(V_\lambda)| = |Fix(V_0)| = |Fix(V_1)|$ for any $\lambda \in [0,1]$. ☐

Corollary 3.4. *If* $m = 4$, *then* $|\mathcal{V}_t^4 \diagup \sim| = 112$.

The following theorem can be considered a reverse to Theorem 3.1.

Theorem 3.4. *Let* $V_0, V_1 \in \mathcal{V}_t^{m-1}$ *and* $A_m^{(0)}$, $A_m^{(1)}$ *be their the corresponding skew-symmetric matrices. If all corresponding even order main subpffaffians of the matrices* $A_m^{(0)}$ *and* $A_m^{(1)}$ *have the same sign, that is*

$$Sign\Big(gp_{i_1 i_2 \dots i_{2k}}^{(0)}\Big) = Sign\Big(gp_{i_1 i_2 \dots i_{2k}}^{(1)}\Big), \ \ \forall i_1, i_2, \dots, i_{2k} \in I,$$

then $|Fix(V_0)| = |Fix(V_1)|$.

Proof. We will prove this theorem by the induction with respect to the dimension m of the simplex S^{m-1}. For small dimensions our assumption is true (see Theorems 3.2 and 3.3). Let us assume that the statement of the theorem is true for dimension $m-1$. Now we prove it for dimension m.

Since the restriction of any transversal Volterra operator to any face Γ_α of the simplex is also transversal Volterra operator (see Theorem 2.1), by the assumption of the induction we get that operators V_0 and V_1 have the same number of fixed points in ∂S^{m-1}, i.e.

$$Fix(V_0) \cap \partial S^{m-1} = Fix(V_1) \cap \partial S^{m-1}.$$

Let us show that the operators V_0 and V_1 have the same number of fixed points in riS^{m-1}.

If m is even, then due to Remark 2.1 there is no fixed point of Volterra q.s.o. in the interior of the simplex. Therefore, we have to prove the theorem only when m is odd. In this case, Theorem 2.2 implies that $|Fix(V) \cap riS^{m-1}| \le 1$ for any $V \in \mathcal{V}_t^{m-1}$. According to Theorem 2.1 $x \in Fix(V) \cap riS^{m-1}$ if and only if $x \in KerA_m \cap riS^{m-1}$.

Due to oddness of m the determinant of A_m equals to 0, but the transversality of the operator V_0 yields that the minor of order $m-1$ is not zero, which means the dimension of image $Im(A_m)$ is $m-1$. Hence, the equality $dim(KerA_m) + dim(ImA_m) = m$ implies that $KerA_m$ is a one dimensional space.

Now we are going to describe $KerA_m$. Keeping in mind that $detA_m$ is zero, one finds

$$\sum_{k=1}^{m} a_{ki}A_{ki} = detA_m = 0, \quad \forall k = \overline{1,m}, \tag{3.1}$$

where A_{ki} is an algebraic completion (i.e. algebraic minor) of entry a_{ki}. It is known [24] that

$$A_{ki} = (-1)^{k+i} gp_{I_k} gp_{I_i}, \tag{3.2}$$

where as before gp_{I_k} and gp_{I_i} are pffaffians of the minors M_{kk} and M_{ii}, respectively, $I_k = I \setminus \{k\}$ and $I_i = I \setminus \{i\}$. It then follows from (3.1) and (3.2) that

$$(-1)^k gp_{I_k} \sum_{i=1}^{m} a_{ki}(-1)^i gp_{I_i} = 0, \quad \forall k = \overline{1,m}.$$

Thanks to $gp_{I_k} \neq 0$, one finds

$$\sum_{i=1}^{m} a_{ki}(-1)^i gp_{I_i} = 0, \quad \forall k = \overline{1,m}. \tag{3.3}$$

This means that for an element defined by

$$x_0 = \left(-gp_{I_1}, gp_{I_2}, \dots, (-1)^i gp_{I_i}, \dots (-1)^m gp_{I_m}\right)$$

one has $A_m(x_0) = 0$, i.e. $x_0 \in KerA_m$. The one-dimensionality of A_m implies that $KerA_m = \{\lambda x_0 : \lambda \in \mathbb{R}\}$. This means that there is an interior fixed point for the Volterra operator V_0 if and only if

$$Sign(-1)^1 gp_{I_1} = Sign(-1)^2 gp_{I_2} = \cdots = Sign(-1)^i p_{I_i} = \cdots = Sign(-1)^m gp_{I_m}$$

and that fixed point is given by

$$x = \frac{1}{GP} x_0, \tag{3.4}$$

where

$$GP = \sum_{i=1}^{m} (-1)^k gp_{I_i}.$$

Now if the condition of the theorem is satisfied, then from (3.4) one concludes that

$$|Fix(V_0) \cap riS^{m-1}| = |Fix(V_1) \cap riS^{m-1}|.$$

Consequently, one gets $|Fix(V_0)| = |Fix(V_0)|$. This completes the proof. □

According to Theorem 2.1 the set of all Volterra q.s.o. geometrically forms a $(m(m-1)/2)$-dimensional cube $\mathcal{V}^{m-1}$ in $\mathbb{R}^{m(m-1)/2}$. Now let us consider the following manifolds

$$\Big\{ V \in \mathcal{V}^{m-1} : gp_{i_1 i_2 \dots i_{2k}}(V) = 0, \ \exists i_1, i_2, \dots, i_{2k} \in I \Big\}.$$

These manifolds divide the cube into several connected components.

From Theorems 3.1 and 3.4 one can prove the following:

Theorem 3.5. *Two Volterra q.s.o. V_0 and V_1 ($V_0, V_1 \in \mathcal{V}_t^{m-1}$) are homotopic if and only if the operators V_0 and V_1 belong to only one connected component of the cube.*

Proof. 'If' part of the proof immediately follows from Theorem 3.1. Therefore, let us prove 'only if' part.

Let us assume that V_0 and V_1 belong to the same connected component. Then from the definition of component one can conclude that such operators can be connected by a continuous path $\{V_\lambda\} \in \mathcal{V}_t^{m-1}$ located in that component. On the other hand, we see that the corresponding all main subpffaffians of all operators V_λ have the same signs. So, thanks Theorem 3.4 one finds that $|Fix(V_\lambda)| = |Fix(V_0)| = |Fix(V_1)|$ which implies that $V_0 \sim V_1$. □

Remark 3.2. Note that in small dimensions ($m \leq 4$) the necessity condition, for homotopy of Volterra operators, is sufficient as well.

Corollary 3.5. *If $V_0 \sim V_1$, then for any face Γ_α of the simplex S^{m-1} one has $V_0 \mid_{\Gamma_\alpha} \sim V_1 \mid_{\Gamma_\alpha}$.*

Proof. Let $I \setminus \Gamma_\alpha = \{i_1, i_2, \dots, i_k\}$. According to Theorem 2.1 the restriction of Vorterra q.s.o. V to Γ_α, i.e. $V_\alpha : \Gamma_\alpha \to \Gamma_\alpha$ is also Volterra q.s.o. Therefore, the corresponding the skew-symmetrical matrix A_α to $V \mid_{\Gamma_\alpha}$ is a matrix which can be obtained from the matrix A_m by eliminating the rows $\{i_1, i_2, \dots, i_m\}$ and the columns $\{i_1, i_2, \dots, i_m\}$. Now if $V_0 \sim V_1$, then from Theorem 3.5 it follows that the operators V_0 and V_1 lie on the same connected component. From the definition of the subpffaffinas one concludes that the operators $V_0 \mid_{\Gamma_\alpha}, V_1 \mid_{\Gamma_\alpha}$ also belong to the same connected component, hence again Theorem 3.5 implies that $V_0 \mid_{\Gamma_\alpha}$ and $V_1 \mid_{\Gamma_\alpha}$ are homotopic. □

Corollary 3.6. *Let $V_0 \sim V_1$. Then for any face Γ_α of the simplex S^{m-1} one has*

(i) $|Fix(V_0) \cap \Gamma_\alpha| = |Fix(V_1) \cap \Gamma_\alpha|$.
(ii) $|Fix(V_0) \cap ri\Gamma_\alpha| = |Fix(V_1) \cap ri\Gamma_\alpha|$.

Proof. (i). Corollary 3.5 yields that $V_0 \mid_{\Gamma_\alpha} \sim V_1 \mid_{\Gamma_\alpha}$ for any $\alpha \subset I$. Therefore, $|Fix(V_0) \cap \Gamma_\alpha| = |Fix(V_0) \cap \Gamma_\alpha|$.

(ii) Now suppose that $\alpha = \{i_1, i_2, \dots, i_k\}$. One can see that

$$\partial\Gamma_\alpha = \bigcup_{n=1}^{k} \Gamma_{\alpha_{i_n}}, \tag{3.5}$$

where $\alpha_{i_n} = \alpha \setminus \{i_n\}$. From (i) one finds that $|Fix(V_0) \cap \Gamma_{\alpha_{i_n}}| = |Fix(V_0) \cap \Gamma_{\alpha_{i_n}}|$ for any $n = \overline{1,k}$. Hence from (3.5) we get $|Fix(V_0) \cap \partial\Gamma_\alpha| = |Fix(V_0) \cap \partial\Gamma_\alpha|$, which implies $|Fix(V_0) \cap ri\Gamma_\alpha| = |Fix(V_0) \cap ri\Gamma_\alpha|$. □

Remark 3.3. The proved corollaries imply that equivalent Volterra operators have the same number of fixed points on every face and its interior as well. Due to these facts one can ask: are there homotopic extensions of a given Volterra operator on a face to whole simplex? Next we are going to study this question.

Let $\alpha \subset I$ and $V_0 : \Gamma_\alpha \to \Gamma_\alpha$ be a transitive Volterra q.s.o. on a face Γ_α. Denote

$$F_{V_0}(\alpha) = \{V \in \mathcal{V}_t^{m-1} : V \mid_{\Gamma_\alpha} \sim V_0\}. \tag{3.6}$$

Remark 3.4. From the definition of F_{V_0} it follows for any $V_1, V_2 \in F_{V_0}(\alpha)$ one has $V_1 \mid_{\Gamma_\alpha} \sim V_2 \mid_{\Gamma_\alpha}$.

Since Γ_α is a $(|\alpha| - 1)$-dimensional simplex, so Corollary 3.5 implies that

Lemma 3.1. *For any $\alpha, \beta \subset I$ one has*

$$\beta \subset \alpha \quad \Rightarrow \quad F_{V_0}(\alpha) \subset F_{V_0}(\beta).$$

In particular, $\forall \alpha \subset I$ one gets

$$F_{V_0}(I) \subset F_{V_0}(\alpha).$$

Let

$$\mathcal{V}_t^{m-1}/_\sim = \{\mathcal{V}_1, \dots, \mathcal{V}_r\}. \tag{3.7}$$

Then from (3.6) and (3.7) one can see that for any $V_0 \in \mathcal{V}_t^{m-1}$ there exists $i \in \{1, 2, \dots, r\}$ such that

$$F_{V_0}(I) = \mathcal{V}_i.$$

Therefore, we are interested in $|\alpha| \le n - 1$. In this case, it is clear that any Volterra operator given on Γ_α can be extended to a transversal Volterra operator defined on the simplex S^{m-1}. Note that such an extension is not unique.

In what follows we shall assume that a Volterra operator V_0 is defined on the whole simplex S^{m-1}, i.e. $V_0 \in \mathcal{V}_t^{m-1}$.

Theorem 3.6. *Let $|\alpha| \leq n-1$. Then there are $i, j \in \{1, 2, \ldots, r\}$, $i \neq j$ such that*

$$F_{V_0} \cap \mathcal{V}_i \neq \emptyset \quad \textit{and} \quad F_{V_0} \cap \mathcal{V}_j \neq \emptyset.$$

Proof. Due to $V_0 \in \mathcal{V}_t^{m-1}/_\sim$ there is $i \in \{1, 2, \ldots, r\}$ such that $V_0 \in \mathcal{V}_i$. As before, by A_m^0 we denote the corresponding skew-symmetric matrix. From $|\alpha| \leq n-1$ we have $I \setminus \alpha \neq \emptyset$. Let $p_0 \in I \setminus \alpha$, $q_0 \in I$. Then $a^0_{p_0q_0}$ is not an element of $A^0_\alpha = A^0_m \mid_{\Gamma_\alpha}$. Without loss of generality we may assume that $a^0_{p_0q_0} > 0$. Now we are going to construct a skew-symmetric matrix $A^1_m = (a^1_{ij})^m_{i,j=1}$ as follows: if $i, j \notin \{p_0, q_0\}$, then we put $a^1_{ij} = a^0_{ij}$. We choose $a^1_{p_0q_0}$ from the segment $[-1, 0)$ (Note that $a^0_{q_0p_0} \in (0, 1]$) such that all pffaffians of the matrix A^1_m is not zero. The existence of such a number comes from that fact that each pffaffian is a polynomial with respect to $a^1_{p_0q_0}$ (since all the rest elements are defined), therefore, its zeros are finite, and such paffaffians are finite as well. So, all pffaffians are not zero except for finite numbers of $[-1, 0)$. According to the construction A^1_m is a skew-symmetric, hence the corresponding Volterra q.s.o. V_1 is transversal. Moreover, $A^1_m \mid_{\Gamma_\alpha} = A^0_m \mid_{\Gamma\alpha}$, i.e., $V_1 \mid_{\Gamma_\alpha} \sim V_0 \mid_{\Gamma_\alpha}$. But V_1 and V_0 are not homotopic, since $a^1_{p_0q_0}$ and $a^0_{p_0q_0}$ have different sighs. This means that the second order pffaffians have different sighs too (see Theorem 3.1). Let $\mathcal{V}_j$ be a set of Volttera operators which are equivalent to V_1. Then the construction shows that $i \neq j$. □

Corollary 3.7. *Note also that if $|\alpha| \leq n-1$, then*

(i) *$F_{V_0}(\alpha)$ is not a subset of any $\mathcal{V}_i$ (here $i \in \{1, 2, \ldots, r\}$);*
(ii) *$F_{V_0}(\alpha)$ is not a linearly connected set.*

Remark 3.5. From Theorem 3.6 we conclude that any transversal Volterra operator given on a face has no unique homotopic extension.

It is clear that

$$F_{V_0}(\alpha) = \bigcup_{i=1}^{r} \Big(F_{V_0}(\alpha) \bigcap \mathcal{V}_i \Big).$$

Theorem 3.7. *If there is some $i \in \{1, 2, \ldots, r\}$ such that $F_{V_0}(\alpha) \bigcap \mathcal{V}_i$ is not empty, then it is linearly connected.*

Proof. Let us assume that $F_{V_0}(\alpha) \bigcap \mathcal{V}_i$ is not empty for some $i \in \{1, 2, \ldots, r\}$. Then we take two elements $V_1, V_2 \in F_{V_0}(\alpha) \bigcap \mathcal{V}_i$. Now we are going to show such element can be connected with a path lying in $F_{V_0}(\alpha) \bigcap \mathcal{V}_i$. Taking into account that $V_1, V_2 \in \mathcal{V}_i$ and Theorem 3.5 we find that there is a path $\{V_\lambda\}_{\lambda \in [1,2]} \subset \mathcal{V}_i$ connecting them. For any $\lambda \in [1, 2]$ one has $V_\lambda \sim V_1$, hence $V_\lambda \mid_{\Gamma_\alpha} \sim V_1 \mid_{\Gamma_\alpha}$. From $V_1 \mid_{\Gamma_\alpha} \sim V_0 \mid_{\Gamma_\alpha}$ we obtain $V_\lambda \mid_{\Gamma_\alpha} \sim V_0 \mid_{\Gamma_\alpha}$, this means that $\{V_\lambda\}_{\lambda \in [1,2]} \subset F_{V_0}(\alpha)$. Therefore, $F_{V_0}(\alpha) \bigcap \mathcal{V}_i$ is linearly connected. □

Corollary 3.8. *For any $\alpha \subset I$ one has*

$$F_{V_0}(\alpha)/_{\sim} = \left\{F_{V_0}(\alpha) \bigcap \mathcal{V}_i\right\}_{i=1}^{r},$$

where for some i the set $F_{V_0}(\alpha) \bigcap \mathcal{V}_i$ can be empty. Therefore,

$$|F_{V_0}(\alpha)/_{\sim}| \leq r.$$

In particular, the equality occurs when $|\alpha| = 1$, i.e.,

$$F_{V_0}(\alpha)/_{\sim} = \{\mathcal{V}_i\}_{i=1}^{r} = \mathcal{V}_t^{m-1}/_{\sim}.$$

From Theorem 3.6 we conclude that if two Volterra operators are homotopic on a face, then they need not be homotopic on the simplex S^{m-1}. But there arises the following problem:

Problem 3.1. How many faces, on which two Volterra operators are homotopic, need to be homotopic of such operators on the simplex S^{m-1}?

Now we are going to show that the formulated problem has negative solution when m is even.

Example 3.2. Consider a case when $m = 4$, then according to Theorem 3.3 we know that two Volterra q.s.o. are homotopic iff the signature of corresponding matrices are the same, and moreover, Pfaffians of their determinants have the same sign. Let us consider two transversal Volttera q.s.o. corresponding to the following matrices

$$A_4^{(1)} = \begin{pmatrix} 0 & 1 & 1 & 1 \\ -1 & 0 & 1 & 1 \\ -1 & -1 & 0 & 1 \\ -1 & -1 & -1 & 0 \end{pmatrix}, \quad A_4^{(2)} = \begin{pmatrix} 0 & \frac{1}{2} & 1 & \frac{1}{2} \\ -\frac{1}{2} & 0 & \frac{1}{2} & 1 \\ -1 & -\frac{1}{2} & 0 & \frac{1}{2} \\ -\frac{1}{2} & -1 & -\frac{1}{2} & 0 \end{pmatrix}.$$

Then one can check that the operators V_1 and V_2 are homotopic on any proper face of the simplex S^3(see Theorem 3.2). Since the pffaffians corresponding to determinants of the matrices $A_4^{(1)}$ and $A_4^{(2)}$ are 1 and $-1/2$, respectively, from Theorem 3.3 we conclude that they are not homotopic on the whole simplex S^3.

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