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Transmuted Lindley-Geometric Distribution and its Applications

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Abstract: A functional composition of the cumulative distribution function of one probability distribution with the inverse cumulative distribution function of another is called the transmutation map. In this article, we will use the quadratic rank transmutation map (QRTM) in order to generate a flexible family of probability distributions taking Lindley-geometric distribution as the base value distribution by introducing a new parameter that would offer more distributional flexibility. It will be shown that the analytical results are applicable to model real world data.

Keywords: Lindley geometric distribution, moments ,Order Statistics,Transmutation map, Maximum Likelihood Estimation, Reliability Function.

1 Introduction and Motivation

The Lindley distribution was originally proposed by Lindley [23] in the context of Bayesian statistics, as a counter example of fudicial statistics. More details on the Lindley distribution can be found in Ghitany et al. [10].

A random variable X is said to have the Lindley distribution with parameter θ if its probability density is defined as

$$f_L(x,\theta) = \frac{\theta^2}{\theta+1}(1+x)e^{-\theta x}\,;\, x>0,\theta>0. \tag{1}$$

The corresponding cumulative distribution function (c.d.f.) is:

$$F_L(x,\theta) = 1-(1+\frac{\theta x}{\theta+1})e^{-\theta x}, x>0,\theta>0. \tag{2}$$

Many authors gives generalized Linldey distribution like Sankaran [27] introduced the discrete Poisson-Lindley, Mahmoudi and Zakerzadeh [14] introduced generalized Lindley distribution, Bakouch et al. [5] introduced extended Lindley (EL) distribution, Adamidis and Loukas [4] introduced exponential geometric (EG) distribution.

Recently, Hojjatollah and Mahmoudi [29] introduced Lindley-geometric distribution where the cdf and pdf of this distribution are given by

$$F_{LG}(x,\theta,p) = \frac{1-(1+\frac{\theta x}{\theta+1})e^{-\theta x}}{1-p(1+\frac{\theta x}{\theta+1})e^{-\theta x}}, x>0,\theta>0, 0<p<1, \tag{3}$$

and

$$f_{LG}(x,\theta,p) = \frac{\theta^2}{\theta+1}(1-p)(1+x)e^{-\theta x}\left[1-p(1+\frac{\theta x}{\theta+1})e^{-\theta x}\right]^{-2}, \tag{4}$$

respectively. In this paper, we introduce a new lifetime distribution by transmuted and compounding Lindley and geometric distributions named transmuted Lindley-geometric distribution. The concept of transmuted explained in the following subsection.

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1.1 Transmutation Map

In this subsection we demonstrate transmuted probability distribution. Let F_1 and F_2 be the cumulative distribution functions, of two distributions with a common sample space. The general rank transmutation is defined as

$$G_{R12}(u) = F_2(F_1^{-1}(u)) \text{ and } G_{R21}(u) = F_1(F_2^{-1}(u)).$$

Note that the inverse cumulative distribution function also known as quantile function is defined as

$$F^{-1}(y) = \inf_{x\in R}\{F(x) \geq y\} \text{ for } y \in [0,1].$$

The functions $G_{R12}(u)$ and $G_{R21}(u)$ both map the unit interval $I = [0,1]$ into itself, and under suitable assumptions are mutual inverses and they satisfy $G_{Rij}(0) = 0$ and $G_{Rij}(0) = 1$. A quadratic Rank Transmutation Map (QRTM) is defined as

$$G_{R12}(u) = u + \lambda u(1-u), |\lambda| \leq 1, \tag{5}$$

from which it follows that the cdf's satisfy the relationship

$$F_2(x) = (1+\lambda)F_1(x) - \lambda F_1(x)^2 \tag{6}$$

which on differentiation yields,

$$f_2(x) = f_1(x)\left[(1+\lambda) - 2\lambda F_1(x)\right] \tag{7}$$

where $f_1(x)$ and $f_2(x)$ are the corresponding pdfs associated with cdf $F_1(x)$ and $F_2(x)$ respectively. An extensive information about the quadratic rank transmutation map is given in Shaw et al. [31]. Observe that at $\lambda = 0$ we have the distribution of the base random variable. The following lemma proved that the function $f_2(x)$ in given (7) satisfies the property of probability density function.

Lemma: $f_2(x)$ given in (7) is a well defined probability density function.

Many authors dealing with the generalization of some well- known distributions. Aryal and Tsokos [1] defined the transmuted generalized extreme value distribution and they studied some basic mathematical characteristics of transmuted Gumbel probability distribution and it has been observed that the transmuted Gumbel can be used to model climate data. Also Aryal and Tsokos [2] presented a new generalization of Weibull distribution called the transmuted Weibull distribution. Recently, Aryal (2013) proposed and studied the various structural properties of the transmuted Log-Logistic distribution, and Khan and King [13] introduced the transmuted modified Weibull distribution which extends recent development on transmuted Weibull distribution by Aryal et al. [2], Merovci [19],[20],[21]introduced the transmuted Rayleigh distribution, transmuted generalized Rayleigh distribution, transmuted Lindley distribution and they studied the mathematical properties and maximum likelihood estimation of the unknown parameters.

1.2 Transmuted Lindley Geometric Distribution

In this section we studied the transmuted Lindley geometric (TLG) distribution. Now using (5)and (6) we have the cdf of transmuted Lindley-geometric (TLG) distribution

$$F_{TLG}(x,\theta,p,\lambda) = \frac{1-(1+\frac{\theta x}{\theta+1})e^{-\theta x}}{1-p(1+\frac{\theta x}{\theta+1})e^{-\theta x}}\left[1+\lambda-\lambda\left(\frac{1-(1+\frac{\theta x}{\theta+1})e^{-\theta x}}{1-p(1+\frac{\theta x}{\theta+1})e^{-\theta x}}\right)\right] \tag{8}$$

where λ is the transmuted parameter. The corresponding probability density function (pdf) of the transmuted Lindley-geometric is given by

$$\begin{aligned} f_{TLG}(x,\theta,p,\lambda) &= f_{LG}(x)\left[(1+\lambda) - 2\lambda F_{LG}(x)\right] \\ &= \frac{\theta^2}{\theta+1}(1-p)(1+x)e^{-\theta x}\left[1-p(1+\frac{\theta x}{\theta+1})e^{-\theta x}\right]^{-2} \\ &\times\left\{(1+\lambda)-2\lambda\left(\frac{1-(1+\frac{\theta x}{\theta+1})e^{-\theta x}}{1-p(1+\frac{\theta x}{\theta+1})e^{-\theta x}}\right)\right\}, \end{aligned} \tag{9}$$

respectively.

Figure 1 and figure 2 illustrates some of the possible shapes of the pdf and cdf of TLG distribution for selected values of the parameters θ, p and λ, respectively.

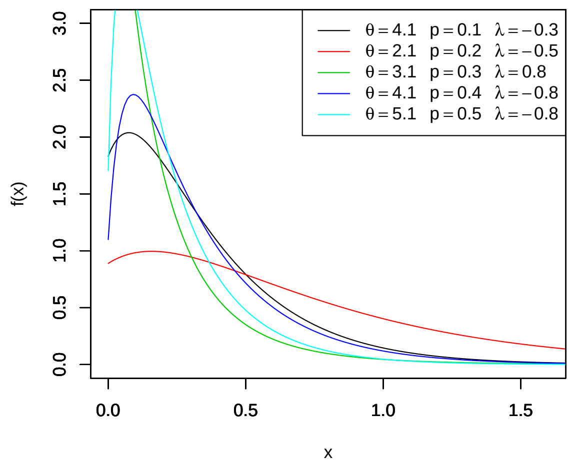

Fig. 1: The pdf's of various TLG distributions.

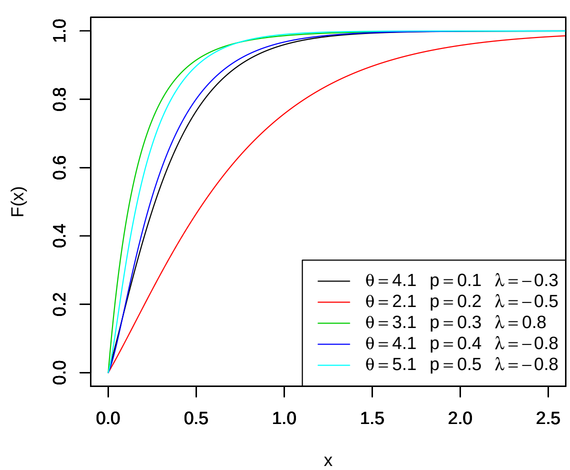

Fig. 2: The cdf's of various TLG distributions.

The reliability function (RF) of the transmuted Lindley-geometric distribution is denoted by $R_{TLG}(x)$ also known as the survivor function and is defined as

$$
\begin{aligned}
R_{TLG}(x) &= 1 - F_{TLG}(x) \\
&= 1 - \frac{1-(1+\frac{\theta x}{\theta+1})e^{-\theta x}}{1-p(1+\frac{\theta x}{\theta+1})e^{-\theta x}} \left[1+\lambda-\lambda\left(\frac{1-(1+\frac{\theta x}{\theta+1})e^{-\theta x}}{1-p(1+\frac{\theta x}{\theta+1})e^{-\theta x}}\right)\right].
\end{aligned}
\tag{10}
$$

Figure 3 illustrates some of the possible shapes of the survival function of transmuted Lindley geometric distribution for selected values of the parameters θ, p and λ, respectively.

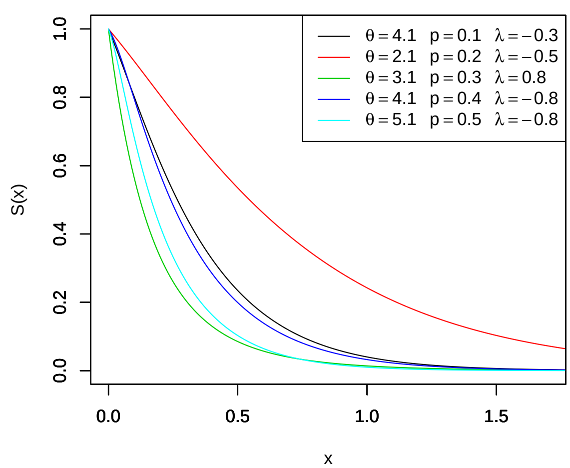

Fig. 3: The survival function of various transmuted Lindely geometric distributions.

It is important to note that $R_{TLG}(x) + F_{TLG}(x) = 1$. One of the characteristic in reliability analysis is the hazard rate function (HF) defined by

$$
h_{TLG}(x) = \frac{f_{TLG}(x)}{1-F_{TLG}(x)} \tag{11}
$$

Figure 4 illustrates some of the possible shapes of the hazard function of transmuted Lindley-geometric distribution for selected values of the parameters θ, p and λ, respectively.

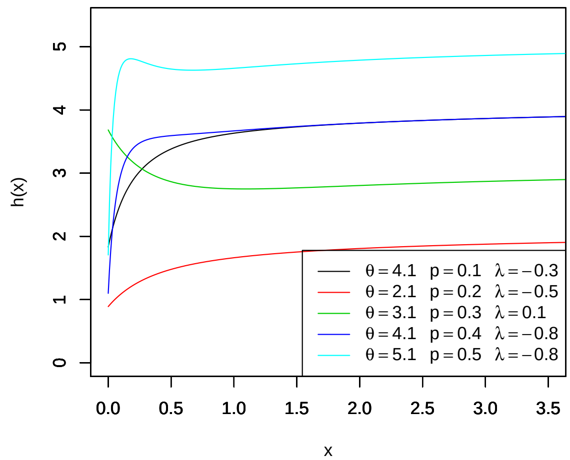

Fig. 4: The survival function of various transmuted Lindely geometric distributions.

It is important to note that the units for $h_{TLG}(x)$ is the probability of failure per unit of time, distance or cycles. These failure rates are defined with different choices of parameters. The cumulative hazard function of the transmuted Lindley-geometric distribution is denoted by $H_{TLG}(x)$ and is defined as

$$H_{TLG}(x) = -\ln\left|\frac{1-(1+\frac{\theta x}{\theta+1})e^{-\theta x}}{1-p(1+\frac{\theta x}{\theta+1})e^{-\theta x}}\left[1+\lambda-\lambda\left(\frac{1-(1+\frac{\theta x}{\theta+1})e^{-\theta x}}{1-p(1+\frac{\theta x}{\theta+1})e^{-\theta x}}\right)\right]\right| \qquad (12)$$

It is important to note that the units for $H_{TLG}(x)$ is the cumulative probability of failure per unit of time, distance or cycles. We can show that. For all choice of parameters the distribution has the decreasing patterns of cumulative instantaneous failure rates.

2 Statistical Properties

This section is devoted to studying statistical properties of the (TLG) distribution.

2.1 Moments

In this subsection we discuss the r_{th} moment for (TLG) distribution. Moments are necessary and important in any statistical analysis, especially in applications. It can be used to study the most important features and characteristics of a distribution (e.g., tendency, dispersion, skewness and kurtosis).

Theorem (3.1).

If X has $TLG\ (\Phi,x)$, $\Phi=(\theta,p,\lambda)$ then the r_{th} moment of X is given by the following

$$\begin{aligned}\mu_r^{'}(x) &= A_{ig}\frac{\Gamma(r+i+1)}{(\theta(j+1))^{r+i+1}}\left[1+\frac{r+i+1}{(\theta(j+1))}\right]\\ &-B_{ij}\left\{\frac{\Gamma(r+i+1)}{(\theta(j+1))^{r+i+1}}\left[1+\frac{r+i+1}{(\theta(j+1))}\right]-\frac{\Gamma(r+i+1)}{(\theta(j+2))^{r+i+1}}\left[1+\frac{r+i+1}{(\theta(j+2))}\right]\right.\\ &\left.-\frac{\theta}{\theta+1}\left(\frac{\Gamma(r+i+2)}{(\theta(j+2))^{r+i+2}}\right)\left[1+\frac{r+i+2}{(\theta(j+2))}\right]\right\},\end{aligned} \qquad (13)$$

where

$$A_{ig}=\frac{\theta^2(1+\lambda)}{\theta+1}(1-p)\sum_{j=0}^{\infty}\sum_{i=0}^{j}\binom{j}{i}(j+1)p^j\left(\frac{\theta}{\theta+1}\right)^i,$$

and

$$B_{ij}=\frac{\lambda\theta^2}{\theta+1}(1-p)\sum_{j=0}^{\infty}\sum_{i=0}^{j}\binom{j}{i}(j+1)(j+2)p^j\left(\frac{\theta}{\theta+1}\right)^i.$$

Proof:

Let X be a random variable with density function (9). The r_{th} ordinary moment of the (TLG) distribution is given by

$$\begin{aligned}\mu_r^{'}(x)&=E(X^r=\int_0^{\infty}x^rf(x,\Phi)dx\\&=\frac{\theta^2(1+\lambda)}{\theta+1}(1-p)\int_0^{\infty}(x^r+x^{r+1})e^{-\theta x}\left[1-p(1+\frac{\theta x}{\theta+1})e^{-\theta x}\right]^{-2}dx\\&-\frac{2\lambda\theta^2}{\theta+1}(1-p)\int_0^{\infty}(x^r+x^{r+1})e^{-\theta x}\left(1-(1+\frac{\theta x}{\theta+1})e^{-\theta x}\right)\left[1-p(1+\frac{\theta x}{\theta+1})e^{-\theta x}\right]^{-3}dx.\end{aligned}\tag{14}$$

using the series expansion

$$(1-z)^{-k}=\sum_{j=0}^{\infty}\frac{\Gamma(k+j)}{\Gamma(k)j!}z^j,\tag{15}$$

where$|z|<1$ and $k>0$.

Equation (14) can be demonstrated by

$$\begin{aligned}\mu_r^{'}(x)=&\frac{\theta^2(1+\lambda)}{\theta+1}(1-p)\sum_{j=0}^{\infty}(j+1)p^j\int_0^{\infty}(x^r+x^{r+1})(1+\frac{\theta x}{\theta+1})^je^{-\theta(j+1)x}dx\\&-\left\{\frac{\lambda\theta^2}{\theta+1}(1-p)\sum_{j=0}^{\infty}(j+1)(j+2)p^j\right.\\&\left.\int_0^{\infty}(x^r+x^{r+1})(1+\frac{\theta x}{\theta+1})^j\left(1-(1+\frac{\theta x}{\theta+1})e^{-\theta x}\right)e^{-\theta(j+1)x}dx\right\},\end{aligned}\tag{16}$$

also applying the binomial expression for $(1+\frac{\theta x}{\theta+1})^j$ where

$$(1+\frac{\theta x}{\theta+1})^j=\sum_{i=0}^{j}\binom{j}{i}\left(\frac{\theta}{\theta+1}\right)^ix^i,\tag{17}$$

substituting from (17) into (16) we get

$$\begin{aligned}\mu_r^{'}(x)=&\left\{\frac{\theta^2(1+\lambda)}{\theta+1}(1-p)\sum_{j=0}^{\infty}\sum_{i=0}^{j}\binom{j}{i}(j+1)p^j\left(\frac{\theta}{\theta+1}\right)^i\right.\\&\left.\int_0^{\infty}(x^{r+i}+x^{r+i+1})e^{-\theta(j+1)x}dx\right\}\\&-\left\{\frac{\lambda\theta^2}{\theta+1}(1-p)\sum_{j=0}^{\infty}\sum_{i=0}^{j}\binom{j}{i}(j+1)(j+2)p^j\left(\frac{\theta}{\theta+1}\right)^i\right.\\&\left.\int_0^{\infty}(x^{r+i}+x^{r+i+1})\left(1-(1+\frac{\theta x}{\theta+1})e^{-\theta x}\right)e^{-\theta(j+1)x}dx\right\}\\&=A_{ig}I_1-B_{ij}I_2\end{aligned}$$

where

$$A_{ig}=\frac{\theta^2(1+\lambda)}{\theta+1}(1-p)\sum_{j=0}^{\infty}\sum_{i=0}^{j}\binom{j}{i}(j+1)p^j\left(\frac{\theta}{\theta+1}\right)^i,$$

$$B_{ij}=\frac{\lambda\theta^2}{\theta+1}(1-p)\sum_{j=0}^{\infty}\sum_{i=0}^{j}\binom{j}{i}(j+1)(j+2)p^j\left(\frac{\theta}{\theta+1}\right)^i,$$

$$\begin{aligned}I_1&=\int_0^{\infty}(x^{r+i}+x^{r+i+1})e^{-\theta(j+1)x}dx\\&=\frac{\Gamma(r+i+1)}{(\theta(j+1))^{r+i+1}}+\frac{\Gamma(r+i+2)}{(\theta(j+1))^{r+i+2}}\\&=\frac{\Gamma(r+i+1)}{(\theta(j+1))^{r+i+1}}\left[1+\frac{r+i+1}{(\theta(j+1))}\right],\end{aligned}$$

and

$$\begin{aligned}I_2&=\int_0^{\infty}(x^{r+i}+x^{r+i+1})\left(1-(1+\frac{\theta x}{\theta+1})e^{-\theta x}\right)e^{-\theta(j+1)x}dx\\&=\frac{\Gamma(r+i+1)}{(\theta(j+1))^{r+i+1}}\left[1+\frac{r+i+1}{(\theta(j+1))}\right]-\frac{\Gamma(r+i+1)}{(\theta(j+2))^{r+i+1}}\left[1+\frac{r+i+1}{(\theta(j+2))}\right]\\&-\frac{\theta}{\theta+1}\left(\frac{\Gamma(r+i+2)}{(\theta(j+2))^{r+i+2}}\right)\left[1+\frac{r+i+2}{(\theta(j+2))}\right],\end{aligned}$$

thus the r_{th} moment is given by

$$\mu_r(x)=\theta\alpha^2\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}(-1)^j\binom{j}{m}\alpha^m\frac{\Gamma(r+m+2)}{(\alpha(j+1))^{r+m+2}}\left[(1+\lambda)\binom{\theta-1}{j}-2\lambda\binom{2\theta-1}{j}\right].$$

Which completes the proof .

We notice that if we put $\lambda=0$, we get the r_{th} moment of Lindley geometric (see Hojjatollah and Mahmoudi (2012)). Based on the first four moments of the (TLG) distribution, the measures of skewness $A(\Phi)$ and kurtosis $k(\Phi)$ of the (TLG) distribution can obtained as

$$A(\Phi)=\frac{\mu_3(\theta)-3\mu_1(\theta)\mu_2(\theta)+2\mu_1^3(\theta)}{\left[\mu_2(\theta)-\mu_1^2(\theta)\right]^{\frac{3}{2}}},$$

and

$$k(\Phi)=\frac{\mu_4(\theta)-4\mu_1(\theta)\mu_3(\theta)+6\mu_1^2(\theta)\mu_2(\theta)-3\mu_1^4(\theta)}{\left[\mu_2(\theta)-\mu_1^2(\theta)\right]^2}.$$

2.2 Moment Generating function

In this subsection we derived the moment generating function of (TLG) distribution.

Theorem (3.2): If X has (TLG) distribution, then the moment generating function $M_X(t)$ has the following form

$$\begin{aligned} M_X(t) = & \frac{A_{ig}\Gamma(i+1)}{(\theta(j+1)-t)^{i+1}}\left[1+\frac{i+1}{(\theta(j+1)-t)}\right] \\ & -B_{ij}\left\{\frac{\Gamma(i+1)}{(\theta(j+1)-t)^{i+1}}\left[1+\frac{i+1}{(\theta(j+1)-t)}\right]\right. \\ & -\frac{\Gamma(i+1)}{(\theta(j+2)-t)^{i+1}}\left[1+\frac{i+1}{(\theta(j+2)-t)}\right] \\ & \left.-\frac{\theta}{\theta+1}\left(\frac{\Gamma(i+2)}{(\theta(j+2)-t)^{i+2}}\right)\left[1+\frac{i+2}{(\theta(j+2)-t)}\right]\right\} \end{aligned} \tag{18}$$

Proof.

We start with the well known definition of the moment generating function given by

$$\begin{aligned} M_X(t) = E(e^{tx}) = & \int_0^\infty e^{tx} f_{TLG}(x,\Phi)dx \\ = & \frac{\theta^2(1+\lambda)}{\theta+1}(1-p)\int_0^\infty (1+x)e^{-x(\theta-t)}\left[1-p(1+\frac{\theta x}{\theta+1})e^{-\theta x}\right]^{-2} dx \\ & -\frac{2\lambda\theta^2}{\theta+1}(1-p)\int_0^\infty (1+x)e^{-x(\theta-t)}\left(1-(1+\frac{\theta x}{\theta+1})e^{-\theta x}\right)\left[1-p(1+\frac{\theta x}{\theta+1})e^{-\theta x}\right]^{-3} dx. \end{aligned} \tag{19}$$

substituting from (15) and (17) into (19) we get

$$\begin{aligned} M_X(t) = & A_{ig}\int_0^\infty (x^i+x^{i+1})e^{-x[\theta(j+1)-t]}dx \\ & -B_{ij}\int_0^\infty (x^i+x^{i+1})e^{-x[\theta(j+1)-t]}\left(1-(1+\frac{\theta x}{\theta+1})e^{-\theta x}\right) \\ = & \frac{A_{ig}\Gamma(i+1)}{(\theta(j+1)-t)^{i+1}}\left[1+\frac{i+1}{(\theta(j+1)-t)}\right] \\ & -B_{ij}\left\{\frac{\Gamma(i+1)}{(\theta(j+1)-t)^{i+1}}\left[1+\frac{i+1}{(\theta(j+1)-t)}\right]\right. \\ & -\frac{\Gamma(i+1)}{(\theta(j+2)-t)^{i+1}}\left[1+\frac{i+1}{(\theta(j+2)-t)}\right] \\ & \left.-\frac{\theta}{\theta+1}\left(\frac{\Gamma(i+2)}{(\theta(j+2)-t)^{i+2}}\right)\left[1+\frac{i+2}{(\theta(j+2)-t)}\right]\right\} \end{aligned} \tag{20}$$

Which completes the proof.

3 Distribution of the order statistics

In this section, we derive closed form expressions for the pdfs of the r_{th} order statistic of the TLG distribution, also, the measures of skewness and kurtosis of the distribution of the r_{th} order statistic in a sample of size n for different choices of $n;r$ are presented in this section. Let $X_1, X_2, ..., X_n$ be a simple random sample from (TLG) distribution with pdf and cdf given by (8) and (9), respectively.

Let $X_1, X_2, ..., X_n$ denote the order statistics obtained from this sample. We now give the probability density function of $X_{r:n}$, say $f_{r:n}(x, \Phi)$ and the moments of $X_{r:n}$, $r = 1, 2, ..., n$. Therefore, the measures of skewness and kurtosis of the distribution of the $X_{r:n}$ are presented. The probability density function of $X_{r:n}$ is given by

$$f_{r:n}(x,\Phi) = \frac{1}{B(r,n-r+1)} [F(x,\Phi)]^{r-1} [1-F(x,\Phi)]^{n-r} f(x,\Phi) \tag{21}$$

where $F(x,\Phi)$ and $f(x,\Phi)$ are the cdf and pdf of the (TLG) distribution given by (8), (9), respectively, and $B(.,.)$ is the beta function, since $0 < F(x,\Phi) < 1$, for $x > 0$, by using the binomial series expansion of $[1-F(x,\Phi)]^{n-r}$, given by

$$[1-F(x,\Phi)]^{n-r} = \sum_{j=0}^{n-r} (-1)^j \binom{n-r}{j} [F(x,\Phi)]^j, \tag{22}$$

we have

$$f_{r:n}(x,\Phi) = \sum_{j=0}^{n-r} (-1)^j \binom{n-r}{j} [F(x,\Phi)]^{r+j-1} f(x,\Phi), \tag{23}$$

substituting from (8) and (9) into (23), we can express the k_{th} ordinary moment of the r_{th} order statistics $X_{r:n}$ say $E(X_{r:n}^k)$ as a liner combination of the k_{th} moments of the (TLG) distribution with different shape parameters. Therefore, the measures of skewness and kurtosis of the distribution of $X_{r:n}$ can be calculated.

4 Estimation and Inference

4.1 Least Squares and Weighted Least Squares Estimators

In this subsection we provide the regression based method estimators of the unknown parameters of the transmuted Lindley-geometric distribution, which was originally suggested by Swain, Venkatraman and Wilson (1988) to estimate the parameters of beta distributions. It can be used some other cases also. Suppose $Y_1, ..., Y_n$ is a random sample of size n from a distribution function $G(.)$ and suppose $Y_{(i)}$; $i = 1, 2, ..., n$ denotes the ordered sample. The proposed method uses the distribution of $G(Y_{(i)})$. For a sample of size n, we have

$$E\left(G(Y_{(j)})\right) = \frac{j}{n+1}, V\left(G(Y_{(j)})\right) = \frac{j(n-j+1)}{(n+1)^2(n+2)}$$
$$\text{and } Cov\left(G(Y_{(j)}), G(Y_{(k)})\right) = \frac{j(n-k+1)}{(n+1)^2(n+2)}; \text{for } j < k,$$

see Johnson, Kotz and Balakrishnan (1995). Using the expectations and the variances, two variants of the least squares methods can be used.

Method 1 (Least Squares Estimators) . Obtain the estimators by minimizing

$$\sum_{j=1}^{n} \left(G(Y_{(j)} - \frac{j}{n+1} \right)^2, \tag{24}$$

with respect to the unknown parameters. Therefore in case of TLG distribution the least squares estimators of θ, p and λ , say , $\widehat{\theta}_{LSE}$, $\widehat{p}_{LSE}$ and $\widehat{\lambda}_{LSE}$ respectively, can be obtained by minimizing

$$\sum_{j=1}^{n} \left[\frac{1-(1+\frac{\theta x}{\theta+1})e^{-\theta x}}{1-p(1+\frac{\theta x}{\theta+1})e^{-\theta x}} \left[1+\lambda-\lambda \left(\frac{1-(1+\frac{\theta x}{\theta+1})e^{-\theta x}}{1-p(1+\frac{\theta x}{\theta+1})e^{-\theta x}} \right) \right] - \frac{j}{n+1} \right]^2$$

with respect to θ, p and λ.

Method 2 (Weighted Least Squares Estimators). The weighted least squares estimators can be obtained by minimizing

$$\sum_{j=1}^{n} w_j \left(G(Y_{(j)} - \frac{j}{n+1} \right)^2, \tag{25}$$

with respect to the unknown parameters, where

$$w_j = \frac{1}{V\left(G(Y_{(j)})\right)} = \frac{(n+1)^2(n+2)}{j(n-j+1)}.$$

Therefore, in case of TLG distribution the weighted least squares estimators of θ, p and λ , say , $\widehat{\theta}_{WLSE}$, $\widehat{p}_{WLSE}$ and $\widehat{\lambda}_{WLSE}$ respectively , can be obtained by minimizing

$$\sum_{j=1}^{n} w_j \left[\frac{1-(1+\frac{\theta x}{\theta+1})e^{-\theta x}}{1-p(1+\frac{\theta x}{\theta+1})e^{-\theta x}} \left[1+\lambda-\lambda\left(\frac{1-(1+\frac{\theta x}{\theta+1})e^{-\theta x}}{1-p(1+\frac{\theta x}{\theta+1})e^{-\theta x}} \right) \right] - \frac{j}{n+1} \right]^2$$

with respect to the unknown parameters only.

4.2 MAximum likelihood estimation

In this subsection we determine the maximum likelihood estimates (MLEs) of the parameters of the (TLG) distribution from complete samples only. Let $X_1, X_2, ..., X_n$ be a random sample of size n from $TLG\ (\theta, p, \lambda, x)$.The likelihood function for the vector of parameters $\Phi = (\theta, p, \lambda)$ can be written as

$$\begin{aligned} Lf(x_{(i)}, \Phi) &= \Pi_{i=1}^{n} f(x_{(i)}, \Phi) \\ &= \left(\frac{\theta^2}{\theta+1} \right)^n (1-p)^n \Pi_{i=1}^{n} (1+x_i)\ e^{-\theta \sum_{i=1}^{n} x_i} \Pi_{i=1}^{n} \left[1-p(1+\frac{\theta x_i}{\theta+1})e^{-\theta x_i} \right]^{-2} \\ &\times \Pi_{i=1}^{n} \left\{ (1+\lambda) - 2\lambda \left(\frac{1-(1+\frac{\theta x_i}{\theta+1})e^{-\theta x_i}}{1-p(1+\frac{\theta x_i}{\theta+1})e^{-\theta x_i}} \right) \right\}. \end{aligned} \tag{26}$$

Taking the log-likelihood function for the vector of parameters $\Phi = (\theta, p, \lambda)$ we get

$$\begin{aligned} \ell = \log L &= 2n\log\theta - n\log(1+\theta) + n\log(1-p) + \sum_{i=1}^{n} \log(1+x_i) - \theta \sum_{i=1}^{n} x_{(i)} \\ &- 2\sum_{i=1}^{n} \log \left[1-p(1+\frac{\theta x_i}{\theta+1})e^{-\theta x_i} \right] \\ &+ \sum_{i=1}^{n} \log \left\{ (1+\lambda) - 2\lambda \left(\frac{1-(1+\frac{\theta x_i}{\theta+1})e^{-\theta x_i}}{1-p(1+\frac{\theta x_i}{\theta+1})e^{-\theta x_i}} \right) \right\}. \end{aligned} \tag{27}$$

The log-likelihood can be maximized either directly or by solving the nonlinear likelihood equations obtained by differentiating (27). The components of the score vector are given by

$$\begin{aligned} \frac{\partial \ell}{\partial p} &= \frac{-n}{1-p} + 2\sum_{i=1}^{n} \frac{(1+\frac{\theta x_i}{\theta+1})e^{-\theta x_i}}{\left[1-p(1+\frac{\theta x_i}{\theta+1})e^{-\theta x_i} \right]} \\ &- 2\lambda \sum_{i=1}^{n} \frac{\left[1-(1+\frac{\theta x_i}{\theta+1})e^{-\theta x_i} \right] \left[\frac{(1+\frac{\theta x_i}{\theta+1})e^{-\theta x_i}}{(1-p(1+\frac{\theta x_i}{\theta+1})e^{-\theta x_i})^2} \right]}{\left\{ (1+\lambda) - 2\lambda \left(\frac{1-(1+\frac{\theta x_i}{\theta+1})e^{-\theta x_i}}{1-p(1+\frac{\theta x_i}{\theta+1})e^{-\theta x_i}} \right) \right\}} = 0, \end{aligned} \tag{28}$$

‘

$$\frac{\partial \ell}{\partial \theta} = \frac{2n}{\theta} - \frac{n}{1+\theta} - \sum_{i=1}^{n} x_i - 2p \sum_{i=1}^{n} \frac{x_i e^{-\theta x_i} \left[(1+\frac{\theta x_i}{\theta+1}) - \frac{1}{(1+\theta)^2} \right]}{\left[1 - p(1+\frac{\theta x_i}{\theta+1}) e^{-\theta x_i} \right]}$$

$$-2\lambda \sum_{i=1}^{n} \frac{(1-p)}{\left\{ (1+\lambda) - 2\lambda \left(\frac{1-(1+\frac{\theta x_i}{\theta+1})e^{-\theta x_i}}{1-p(1+\frac{\theta x_i}{\theta+1})e^{-\theta x_i}} \right) \right\}} \left[\frac{x_i e^{-\theta x_i} \left[(1+\frac{\theta x_i}{\theta+1}) - \frac{1}{(1+\theta)^2} \right]}{\left[1 - p(1+\frac{\theta x_i}{\theta+1}) e^{-\theta x_i} \right]^2} \right] = 0 \quad (29)$$

and

$$\frac{\partial \ell}{\partial \lambda} = \sum_{i=1}^{n} \frac{1 - 2\left(\frac{1-(1+\frac{\theta x_i}{\theta+1})e^{-\theta x_i}}{1-p(1+\frac{\theta x_i}{\theta+1})e^{-\theta x_i}} \right)}{\left\{ (1+\lambda) - 2\lambda \left(\frac{1-(1+\frac{\theta x_i}{\theta+1})e^{-\theta x_i}}{1-p(1+\frac{\theta x_i}{\theta+1})e^{-\theta x_i}} \right) \right\}} = 0. \quad (30)$$

We can find the estimates of the unknown parameters by maximum likelihood method by setting these above non-linear equations (29)- (30) to zero and solve them simultaneously. Therefore, we have to use mathematical package to get the MLE of the unknown parameters. Applying the usual large sample approximation, the MLE $\hat{\Phi}$ can be treated as being approximately trivariate normal and variance-covariance matrix equal to the inverse of the expected information matrix, i.e.

$$\sqrt{n}(\hat{\Phi} - \Phi) \to N\left(0, nI^{-1}(\Phi)\right),$$

where $I^{-1}(\Phi)$ is the limiting variance-covariance matrix of $\hat{\Phi}$. The elements of the 3×3 matrix $I(\Phi)$ can be estimated by $I_{ij}(\hat{\Phi}) = -\ell_{\Phi_i \Phi_j}\big|_{\Phi=\hat{\Phi}}$, $i, j \in \{1,2,3\}$.

Approximate two sided $100(1-\alpha)\%$ confidence intervals for θ, p and for λ are, respectively, given by

$$\hat{\theta} \pm z_{\alpha/2} \sqrt{I_{11}^{-1}(\hat{\theta})}, \hat{p} \pm z_{\alpha/2} \sqrt{I_{22}^{-1}(\hat{p})}$$

and

$$\hat{\lambda} \pm z_{\alpha/2} \sqrt{I_{33}^{-1}(\hat{\lambda})},$$

where z_α is the upper αth quantile of the standard normal distribution. Using R we can easily compute the Hessian matrix and its inverse and hence the standard errors and asymptotic confidence intervals.

5 Application

In this section, we use a real data set to show that the transmuted Lindley distribution can be a better model than one based on the Lindley geometric distribution and Lindley distribution. The data set given in Table 1 represents the waiting times (in minutes) before service of 100 bank customers.

Table 1: The waiting times (in minutes) before service of 100 bank customers.

0.8	0.8	1.3	1.5	1.8	1.9	1.9	2.1	2.6	2.7
2.9	3.1	3.2	3.3	3.5	3.6	4.0	4.1	4.2	4.2
4.3	4.3	4.4	4.4	4.6	4.7	4.7	4.8	4.9	4.9
5.0	5.3	5.5	5.7	5.7	6.1	6.2	6.2	6.2	6.3
6.7	6.9	7.1	7.1	7.1	7.1	7.4	7.6	7.7	8.0
8.2	8.6	8.6	8.6	8.8	8.8	8.9	8.9	9.5	9.6
9.7	9.8	10.7	10.9	11.0	11.0	11.1	11.2	11.2	11.5
11.9	12.4	12.5	12.9	13.0	13.1	13.3	13.6	13.7	13.9
14.1	15.4	15.4	17.3	17.3	18.1	18.2	18.4	18.9	19.0
19.9	20.6	21.3	21.4	21.9	23.0	27.0	31.6	33.1	38.5

Table 2: Estimated parameters of the Lindley, Lindley-geometric and transmuted Lindley geometric distribution for the waiting times (in minutes) before service of 100 bank customers.

Model	Parameter Estimate	Standard Error	$-\ell(\cdot;x)$
Lindley	$\hat{\theta}=0.186$	0.013	319.037
Lindley	$\hat{\theta}=0.202$	0.034	318.913
Geometric	$\hat{p}=-0.242$	0.5270	
Transmuted	$\hat{\theta}=0.171$	0.0351	317.207
Lindley	$\hat{p}=0.657$	0.181	
Geometric	$\hat{\lambda}=-0.954$	0.192	

The variance covariance matrix of the MLEs under the transmuted Lindley geometric distribution is computed as

$$I(\hat{\theta})^{-1}=\begin{pmatrix} 0.001 & -0.005 & 0.002 \\ -0.005 & 0.032 & -0.020 \\ 0.002 & -0.020 & 0.037 \end{pmatrix}.$$

Thus, the variances of the MLE of θ, p and λ is $var(\hat{\theta})=0.0012, var(\hat{p})=0.0326, var(\hat{a})=0.0368$. Therefore, 95% confidence intervals for θ, p and λ are $[0.102, 0.240], [0.302, 1]$, and $[-0.577, 1]$ respectively.

Table 3: Criteria for comparison.

Model	K-S	-2ℓ	AIC	AICC
Lindley	0.0677	638.1	640.1	640.1
Lindley Geometric	0.0557	637.8	641.8	642
TLG	0.0017	634.414	640.414	640.664

In order to compare the two distribution models, we consider criteria like K-S, -2ℓ, AIC (Akaike information criterion)and AICC (corrected Akaike information criterion) for the data set. The better distribution corresponds to smaller K-S, -2ℓ, AIC and AICC values:‘

$$\text{AIC}=2k-2\ell, \quad \text{and} \quad \text{AICC}=\text{AIC}+\frac{2k(k+1)}{n-k-1},$$

where k is the number of parameters in the statistical model, n the sample size and ℓ is the maximized value of the log-likelihood function under the considered model. Also, here for calculating the values of KS we use the sample estimates of θ, α, a, b and c. Table 2 shows the MLEs under both distributions, Table 3 shows the values of K-S, -2ℓ, AIC and AICC values. The values in table 3 indicate that the transmuted Lindley geometric distribution leads to a better fit than the Lindley geometric distribution and Lindely distribution.

A density plot compares the fitted densities of the models with the empirical histogram of the observed data (Fig. 4). The fitted density for the transmuted Linldey geometric model is closer to the empirical histogram than the fits of the Lindley geometric and Lindley sub-models.

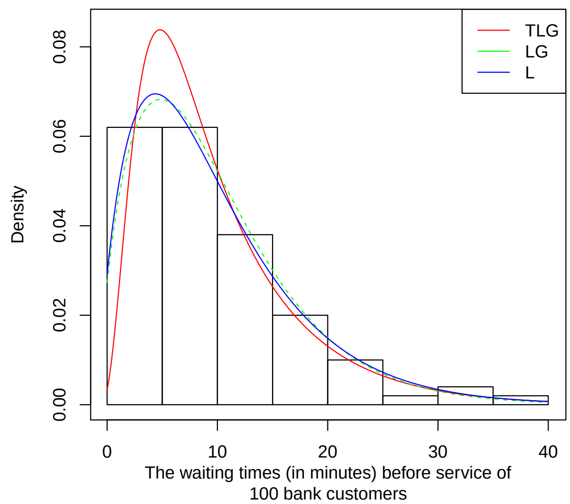

Fig. 5: Estimated densities of the models for the waiting times (in minutes) before service of 100 bank customers.

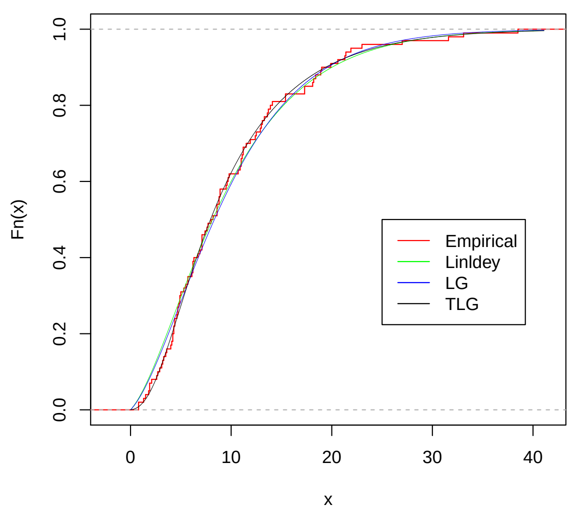

Fig. 6: Empirical, fitted Lindley, Lindley geometric and transmuted Lindley geometric cdf of the the waiting times (in minutes) before service of 100 bank customers.

6 Conclusion

Here we propose a new model, the so-called the transmuted Lindley geometric distribution which extends the Lindley geometric distribution in the analysis of data with real support. An obvious reason for generalizing a standard distribution is because the generalized form provides larger flexibility in modeling real data. We derive expansions for moments and for the moment generating function. The estimation of parameters is approached by the method of maximum likelihood, also the information matrix is derived. An application of the transmuted Lindley geometric distribution to real data show that the new distribution can be used quite effectively to provide better fits than Lindley geometric and Lindley distribution.

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