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Numerical Solution of Duffing Equation by the Differential Transform Method

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Abstract: In this article, Differential transform method is presented for solving Duffing equations.We apply these method to three examples. First Duffing equation has been converted to power series by one-dimensional differential transformation,Then the numerical solution of equation was put into Multivariate Pad series form.Thus, we have obtained numerical solution differential equation of Duffing. These examples are prepared to show the efficiency and simplicity of the method.

Keywords: Duffing Equation, Power series, Pade Approximation, Differential Transform Method.

1 Introduction

The Duffing equation describes by second order ordinary differential equation with the common form

$$x^{''} + px^{'} + p_1 x + p_2 x^3 = f(t), \quad (1.1)$$

$$x(0) = \alpha, x'(0) = \beta, \quad (1.2)$$

Where p, p_1, p_2, α and β are real constants. Mathematical modeling of many frontier physical systems leads to nonlinear ordinary differential equations (NODE). One of the most common physical NODE's, governs many oscillative systems, is the Duffing equations. The Duffing equations can be found in a wide variety of engineering and scientific applications. In recent years, numerous works have focused on the development of more advanced and efficient methods for Duffing equations such as Laplace decomposition algorithm [2], Restarted Adomian decomposition method [1]. Differential transform method (DTM) is based on Taylor series expansion [5] and [6]. In 1986, the differential transform method (DTM) was first introduced by Zhou [7] to solve linear and nonlinear initial value problems associated with electrical circuit analysis. The differential transform method obtains an analytical solution in the form of a polynomial. It is different from the traditional high order Taylor's series method, which requires symbolic competition of the necessary derivatives of the data functions. All of the previous applications of the differential transform method deal with solutions without discontinuity. As the DTM is more effective than the other methods, we further apply it to solve the The Duffing equations. In this paper, we apply these method to three examples. First, differential equation of Duffing has been converted to power series by one-dimensional differential transformation Then the numerical solution of equation was put into Pade series form [10]. The Pade approximation method was used to accelerate the convergence of the power series solution. Thus, we obtain numerical solution differential equation of Duffing.

2 One-Dimensional Differential Transform

Differential transform of function $y(x)$ is defined as follows:

$$Y(k) = \frac{1}{k!}\left[\frac{d^k y(x)}{dx^k}\right]_{x=0}, \quad (2.1)$$

In equation (2.1), $y(x)$ is the original function and $Y(k)$ is the transformed function, which is called the T-function. Differential inverse transform of $Y(k)$ is defined as

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$$y(x)=\sum_{k=0}^{\infty}x^kY(k), \tag{2.2}$$

from equation (2.1) and (2.2), we obtain

$$y(x)=\sum_{k=0}^{\infty}\frac{x^k}{k!}\left[\frac{d^ky(x)}{dx^k}\right]_{x=0}, \tag{2.3}$$

Equation (2.3) implies that the concept of differential transform is derived from Taylor series expansion, but the method does not evaluate the derivatives symbolically. However, relative derivatives are calculated by an iterative way which are described by the transformed equations of the original functions. In this study we use the lower case letter to represent the original function and upper case letter represent the transformed function. From the definitions of equations (2.1) and (2.2), it is easily proven that the transformed functions comply with the basic mathematics operations shown in Table 1. In actual applications, the function $y(x)$ is expressed by a finite series and equation (2.2) can be written as

$$y(x)=\sum_{k=0}^{m}x^kY(k), \tag{2.4}$$

Equation (2.3) implies that $y(x)=\sum_{k=m+1}^{\infty}x^kY(k)$ is negligibly small. In fact, m is decided by the convergence of natural frequency in this study.

Theorem 1. if

$$y(t)=u_1(t)u_2(t)...u_{n-1}(t)u_n(t),$$

then

$$Y(k)=\sum_{l_{n-1}=0}^{k}\sum_{l_{n-2}=0}^{l_{n-1}}...\sum_{l_2=0}^{l_3}\sum_{l_1=0}^{l_2}U_1(l_1)$$

$$U_2(l_2-l_1)...U_{n-1}(l_{n-1}-l_{n-2})U_n(k-l_{n-1}),$$

Table 1 The fundamental operations of one-dimensional DTM

Original function	Transformed function
$y(x)=u(x)\pm v(x)$	$Y(k)=U(k)\pm V(k)$
$y(x)=exp(x)$	$Y(k)=\frac{1}{k!}$
$y(x)=\frac{d^jw(x)}{dx^j}$	$Y(k)=(k+1)...(k+j)W(k+j)$
$y(x)=u(x)v(x)$	$Y(k)=\sum_{r=0}^{k}U(r)V(k-r)$
$y(x)=cos(wx+\alpha)$	$\frac{w^k}{k!}cos(\frac{k\pi}{2}+\alpha)$

3 Pade Approximation

Suppose that we are given a power series $\sum_{i=0}^{\infty}a_ix^i$, representing a function $f(x)$, so that

$$f(x)=\sum_{i=0}^{\infty}a_ix^i, \tag{3.1}$$

A Pade approximation is a rational fraction

$$[L/M]=\frac{p_0+p_1x+...+p_Lx^L}{q_0+q_1x+...+q_Mx^M}, \tag{3.2}$$

which has a Maclaurin expansion which agress with (3.1) as for as possible, Notice that in (3.2) there are $L+1$ numerator coefficients and $M+1$ denominator coefficients. There is a more or less irrelevant common factor between them, and for definitenees we take $q_0=1$. This choice turns out to be an essential part of the precise definition and (3.2) is our conventional notation with this choice for q_0. So there are $L+1$ independent numerator coefficients and M independent numerator coefficients, making $L+M+1$ unknown coefficients in all. This number suggest that normally the $[L/M]$ ought to fit the power series (3.1) through the orders $1,x,x^2,...,x^{L+M}$ in the notation of formal power series.

$$\sum_{i=0}^{\infty}a_ix^i=\frac{p_0+p_1x+...+p_Lx^L}{q_0+q_1x+...+q_Mx^M}+O(x^{L+M+1}). \tag{3.3}$$

Multiply the both side of (3.3) by the denominator of right side in (3.3) and compare the coefficients of both sides (3.3), we have

$$a_l+\sum_{k=1}^{M}a_{l-k}q_k=p_l,\ (l=0,...,M), \tag{3.4}$$

$$a_l+\sum_{k=1}^{L}a_{l-k}q_k=p_l,\ (l=M+1,...,M+L). \tag{3.5}$$

Solve the linear equation in (3.5), we have $q_k,(k=1,...,L)$. And substitute q_k into (3.4), we have $p_l,(L=0,...,M)$. Therefore, we have constructed a $[L\setminus M]$ Pade approximation, which agress with $\sum_{i=0}^{\infty}a_ix^i$ through order x^{L+M}. if $M\leq L\leq M+2$, where M and L are the degree of numerator and denominator in Pade series, respectively, then Pade series gives an A-stable formula for an ordinary differential equation.

4 Applications

Example 1. (see Table 2 and Figure 1). We first considered the Duffing equation

$$x^{''}+x^{'}+x+x^3=cos^3(t)-sin(t), \quad (4.1)$$

with initial values

$$x(0)=1, x'(0)=0, \quad (4.2)$$

With the exact solution $x(t)=cos(t)$. the Duffing equation Considering the Maclaurin series of the excitation term

$$cos^3(t)-sin(t)\approx 1-t-\frac{3t^2}{2}+\frac{t^3}{6}+\frac{7t^4}{8}-\frac{t^5}{120}-\frac{61t^6}{240}. \quad (4.3)$$

Substituting equation (4.3) into equation (4.1), we get

$$x^{''}+x^{'}+x+x^3=1-t-\frac{3t^2}{2}+\frac{t^3}{6}+7\frac{t^4}{8}-\frac{t^5}{120}-\frac{61t^6}{240}. \quad (4.4)$$

By using the fundamental operations of differential transformation method in Table 1, we obtained the following recurrence relation for equation (4.4):

$$X(k+2)=\frac{1}{(k+1)(k+2)}[-(k+1)X(k+1)-X(k) \quad (4.5)$$

$$-\sum_{k_2=0}^{k}\sum_{k_1=0}^{k_2}X(k_1)X(k_2-k_1)X(k-k_1)$$

$$+\delta(k)-\delta(k-1)-\frac{3}{2}\delta(k-2)+\frac{1}{6}\delta(k-3)$$

$$+\frac{7}{8}\delta(k-4)-\frac{1}{120}\delta(k-5)-\frac{61}{240}\delta(k-6)],$$

From the initial condition (4.2), we have

$$X(0)=1, X(1)=0, \quad (4.6)$$

The values $X(k)$, in $k=0,1,2,3,\ldots$ of equation (4.5) and (4.6) can be evaluated as follows:

$$X(2)=\frac{-1}{2}, X(3)=0, X(4)=\frac{1}{24}, X(5)=0, X(6)=\frac{-1}{720}, \quad (4.7)$$

$$X(7)=0, X(8)=\frac{1}{40320}, X(9)=0, X(10)=\frac{-1}{3628800}, \ldots$$

By using the inverse transformation rule for one dimensional in equation (2.2), the following solution can be obtained:

$$x(t)=\sum_{k=0}^{\infty}t^kX(k)=X(0)+tX(1)+t^2X(2)+t^3X(3)+t^4X(4)+\ldots \quad (4.8)$$

$$=1-\frac{1}{2}t^2+\frac{1}{24}t^4-\frac{1}{720}t^6+\frac{1}{40320}t^8-\frac{1}{3628800}t^{10}+\ldots$$

$$=1-\frac{1}{2!}t^2+\frac{1}{4!}t^4-\frac{1}{6!}t^6+\frac{1}{8!}t^8-\frac{1}{10!}t^{10}+\ldots=cos(t).$$

Which $x(t)$ is exact solution. Power series $x(t)$ can be transformed into Pade series

$$P[5/4]=\frac{1-0.4563492063492063t^2+0.0207010582010582t^4}{1+0.0436507936507937t^2+8.597883597883598\times10^{-4}t^4}, \quad (4.9)$$

Table 2 Comparison of numerical solution of $x(t)$ and Pade Approximation $P[5/4]$

t	$x(t)$	$P[5/4]$	$\lvert x(t)-P[5/4]\rvert$
0.1	0.9950041653	0.9950041652	1×10^{-10}
0.2	0.9800665779	0.9800665773	6×10^{-10}
0.3	0.9553364891	0.9553364898	7×10^{-10}
0.4	0.9210609941	0.9210609937	4×10^{-10}
0.5	0.8775825619	0.8775825624	5×10^{-10}
0.6	0.8253356149	0.8253356166	1.7×10^{-9}
0.7	0.7648421873	0.7648421979	1.06×10^{-8}
0.8	0.6967067093	0.6967067492	3.99×10^{-8}
0.9	0.6216099683	0.6216100956	1.273×10^{-7}
1.0	0.5403023059	0.5403026658	3.599×10^{-7}

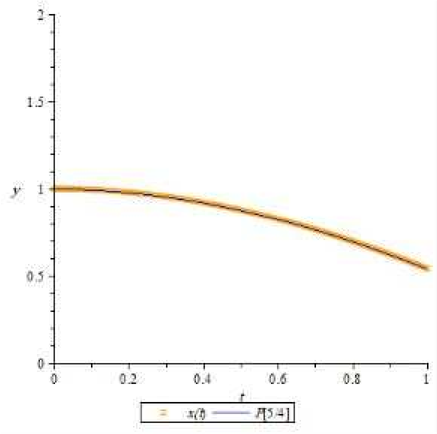

Figure 1. Values of $x(t)$ and its $P[5/4]$ Pade approximant.

Example 2. (see Table 3 and Figure 2). Now, we consider a further version of Duffing equation as follows:

$$x'' + 2x' + x + 8x^3 = e^{-3t}, \tag{4.10}$$

with initial values

$$x(0) = \frac{1}{2}, x'(0) = \frac{-1}{2}, \tag{4.11}$$

the exact solution $x(t) = \frac{1}{2}e^{-t}$.
Taking the one dimensional differential transform of (4.10), we can obtain:

$$X(k+2) = \frac{1}{(k+1)(k+2)}[-2(k+1)X(k+1) - X(k) \tag{4.12}$$

$$-8\sum_{k_2=0}^{k}\sum_{k_1=0}^{k_2} X(k_1)X(k_2-k_1)X(k-k_1) + \frac{(-3)^k}{k!}],$$

From the initial condition (4.11), we have

$$X(0) = \frac{1}{2}, X(1) = \frac{-1}{2}, \tag{4.13}$$

For each k, substituting equation(4.13)into equation (4.12), and via the recursive method,the values $X(k)$, can be evaluated as follows:

$$X(2) = \frac{1}{4}, X(3) = \frac{-1}{12}, X(4) = \frac{1}{48}, X(5) = \frac{-1}{240}, \tag{4.14}$$

$$X(6) = \frac{1}{1440}, X(7) = \frac{-1}{10080}, \ldots,$$

By using the inverse transformation rule for one dimensional in equation (2.2), the following solution can be obtained:

On rearranging the solution, we get the following closed form solution:

$$\sum_{k=0}^{\infty} t^k X(k) = X(0) + tX(1) + t^2X(2) + t^3X(3) + t^4X(4) + \frac{1}{48}t^4 \tag{4.15}$$

$$+t^5X(5) + t^6X(6) + \ldots = \frac{1}{2} - \frac{1}{2}t + \frac{1}{4}t^2 - \frac{1}{12}t^3$$

$$-\frac{1}{240}t^5 + \frac{1}{1440}t^6 - \frac{1}{10080}t^7 + \ldots = \frac{1}{2}(1-t$$

$$+\frac{1}{2!}t^2 - \frac{1}{3!}t^3 + \frac{1}{4!}t^4 - \frac{1}{5!}t^5 + \frac{1}{6!}t^6 - \frac{1}{7!}t^7 + \ldots),$$

That is,

$$x(t) = \frac{1}{2}e^{-t}. \tag{4.16}$$

Which $x(t)$ is exact solution. Power series $x(t)$ can be transformed into Pade series

$$P[5/4] = (0.5 - 0.2777777777777778t + 0.0694444444444444t^2 \tag{4.17}$$

$$-0.0099206349206349t^3 + 8.267195767195767 \times 10^{-4}t^4$$

$$-3.306878306878307 \times 10^{-5}t^5)/(1 + 0.4444444444444444t$$

$$+0.0833333333333333t^2 + 0.0079365079365079t^3$$

$$+3.306878306878307 \times 10^{-4}t^4),$$

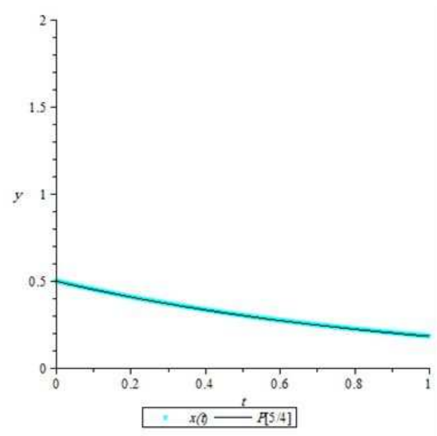

Figure 2. Values of $x(t)$ and its $P[5/4]$ Pade approximant.

Table 3 Comparison of numerical solution of $x(t)$ and Pade Approximation $P[5/4]$

t	$x(t)$	$P[5/4]$	$\|x(t)-P[5/4]\|$
0.1	0.4524187090	0.4524187092	2×10^{-10}
0.2	0.4093653764	0.4093653767	3×10^{-10}
0.3	0.3704091103	0.3704091102	1×10^{-10}
0.4	0.3351600229	0.3351600228	1×10^{-10}
0.5	0.3032653301	0.3032653298	3×10^{-10}
0.6	0.2744058181	0.2744058180	1×10^{-10}
0.7	0.2482926519	0.2482926520	1×10^{-10}
0.8	0.2246644819	0.2246644819	0
0.9	0.2032848295	0.2032848297	2×10^{-7}
1.0	0.1839397196	0.1839397202	6×10^{-7}

Example 3. (see Table 4 and Figure 3).Consider the non dimensional Duffing equation[9]

$$x'' + kx + \varepsilon x^3 = |k-1|cos(t), \tag{4.18}$$

with initial values

$$x(0) = 0, x'(0) = 0, \tag{4.19}$$

The solution depends on the two dimensionless parameters k and ε .

For the case $k = 30$, $\varepsilon = 0$ we have

$$x'' + 30x = 29cos(t), \tag{4.20}$$

By using the fundamental operations of differential transformation method in Table 1, we obtained the following recurrence relation for equation (4.20):

$$X(k+2) = \frac{1}{(k+1)(k+2)}\left[-30X(k) + \frac{29}{k!}cos\left(\frac{k\pi}{2}\right)\right], \tag{4.21}$$

From the initial condition (4.19), we have

$$X(0) = 0, X(1) = 0, \tag{4.22}$$

The values $X(k)$, in $k = 0,1,2,3,\ldots$ of equation(4.21)and(4.22)can be evaluated as follows:

$$X(2) = \frac{29}{2}, X(3) = 0, X(4) = \frac{-899}{24}, X(5) = 0, \tag{4.23}$$

$$X(6) = \frac{26999}{720}, X(7) = 0, X(8) = \frac{-809999}{40320}, X(9) = 0,$$

$$X(10) = \frac{24299999}{3628800}, X(11) = 0, X(12) = \frac{-104142857}{68428800},$$

$$X(13) = 0, X(14) = \frac{21869999999}{87178291200}, X(15) = 0,$$

$$X(16) = \frac{-656099999999}{20922789888000}, X(17) = 0,$$

$$X(18) = \frac{401693877551}{130660687872000}, X(19) = 0, \ldots$$

By using the inverse transformation rule for one dimensional in equation (2.2), the following solution can be obtained:

$$x(t) = \sum_{k=0}^{\infty} t^k X(k) = X(0) + tX(1) + t^2X(2) + t^3X(3) + t^4X(4) + \ldots \tag{4.24}$$

$$= \frac{29}{2}t^2 - \frac{899}{24}t^4 + \frac{26999}{720}t^6 - \frac{809999}{40320}t^8 + \frac{24299999}{3628800}t^{10}$$

$$-\frac{104142857}{68428800}t^{12} + \frac{21869999999}{87178291200}t^{14} - \frac{656099999999}{20922789888000}t^{16}$$

$$+\frac{401693877551}{130660687872000}t^{18} - \ldots.$$

Power series $x(t)$ can be transformed into Pade series

$$P[5/4] = \frac{14.5t^2 - 18.67370920505477t^4}{1 + 1.295491319191625t^2 + 0.7605747968005868t^4}, \tag{4.25}$$

Table 4 Comparison of numerical solution of $x(t)$ and Pade Approximation $P[5/4]$

t	$x(t)$	$P[5/4]$	$\lvert x(t)-P[5/4]\rvert$
0.1	0.1412914651	0.1412914643	8×10^{-10}
0.2	0.5224158288	0.5224149658	8.630×10^{-7}
0.3	1.027644677	1.027600040	0.000044637
0.4	1.502172577	1.501490977	0.000681600
0.5	1.797479057	1.792239670	0.005239387
0.6	1.814878675	1.789125527	0.025753148
0.7	1.534516826	1.442409794	0.092107032
0.8	1.021328053	0.7620357587	0.2592922943
0.9	0.4062256777	-0.1988810030	0.6051066807
1.0	-0.1518890375	-1.365712996	1.213823958

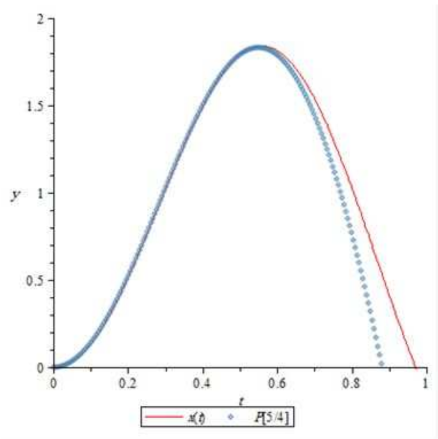

Figure 3. Values of $x(t)$ and its $P[5/4]$ Pade approximant.

5 Conclusion

In this study, the differential transform method is successfully expanded for the solution of Duffing equations. Since the Differential transform method (DTM) gives rapidly converging series solutions, the differential transform method is more effective than other methods. The accuracy of the obtained solution can be improved by taking more terms in the solution. Exact closed form solution is obtained for all examples presented in this paper.

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