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Entropy of a Two-Level Atom Driven by a Detuned Monochromatic Laser Field and Damped by a Squeezed Vacuum

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In this paper we study the entropy of a two-level atom driven by a detuned monochromatic laser field and damped by a squeezed vacuum. We obtain an exactly analytic solution of the model, by means of which we identify and numerically demonstrate the region of parameters where significantly large entanglement can be obtained. Also, the marginal distribution of the atomic Wehrl density is studied.

Keywords: Entropy, monochromatic laser field, quantum entropy.

1 Introduction

The interaction between a two-level atom and the radiation field is a quantum optical problem that lies at the heart of many problems in laser physics and quantum optics [1−3]. The fundamental difference between quantum and classical physics is the possible existence of nonclassical correlations between distinct quantum systems. The physical property responsible for the nonclassical correlations is called entanglement. From the beginning of the nineties, the field of quantum information theory opened up and expanded rapidly.

Quantum entanglement began to be seen not only as a puzzle, but also as a resource to be manipulated for communication, information processing and quantum computing, such as in the investigation of quantum teleportation, dense coding, decoherence in quantum computers and the evaluation of quantum cryptographic schemes [4−8]. A number of entanglement measures have been discussed in the literature, such as the von Neumann reduced entropy, the relative entropy of entanglement [9−11], the so-called entanglement of distillation and the entanglement of formation [12]. Several authors proposed physically motivated postulates to characterize entanglement measures [9−11], [13−17]. These postulates (although they vary from author to author in the details) are based on the concepts of the operational formulation of quantum mechanics [18]. A method using quantum mutual entropy to measure the degree of entanglement in the time development of the Jaynes-Cummings model has been adopted in [19], which is called DEM (degree of entanglement due to mutual entropy). The entanglement in the time development of the JC-model with squeezed state has been investigated [20], where it was shown that the entanglement can be controlled by means of squeezing.

In this paper, we aim at extending the previously cited treatment to study the problem of a two-level atom interacting with a single-mode including acceptable kinds of nonlinearities of both the field and the intensity-dependent atom-field coupling. We suppose that, inside the cavity a single-mode is coupled with both the atom as well as the nonlinear medium. The organization of this paper is as follows: in section 2 we introduce our Hamiltonian model and give exact expression for the density matrix operator $\rho(t)$. In section 3 we employ the analytical results obtained in section 2 to investigate the entanglement degree due to the quantum field entropy followed by a summary in section 4.

2 The Model

We consider here a general situation of the interaction between a two-level atom in a squeezed vacuum with a broad bandwidth, in which arbitrary values of both the Rabi frequency and the detuning are considered. The model of Hamiltonian consists of a two-level atom driven by a detuned monochromatic laser field and damped by a squeezed vacuum with a broad bandwidth is given by

$$\hat{H} = \hat{H}_A + \hat{H}_F + \hat{H}_L + \hat{H}_I. \tag{2.1}$$

The atomic part of the Hamiltonian has the following form

$$\hat{H}_A = \frac{h}{2}\omega_A\sigma_z = \frac{h}{2}\Delta\sigma_z + \frac{h}{2}\omega_L\sigma_z \tag{2.2}$$

is the Hamiltonian of the atom,

$$\hat{H}_F = h\int_0^\infty \omega\, b^\dagger(\omega)\, b(\omega)\mathrm{d}\omega\,, \tag{2.3}$$

where $b(\omega)$ and $b^\dagger(\omega)$, respectively, are the annihilation and the creation operators for the field mode.

$$\hat{H}_L = \frac{h\Omega}{2}\left[S_+exp(-i\omega_L - i\phi_L) + S_-exp(i\omega_L + i\phi_L)\right]$$

is the interaction between the atom and the classical laser field,

$$H_I = ih\int_0^\infty K(\omega)\,[b^\dagger(\omega)S_- - S_+b(\omega)]\mathrm{d}\omega \tag{2.4}$$

is the interaction between the atom with the reservoir. In the above equations, $K(\omega)$ is the coupling of the atom to the vacuum modes, $\Delta = \omega_A - \omega_L$ is the detuning parameter of the driving laser field frequency ω_L and S_+, S_- and S_z are the Pauli pseudo-spin operators describing the two-level atom and satisfy the relations $[S_\pm, S_z] = \mp S_-$, $[S_+, S_-] = 2S_z$. We assume that the reservoir is in a squeezed vacuum state in which the operators $b(\omega)$ and $b^\dagger(\omega)$ satisfy the relations

$$\begin{aligned} T_{\Gamma_R}[b(\omega)\, b^+(\omega')] &= [N+1]\delta(\omega-\omega')\,, \\ T_{\Gamma_R}[b^+(\omega)\, b(\omega')] &= N\delta(\omega-\omega')\,, \\ T_{\Gamma_R}[b(\omega)\, b(\omega')] &= M(\omega)e^{i\phi_s}\delta(2\omega_s-\omega-\omega')\,, \end{aligned} \tag{2.5}$$

where ω_s is the carrier frequency of the squeezed vacuum and the squeezed phase ϕ_s. For a broadband squeezed vacuum, N and M are the independent of frequency ω and obey the following relation $M = \eta\sqrt{N(N+1)} \quad (0 \leq \eta \leq 1)$.

The quantity η measures the degree of the two-photon correlation in the squeezed vacuum. The value $\eta = 1$ indicates an ideal squeezed vacuum, i.e., one which shows the maximum degree of squeezing possible for a given N, whilst $\eta = 0$ corresponds to no squeezing at all (a chaotic field). Theory predicts that the squeezed output of an ideal parametric oscillator is characterized by $\eta = 1$.

In order to derive the master equation we perform a two unitary transform. We use the atomic Hamiltonian (2.2) and the free field Hamiltonian (2.3) to transform to the frame rotating with the laser frequency ω_L and the interaction picture with respect to the reservoir modes. The rotating frame is also shifted in phase by ϕ_L, i.e., the raising and lowering operators which absorbs the phase factor according to the relations

$$S_-\, exp(i\phi_L) \longrightarrow S_-\,, \quad S_+\, exp(-i\phi_L) \longrightarrow S_+.$$

The total Hamiltonian takes the form

$$H_o + H_I(t)\,,$$

where

$$H_o = \frac{h}{2}\Delta\sigma_z + \frac{h\Omega}{2}(S_+ + S_-)\,, \tag{2.6}$$

and

$$H_I(t) = i\,h\int_0^\infty \left\{K(\omega)S_+b(\omega)e^{i\phi_L+i(\omega_L-\omega)t} - b^\dagger(\omega)S_-e^{-i\phi-i(\omega_L-\omega)}\right\}\mathrm{d}\omega\,. \tag{2.7}$$

The master equation of the reduced density operator for the atom in a frame rotating with the laser frequency ω_L is of the form,

$$\begin{aligned}\frac{\mathrm{d}\rho}{\mathrm{d}t} &= -i\left[\frac{\Delta}{2}S_z + \frac{\Omega}{2}\,(S_+ + S_-\rho)\;\right] - \frac{\gamma}{2}(N+1)(S_+S_-\rho + \rho S_+S_- - 2S_-\rho S_+) \\ &- \frac{\gamma}{2}N(S_-S_+\rho + \rho S_-S_+ - 2S_+\rho S_-) - \gamma|M|\,e^{i\phi_S}\,e^{-2i(\omega_S-\omega_L)t}\,(S_-\rho S_-) \\ &- \gamma|M|\,e^{-i\phi_S}\,e^{2i(\omega_S-\omega_L)t}\,(S_+\rho S_+)\,, \end{aligned} \tag{2.8}$$

where γ is the spontaneous decay rate of the atom into the standard vacuum modes, ω_S is the carrier frequency of squeezed vacuum, $S^+ = |e><g|$ and $S^- = |g><e|$ are the atomic raising and lowering operators, respectively, with $|e>$ and $|g>$ are the excited and ground state of the atom, respectively. For simplicity, we assume that the carrier frequency is coincident with the laser frequency ω_L.

The master equation (2.8) leads to a closed set of three equations of motion for the expectation values of the atomic operators (optical Bloch equations), which are given by

$$\begin{aligned}\frac{\mathrm{d}\rho_x}{\mathrm{d}t} &= -\xi_x\rho_x - [\Delta + \gamma M\sin\Phi]\rho_y\,, \\ \frac{\mathrm{d}\rho_y}{\mathrm{d}t} &= -\xi_y\rho_y + [\Delta - \gamma M\sin\Phi]\rho_x - \Omega\rho_z\,, \\ \frac{\mathrm{d}\rho_z}{\mathrm{d}t} &= -\xi_z\rho_z + \Omega\rho_y - \gamma\,, \end{aligned} \tag{2.9}$$

where

$$\rho_x = \rho_{eg} + \rho_{ge}\,, \quad \rho_y = \rho_{ge} - \rho_{eg} \quad \text{and} \quad \rho_z = \rho_{ee} - \rho_{gg} \tag{2.10}$$

with

$$\xi_x = \Gamma + \gamma M cos\Phi\,, \quad \xi_y = \Gamma - \gamma M cos\Phi\,, \quad \xi_z = \xi_x + \xi_y = 2\Gamma\,,$$

where $\Gamma = \gamma(N + 1/2)$ and $S_z = \frac{1}{2}(|e><e| - |g><g|)$ is the atomic inversion. We denote by Ω is Rabi frequency and ϕ_L is the laser phase. We also denote by $S_x = \frac{1}{2}(S_- + S_+)$ the in-phase (X) and $S_y = (i/2)(S_- - S_+)$ out-phase (Y) quadrature components of the atomic polarization, respectively. $\Phi = 2\phi_L - \phi_S$ is the relative phase between laser field and squeezed vacuum modes. The modified decay rates of the X and Y components of the atomic polarization are ξ_x and ξ_y, respectively, whilst ξ_z is the decay rate of the atomic population inversion. After a tedious algebraic calculations, the solution of optical Bloch equation can be written as

$$\begin{aligned}\rho_x &= \left[\Re_1(t) - \xi_x\Re_2(t) - (\xi_x^2 - \alpha\beta)\Re_3(t)\right]\rho_x(0) + \alpha\left[(\xi_x + \xi_y)\Re_3(t) - \Re_2(t)\right]\rho_y(0) \\ &+ \alpha\Omega\Re_3(t)\rho_z(0) - \alpha\gamma\Omega\int_0^t \Re_3(t)\mathrm{d}t\,, \end{aligned} \tag{2.11}$$

$$\begin{aligned}
\rho_y &= \beta\Big[\Re_2(t) - (\xi_x + \xi_y)\Re_3(t)\Big]\rho_x(0) \\
&+ \Big[\Re_1(t) - \xi_y\Re_2(t) + (\xi_y^2 - \alpha\beta - \Omega^2)R_3(t)\Big]\rho_y(0) \\
&+ \Big[\Omega(\xi_y + \xi_z)\Re_3(t) - \Omega\Re_2(t)\Big]\rho_z(0) - \gamma\Omega\int_0^t \Big[\Re_2(t) - (\xi_y + \xi_z)\Re_3(t)\Big]\,\mathrm{d}t\,, \qquad (2.12)
\end{aligned}$$

$$\begin{aligned}
\rho_z &= \Re_3(t)\Omega\beta\rho_x(0) + \Omega\Big[\Re_2(t) - (\xi_y + \xi_z)\Re_3(t)\Big]\rho_y(0) \\
&+ \Big[(\xi_z^2 - \Omega^2)\Re_3(t) - \xi_z\Re_2(t) + \Re_1(t)\Big]\rho_z(0) \\
&- \gamma\int_0^t \Big[\Re_1(t) - \xi_z\Re_2(t) + (\xi_z^2 - \Omega^2)\Re_3(t)\Big]\,\mathrm{d}t\,, \qquad (2.13)
\end{aligned}$$

where $\Re_1(t)$, $\Re_2(t)$ and $\Re_3(t)$ are given by

$$\begin{pmatrix}\Re_1(t)\\ \Re_2(t)\\ \Re_3(t)\end{pmatrix} = \frac{1}{D}\begin{pmatrix}\mu_2\mu_3(\mu_3-\mu_2) & \mu_1\mu_3(\mu_1-\mu_3) & \mu_1\mu_2(\mu_2-\mu_1)\\ \mu_3^2-\mu_2^2 & \mu_1^2-\mu_3^2 & \mu_2^2-\mu_1^2\\ \mu_3-\mu_2 & \mu_1-\mu_3 & \mu_2-\mu_1\end{pmatrix}\begin{pmatrix}exp(\mu_1(t))\\ exp(\mu_2(t))\\ exp(\mu_3(t))\end{pmatrix}, \qquad (2.14)$$

where

$$\begin{aligned}
D &= (\mu_1-\mu_2)(\mu_2-\mu_3)(\mu_3-\mu_1)\,,\\
\alpha &= \Delta + \gamma|M|\sin\Phi\,, \quad \beta = \Delta - \gamma|M|\sin\Phi\,,\\
\mu_1 &= A^{1/3} + B^{1/3} - \frac{X_1}{3}\,,\\
\mu_{2,3} &= \frac{1}{2}(A^{1/3} + B^{1/3}) \pm \frac{i\sqrt{3}}{2}(A^{1/3} - B^{1/3}) - \frac{X_1}{3}\,,
\end{aligned}$$

where A and B are given by

$$\begin{aligned}
A &= -\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\,, \quad B = -\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\,, \quad p = X_2 - \frac{X_1^2}{3}\,,\\
q &= X_3 - \frac{X_1X_2}{3} + \frac{2X_1^3}{27}\,, \quad X_1 = 4\Gamma\,,\\
X_2 &= 5\Gamma^2 + \Omega^2 + \Delta^2 - \gamma^2|M|^2\,,\\
X_3 &= 2\Gamma(\Gamma^2 + \Delta^2 - \gamma^2|M|^2) + \Omega^2(\Gamma + \gamma M\cos\Phi)\,.
\end{aligned}$$

3 Entanglement

The entanglement can be described by the linear entropy or the von-Neumann entropy. The most prominent choice of pure state entanglement measures is the von-Neumann entropy

$$S\left(\rho_{A(F)}\right) = -tr\left(\rho_{A(F))} \ln \rho_{A(F)}\right),$$

of the reduced density matrix, often simply called the entanglement $E(a) = S\left(\rho_{A(F)}\right)$ of the pure state $|a>$. We work with the linear entropy which is convenient to calculate, which is given by

$$S_A(t) = 1 - tr_A\left(\rho_A^2(t)\right), \tag{3.1}$$

which ranges from 0 for a pure state to 1 for a maximally entangled state and tr_A denotes the trace over the subsystem A. The linear entropy is generally a simpler quantity to calculate than the von-Noumann entropy as there is no need for diagonalization and can be considered as a very useful operational measure of the atomic state purity.

Supplemental to the analytical solution presented in the above section for the quantum entropy, we shall devote the present section to analyze the numerical results of the quantum entanglement. Here we would like to point out that in order to ensure an excellent accuracy, the behavior of the function $S_A(t)$ has been determined with great precision, where a resolution of 10^3 point per unit of scaled time has been employed for regions exhibiting strong fluctuation. For the case $\lambda t = 0$, we get almost zero values for the quantum entropy which means a pure state (see figure 3.1). Experimentally it is well known that the quantity

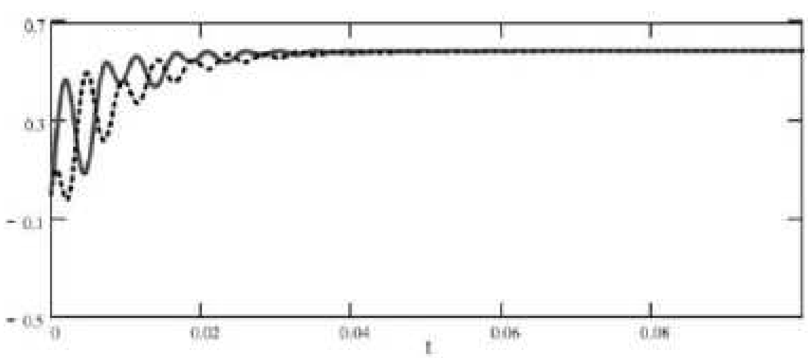

Figure 3.1: The quantum entropy $S_A(t)$ as a function of the scaled time λt.

which is often measured; is the probability of the atom staying in its initial state such as the system is detuned from exact resonance. In figure 3.2 we have plotted the linear entropy for the present system. It is remarkable to point out that, the first maximum of the linear entropy at $t > 0$ is achieved in oscillating way. Also it is noticed that in different values of N, we see a gradual decrease in the amplitudes of the Rabi oscillations.

An alternative approach, if we wish to discuss the temporal behavior of the atomic Wehrl density $S_q(\pi/2, \pi/4, t)$ associated with a particular preparation of the initial state of the atom and the field, we must compare its behavior with the temporal behavior of the linear entropy.

Figure 3.1 displays the evolution of the linear entropy when the mean photon number $\overline{n} = 25$. In this case the maximum value of the linear entropy S_A is given by 1, in general we can say $0 \leq S_A \leq 1$ and the maximum value is achieved just after the onset of the interaction time goes on, then the linear entropy attains its minimum value, so the field and the atom return to the pure state.

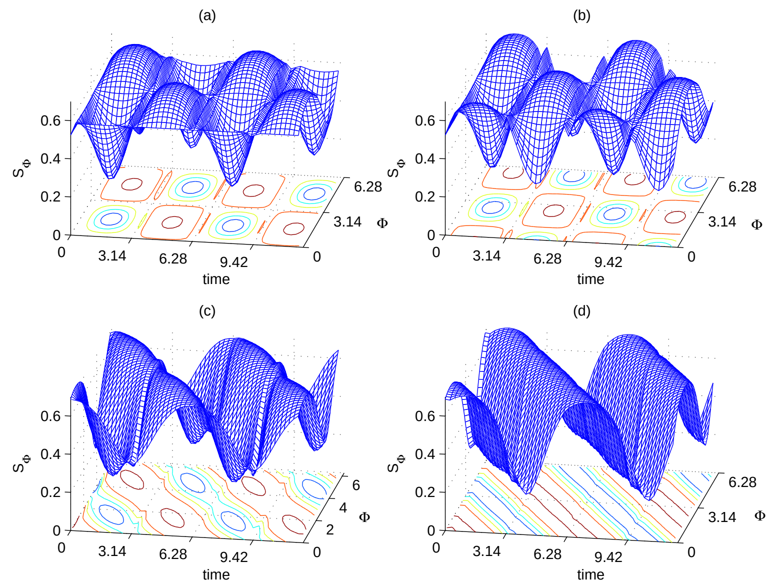

Figure 3.2: The marginal distribution of the atomic Wehrl density S_Φ against scaled time λt and Φ of a two-level atom interacting with field initially in correlated two-mode squeezed vacuum state for $\Delta = \varkappa = 0$ (a) $\theta = \phi = \varphi_S = 0$, (b) $\theta = \phi = 0$, $\varphi_S = \pi/3$, (c) $\theta = \pi/2$, $\phi = \pi/4$, $\varphi_S = 0$ and (d) $\theta = \pi/2$, $\phi = \pi/4$, $\varphi_S = \pi/3$

In figure 3.2, the marginal distribution of the atomic Wehrl density S_Φ against the scaled time λt and Φ. We consider the initial state of the field is a two-mode squeezed vacuum states. The effcet of the parameter φ_S is observed in figure 3.2b. The symmetric spliting observed in figure 3.2a is no longer exist once the superposition parameters θ and ϕ are taken into account (see figures 3.2c and 3.2d).

4 Summary

The quantum linear entropy has been calculated and examined computationally for the model of a single two-level driven by an off-resonant driving laser field and in the presence of a broadband squeezed vacuum reservoir. Regions of entanglements clue are identified for some parameters.

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