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Progressively Censored Data from The Weibull Gamma Distribution Moments and Estimation

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Abstract: In this paper, we establish new recurrence relations satisfied by the single and product moments of the progressively type-II right censored order statistics from non truncated and truncated Weibull gamma distribution (WGD), and derive approximate moments of progressively type-II right censored order statistics from this distribution. Using these moments to derive the best linear unbiased estimates $(BLUE's)$ and maximum likelihood estimates $(MLE's)$ of the location and scale parameters from the Weibull gamma distribution. In addition, we use Monte-Carlo simulation method to obtain the (MSE) of $(BLUE's)$ and $(MLE's)$ and make comparison between them. Finally, a numerical examples based on simulation and real data to illustrate the inference procedures developed in this distribution are presented.

Keywords: Recurrence relations, Single moments, Product moments, Truncated form, Best linear unbiased estimates $(BLUE's)$, Maximum likelihood estimates $(MLE's)$, Monte-Carlo Method, Numerical examples.

1 Introduction

Progressive type-II censored sampling is an important method of obtaining data in lifetime studies. Live units removed early on can be readily used in other tests, thereby saving cost to the experimenter, and a compromise can be achieved between time consumption and the observation of some extreme values.

Let us consider the following type-II right censoring scheme: Suppose N units are placed on test at time zero. Immediately following the first failure, R_1 surviving items are removed from the test at random. Then, immediately following the second observed failure, R_2 surviving items are removed from the test at random. This process continues until, at the time of the m^{th} observed failure, the remaining $R_m = n - R_1 - R_2 - \cdots R_{m-1} - m$ items are all removed from the experiment.

In this scheme, $R_1, R_2, \cdots, R_m$ are pre-determined. Thus, here the censoring times $(T_i's)$ are random, but the numbers of items to fail before each censoring time are fixed. The resulting m ordered values which are obtained are referred to as progressively type-II right censored order statistics. [6, Balakrishnan and Aggarwala (2000)]

If the failure times are based on an absolutely continuous distribution function $F(x)$ with probability density function $f(x)$, the joint probability density function of progressively censored failure times $X_{1:m:n}, X_{2:m:n}, \cdots, X_{m:m:n}$ is given by:

$$f_{X_{1:m:n}, X_{2:m:n}, \cdots, X_{m:m:n}}(x_1, x_2, \cdots, x_m) = A_{n,R_{m-1}} \prod_{i=1}^{m} f(x_i)\,[1 - F(x_i)]^{R_i}, \quad -\infty < x_1 < x_2 < \cdots < x_m < \infty, \tag{1}$$

where $f(.)$ and $F(.)$ are, respectively, the pdf and the cdf of the random variable X.

$$A_{n,R_{m-1}} = n(n - R_1 - 1)\cdots(n - R_1 - R_2 - \cdots R_{m-1} - m + 1) \text{ and } A_{n,R_o} = n. \tag{2}$$

[6, Balakrishnan and Aggarwala (2000)].

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In this paper, we derive new recurrence relations satisfied by the single and product moments of the progressively type-II right censored order statistics from the Weibull gamma distribution (WGD) and its truncated form.

Numerous authors have been developed new recurrence relations satisfied by the single and product moments of the progressively type-II right censored order statistics for different distributions, see, for example [2, Arnold et al. (1992)], [10, Balakrishnan and Sandhu (1995)], [5, Balakrishnan and Aggarwala (1998)], [6, Balakrishnan and Aggarwala (2000)], [1, Abd El-Baset A. and Mohammed A. (2003)], [14, David and Nagaraja (2003)], [13, Balakrishnan et al. (2004)], [4, Balakrishnan (2007)] and [11, Balakrishnan et al. (2011)]. Approximate moments of progressively type-II right censored order statistics from the Weibull gamma distribution (WGD) are derived. These moments are used to derive the best linear unbiased estimates $(BLUE's)$ and maximum likelihood estimates $(MLE's)$ of the location and scale parameters from the Weibull gamma distribution. Several interesting mathematical results for inference procedures have been developed by the authors, see for examples: [18, Lindely (1969)], [6, Balakrishnan and Aggarwala (2000)], [12, Balakrishnan et al. (2002)], [8, Balakrishnan and Basak (2003)], [14, David and Nagaraja (2003)], [9, Balakrishnan and Rao (2003)],[15, Fernandez (2004)], [4, Balakrishnan (2007)], [3, Asgharzadeh (2006)], [19, Mahmoud and Mohie El-Din (2006)] and [21, Raqab et al. (2010)]. In addition, we use Monte-Carlo simulation method to make comparison between the (MSE) of $(BLUE's)$ and $(MLE's)$. Finally, a numerical examples based on simulation and real data to illustrate the inference procedures developed in this distribution are presented.

Let $X_{1:m:n}, X_{2:m:n}, \cdots, X_{m:m:n}$ be the progressively type-II right censored order statistics of size m from the sample of size n with censoring scheme $(R_1, R_2, \cdots, R_m)$ be from the Weibull gamma distribution whose probability function is given by:

$$f(x) = \frac{c}{\delta}\beta x^{c-1}\left(1+\frac{1}{\delta}x^c\right)^{-(\beta+1)}, x \geq 0, \beta, \delta, c > 0, \tag{3}$$

and distribution function is given by

$$F(x) = 1 - \left[1+\frac{1}{\delta}x^c\right]^{-\beta}, x \geq 0, \beta, \delta, c > 0, \tag{4}$$

also, the characterizing differential equations are given by:

$$f(x) = \frac{c}{\delta}\beta x^{c-1}\left[1-F(x)\right]^{\left(1+\frac{1}{\beta}\right)}, \tag{5}$$

$$x\left(1+\frac{\delta}{x^c}\right) f(x) = c\beta\left[1-F((x))\right]. \tag{6}$$

Making use of Equations (5) and (6) the following recurrence relations for the single and product moments of progressively type-II censored order statistics from Weibull gamma distribution have been derived.

2 Recurrence Relations for the Single Moments

Let $X_{1:m:n}, X_{2:m:n}, \cdots, X_{m:m:n}$ be the progressively type-II right censored order statistics of size m from the sample of size n with censoring scheme $(R_1, R_2, \cdots, R_m)$ be from the Weibull gamma distribution whose probability function is given by (3) and distribution function is given by (4). The single moments of the progressively type-II can be written as:

$$\mu_{i:m:n}^{(R_1,\cdots,R_m)^{(k)}} = A_{n,R_{m-1}} \int\int \int_{0<x_1<x_2<\cdots<x_m<\infty} x_i^k f(x_1)\left[1-F(x_1)\right]^{R_1} f(x_2) \times\left[1-F(x_2)\right]^{R_2}\cdots f(x_m)\left[1-F(x_m)\right]^{R_m} dx_1\cdots dx_m, \tag{7}$$

where $A_{n,R_{m-1}}$is defined in (2). [6, Balakrishnan and Aggarwala (2000)].

The single moments of progressively type-II right censored order statistics given by (7) satisfied the following recurrence relations.

Relation 1*For* $2 \leq m \leq n$, $\beta \leq 1$, $k, c > 0$ *and* $\delta > 0$

$$\mu_{1:m:n+\frac{1}{\beta}}^{\left(R_1+\frac{1}{\beta},R_2,\cdots,R_m\right)^{(k+c)}} = \frac{\left(n+\frac{1}{\beta}\right)}{n(R_1+\frac{1}{\beta}+1)}\left[\frac{\delta(k+c)}{c\beta}\mu_{1:m:n}^{(R_1,R_2\cdots,R_m)^{(k)}} - \frac{n(n-R_1-1)}{(n+\frac{1}{\beta})}\mu_{1:m-1:n+\frac{1}{\beta}}^{\left(R_1+R_2+\frac{1}{\beta}+1,R_3,\cdots,R_m\right)^{(k+c)}}\right], \tag{8}$$

and for $s=1$, $n=1,2,\cdots$, $k\geqslant 0, c>0$ *and* $\beta\leq 1$,

$$\mu_{1:1:n+\frac{1}{\beta}}^{\left(n+\frac{1}{\beta}-1\right)^{(k+c)}} = \frac{\delta\,(k+c)}{nc\beta}\mu_{1:1:n}^{(n-1)^{(k)}}. \tag{9}$$

*Proof.*From (7), we write

$$\mu_{1:m:n}^{(R_1,\cdots,R_m)^{(k)}} = A_{n,R_{m-1}}\int\int\cdots\int\limits_{0<x_2<\cdots<x_m<\infty} I(x_2)\,f(x_2)$$
$$\times\left[1-F(x_2)\right]^{R_2}\cdots f(x_m)\left[1-F(x_m)\right]^{R_m}dx_2\cdots dx_m, \tag{10}$$

where $A_{n,R_{m-1}}$ is defined in (1.2) and

$$I(x_2) = \int_0^{x_2} x_1^k\left[1-F(x_1)\right]^{R_1} f(x_1)dx_1, \tag{11}$$

which upon using (5) and integrating by parts, we get

$$I(x_2) = \frac{c\beta}{\delta}\int_0^{x_2} x_1^{k+c-1}\left[1-F(x_1)\right]^{R_1+\frac{1}{\beta}+1}dx_1$$
$$= \frac{c\beta}{\delta}\left[\frac{x_2^{k+c}}{k+c}\left[1-F(x_2)\right]^{R_1+\frac{1}{\beta}+1} + \left(\frac{R_1+\frac{1}{\beta}+1}{k+c}\right)\int_0^{x_2} x_1^{k+c}\left[1-F(x_1)\right]^{R_1+\frac{1}{\beta}} f(x_1)dx_1\right], \tag{12}$$

using (12) in (10), and simplifying the resulting equation, we get

$$\mu_{1:m:n}^{(R_1,R_2,\cdots,R_m)^{(k)}} = \frac{c\beta}{\delta\,(k+c)}\left[\frac{n(n-R_1-1)}{(n+\frac{1}{\beta})}\mu_{1:m-1:n+\frac{1}{\beta}}^{\left(R_1+R_2+\frac{1}{m}+1,R_3,\cdots,R_m\right)^{(k+c)}}\right.$$
$$\left.+\frac{n(R_1+\frac{1}{\beta}+1)}{(n+\frac{1}{\beta})}\mu_{1:m:n+\frac{1}{\beta}}^{\left(R_1+\frac{1}{\beta},R_2,\cdots,R_m\right)^{(k+c)}}\right], \tag{13}$$

thus, we get (8).

Now for $m=1$, $n=1,2,\cdots$ and $k\geq 0, c>0$ using (7), we have

$$\mu_{1:1:n}^{(R_1)^{(k)}} = A_{n,R_0}\int_0^{\infty} x_1^k f(x_1)\left[1-F(x_1)\right]^{R_1}dx_1$$
$$= \frac{nc\beta}{\delta}\int_0^{\infty} x_1^{k+c-1}\left[1-F(x_1)\right]^{R_1+\frac{1}{\beta}+1}dx_1$$
$$= \frac{nc\beta}{\delta\,(k+c)}\mu_{1:m:n+\frac{1}{\beta}}^{\left(R_1+\frac{1}{\beta}\right)^{(k+c)}}, \tag{14}$$

and hence (9) is proved.

Relation 2*For* $2\leq i\leq m-1$, $k\geqslant 0, c>0$, $\beta\leq 1$ *and* $m\leq n+\frac{1}{\beta}+1$,

$$\mu_{i:m:n+\frac{1}{\beta}}^{\left(R_1,\cdots,R_i+\frac{1}{\beta},\cdots,R_m\right)^{(k+c)}} = \frac{A_{n+\frac{1}{\beta},R_{i-1}}}{A_{n,R_{i-1}}}\frac{1}{\left(R_i+\frac{1}{\beta}+1\right)}\left[\frac{\delta\,(k+c)}{c\beta}\mu_{i:m:n}^{(R_1,R_2\cdots,R_m)^{(k)}}\right.$$
$$-\frac{A_{n,R_i}}{A_{n+\frac{1}{\beta},R_{i-1}}}\mu_{i:m-1:n+\frac{1}{\beta}}^{\left(R_1,\cdots,R_i+R_{i+1}+\frac{1}{\beta}+1,\cdots,R_m\right)^{(k+c)}}$$
$$\left.+\frac{A_{n,R_{i-1}}}{A_{n+\frac{1}{\beta},R_{i-2}}}\mu_{i-1:m-1:n+\frac{1}{\beta}}^{\left(R_1,\cdots,R_{i-1}+R_i+\frac{1}{\beta}+1,\cdots,R_m\right)^{(k+c)}}\right]. \tag{15}$$

Relation 3 *For* $2 \le m \le n$, $k \geqslant 0, c > 0$ *and* $\beta \le 1$,

$$\mu_{m:m:n+\frac{1}{\beta}}^{\left(R_1,R_2,\cdots,R_m+\frac{1}{\beta}\right)^{(k+c)}} = \frac{A_{n+\frac{1}{\beta},R_{m-1}}}{(R_m+\frac{1}{\beta}+1)}\left[\frac{\delta\,(k+c)}{c\beta A_{n,R_{m-1}}}\mu_{m:m:n}^{(R_1,R_2,\cdots,R_m)^{(k)}} + \frac{1}{A_{n+\frac{1}{\beta},R_{m-2}}}\mu_{m-1:m-1:n+\frac{1}{\beta}}^{\left(R_1,R_2,\cdots,R_{m-1}+R_m+\frac{1}{\beta}+1\right)^{(k+c)}}\right]. \tag{16}$$

Relation 4 *For* $1 \le m \le n$, $R_i > -1$ *and* $k \geqslant c$,

$$\mu_{1:m:n}^{(R_1,R_2,\cdots,R_m)^{(k)}} = \frac{c\beta\,(R_1+1)}{[c\beta(R_1+1)-k]}\left[\frac{-(n-R_1-1)}{(R_1+1)}\mu_{1:m-1:n}^{(R_1+R_2+1,R_3,\cdots,R_m)^{(k)}} + \frac{\delta k}{c\beta(R_1+1)}\mu_{1:m:n}^{(R_1,R_2,\cdots,R_m)^{(k-c)}}\right], \tag{17}$$

for $m = 1$, $n = 1, 2, \cdots$ *and* $k \geqslant c$,

$$\mu_{1:1:n}^{(R_1)^{(k)}} = \frac{k}{n\beta - k}\mu_{1:1:n}^{(R_1)^{(k-c)}}. \tag{18}$$

Relation 5 *For* $2 \le i \le m-1$, $m \le n$ *and* $k \geqslant c$,

$$\mu_{i:m:n}^{(R_1,\cdots,R_m)^{(k)}} = \frac{c\beta}{c\beta(R_i+1)-k}\left[(n-R_1-\cdots-R_{i-1}-i+1)\times\mu_{i-1:m-1:n}^{(R_1,\cdots,R_{i-1}+R_i+1,\cdots,R_m)^{(k)}} - (n-R_1-\cdots R_i-i)\times\mu_{i:m-1:n}^{(R_1,\cdots,R_i+R_{i+1}+1,\cdots,R_m)^{(k)}} + \frac{\delta k}{c\beta}\mu_{i:m:n}^{(R_1,\cdots,R_m)^{(k-c)}}\right]. \tag{19}$$

Relation 6 *For* $2 \le m \le n$, $R_m > -1$ *and* $k \geqslant c$,

$$\mu_{m:m:n}^{(R_1,R_2\cdots,R_m)^{(k)}} = \mu_{m-1:m-1:n}^{(R_1,R_2,\cdots,R_{m-1}+R_m+1)^{(k)}} + \frac{k}{c\beta(R_m+1)}\left[\delta\mu_{m:m:n}^{(R_1,R_2,\cdots,R_m)^{(k-c)}} + \mu_{m:m:n}^{(R_1,R_2,\cdots,R_m)^{(k)}}\right]. \tag{20}$$

Relation 7 *For* $2 \le m \le n, R_1 > -1, k > -1$ *and* $k \geqslant c-1$,

$$\mu_{1:m:n}^{(R_1,\cdots,R_m)^{(k+1)}} = \frac{(k+1)}{(k+1)-c\beta(R_1+1)}\left[\frac{c\beta}{(k+1)}(n-R_1-1)\times\mu_{1:m-1:n}^{(R_1+R_2+1,R_3,\cdots R_m)^{(k+1)}} - \delta\mu_{1:m:n}^{(R_1,R_2,\cdots,R_m)^{(k-c+1)}}\right]. \tag{21}$$

Relation 8 *For* $2 \le i \le m-1$, $m \le n$, $R_i > -1$ *and* $k \geqslant c-1$

$$\mu_{i:m:n}^{(R_1,\cdots,R_m)^{(k+1)}} = \frac{(k+1)}{(k+1)-c\beta\,(R_i+1)}\left\{\frac{c\beta}{k+1}(n-R_1-\cdots R_i-i)\times\mu_{i:m-1:n}^{(R_1,\cdots,R_i+R_{i+1}+1,\cdots,R_m)^{(k+1)}} - \frac{c\beta}{k+1}(n-R_1-\cdots R_{i-1}-i+1)\times\mu_{i-1:m-1:n}^{(R_1,\cdots,R_{i-1}+R_i+1,\cdots,R_m)^{(k+1)}} - \delta\mu_{i:m:n}^{(R_1,R_2,\cdots,R_m)^{(k-c+1)}}\right]. \tag{22}$$

Relation 9 *For* $2 \le m \le n$, $R_m > -1$ *and* $k \geqslant 1$,

$$\mu_{m:m:n}^{(R_1,\cdots,R_m)^{(k+1)}} = \frac{k+1}{(k+1)-c\beta\,(R_m+1)}\left[\frac{-c\beta}{(k+1)}(n-R_1-\cdots R_{m-1}-m+1)\times\mu_{m-1:m-1:n}^{(R_1,\cdots,R_{m-1}+R_m+1)^{(k+1)}} - \delta\mu_{m:m:n}^{(R_1,\cdots,R_m)^{(k-c+1)}}\right]. \tag{23}$$

3 Recurrence Relations for the Product Moments

For any continuous distribution, we can write the (i,j)-th product moment of the progressively type-II right censored order statistics from (1) as:

$$\mu_{r,s:s:n}^{(R_1,R_2,\cdots,R_s)^{(i,j)}} = E\left[X_{r:s:n}^{(R_1,\cdots,R_s)^{(i)}}\left\{X_{s:s:n}^{(R_1,\cdots,R_s)^{(j)}}\right\}\right] \tag{24}$$
$$= A_{n,R_{s-1}}\int\int\underset{0<x_1<x_2<\cdots<x_s<\infty}{\int} x_r^i x_s^j f(x_1)\left[1-F(x_1)\right]^{R_1}$$
$$\times f(x_2)\left[1-F(x_2)\right]^{R_2}\cdots f(x_s)\left[1-F(x_s)\right]^{R_s} dx_1\cdots dx_s.$$

[6, Balakrishnan and Aggarwala (2000)], where $A_{n,R_{s-1}}$is defined in (2). The product moments defined in (24) satisfied the following recurrence relations.

Relation 10*For* $1\le i<j\le m-1$ *and* $m\le n$

$$\mu_{i,j:m:n+\frac{1}{\beta}}^{\left(R_1,\cdots,R_j+\frac{1}{\beta},\cdots,R_m\right)} = \frac{\delta}{\left(R_j+\frac{1}{\beta}+1\right)}\frac{A_{n+\frac{1}{\beta},R_{j-1}}}{A_{n,R_{j-1}}}\left[\frac{1}{\beta}\mu_{i:m:n}^{(R_1,\cdots,R_m)}\right.$$
$$-\frac{1}{\delta}\frac{A_{n,R_j}}{A_{n+\frac{1}{\beta},R_{j-1}}}\mu_{i,j:m-1:n+\frac{1}{\beta}}^{\left(R_1,\cdots,R_j+R_{j+1}+\frac{1}{\beta}+1,\cdots,R_s\right)^{(1,c)}}$$
$$\left.+\frac{1}{\delta}\frac{A_{n,R_{j-1}}}{A_{n+\frac{1}{\beta},R_{j-2}}}\mu_{i,j-1:m-1:n+\frac{1}{\beta}}^{\left(R_1,\cdots,R_{j-1}+R_j+\frac{1}{\beta}+1,\cdots,R_m\right)^{(1,c)}}\right]. \tag{25}$$

*Proof.*From (24), we have

$$\mu_{i:m:n}^{(R_1,\cdots,R_s)} = E\left[X_{i:m:n}^{(R_1,\cdots,R_s)}\left\{X_{j:m:n}^{(R_1,\cdots,R_s)}\right\}^0\right]$$
$$= A_{n,R_{m-1}}\int\cdots\underset{0<x_1<x_2<\cdots<x_{j-1}<x_{j+1}<\cdots<x_m<\infty}{\int}\cdots\int x_i I(x_j)$$
$$\times f(x_1)\left[1-F(x_1)\right]^{R_1}\cdots f(x_{j-1})\left[1-F(x_{j-1})\right]^{R_{i-1}}$$
$$\times f(x_{j+1})\left[1-F(x_{j+1})\right]^{R_{j+1}}\cdots f(x_{m-1})\left[1-F(x_{m-1})\right]^{R_{s-1}}$$
$$\times\cdots f(x_m)\left[1-F(x_m)\right]^{R_m} dx_1 dx_2\cdots dx_{j-1}dx_{j+1}\cdots dx_m, \tag{26}$$

where

$$I(x_j) = \int_{x_{j-1}}^{x_{j+1}} x_j^0\left[1-F(x_j)\right]^{R_j} f(x_j)dx_j, \tag{27}$$

using (5) in (27) and integrating by parts, we get

$$I(x_j) = \frac{c\beta}{\delta}\int_{x_{j-1}}^{x_{j+1}} x_j^{c-1}\left[1-F(x_j)\right]^{R_j+\frac{1}{\beta}+1}dx_j$$
$$= \frac{\beta x_{j+1}^c}{\delta}\left[1-F(x_{j+1})\right]^{R_j+\frac{1}{\beta}+1} - \frac{\beta x_{j-1}^c}{\delta}\left[1-F(x_{j-1})\right]^{R_j+\frac{1}{\beta}+1}$$
$$+\beta\left(\frac{R_j+\frac{1}{\beta}+1}{\delta}\right)\int_{x_{j-1}}^{x_{j+1}} x_j^c\left[1-F(x_j)\right]^{R_j+\frac{1}{\beta}} f(x_j)dx_j, \tag{28}$$

using (28) in (27), and rearrangement, (25) is obtained.

Relation 11 *For* $1 \le i \le m-1$ *and* $m \le n$,

$$\mu_{i,m:m:n+\frac{1}{\beta}}^{\left(R_1,\cdots,R_m+\frac{1}{\beta}\right)^{(1,c)}} = \frac{1}{(R_m+\frac{1}{\beta}+1)} \frac{A_{n+\frac{1}{\beta},R_{s-1}}}{A_{n,R_{m-1}}} \left[\frac{\delta}{\beta} \mu_{i:m:n}^{(R_1,\cdots,R_m)} + \frac{A_{n,R_{m-1}}}{A_{n+\frac{1}{\beta},R_{m-2}}} \mu_{i,m-1:m-1:n+\frac{1}{\beta}}^{\left(R_1,\cdots,R_{m-1}+R_m+\frac{1}{\beta}+1\right)^{(1,c)}} \right]. \tag{29}$$

Relation 12 *For* $1 \le i < j \le n$, $m \le n, c \leqslant 1$ *and* $R_j > -1$,

$$\mu_{i,j:m:n}^{(R_1,\cdots,R_m)} = \frac{(n-R_1-\cdots R_{j-1}-j+1)}{(R_j+1)} \mu_{i,j-1:m-1:n}^{(R_1,\cdots,R_{j-1}+R_j+1,\cdots,R_m)} - \frac{(n-R_1-\cdots R_j-j)}{(R_j+1)} \mu_{i,j:m-1:n}^{(R_1,\cdots,R_j+R_{j+1}+1,\cdots,R_m)} + \frac{1}{c\beta\,(R_j+1)} \left[\delta \mu_{i,j:m:n}^{(R_1,\cdots,R_m)^{(1,-c+1)}} + \mu_{i,j:m:n}^{(R_1,\cdots,R_m)} \right]. \tag{30}$$

Relation 13 *For* $1 \le i \le m-1, R_{i+1} > -1$ *and* $m \le n$,

$$\mu_{i,i+1:m:n}^{(R_1,\cdots,R_m)} = \left[\frac{(n-R_1-\cdots .R_i-i)}{(R_{i+1}+1)} \mu_{i:m-1:n}^{(R_1,\cdots,R_i+R_{i+1}+1,\cdots,R_m)^{(2)}} - \frac{(n-R_1-\cdots R_{i+1}-i-1)}{(R_{i+1}+1)} \mu_{i,i+1:m-1:n}^{(R_1,\cdots,R_{i+1}+R_{i+2}+1,\cdots,R_m)} + \frac{\delta}{c\beta\,(R_{i+1}+1)} \left[\mu_{i,i+1:m:n}^{(R_1,\cdots,R_m)^{(1,-c+1)}} + \mu_{i,i+1:m:n}^{(R_1,\cdots,R_m)} \right] \right]. \tag{31}$$

Relation 14 *For* $1 \le i \le m-1, m \le n$ *and* $R_m > -1$,

$$\mu_{i,m:m:n}^{(R_1,\cdots,R_m)} = \frac{1}{c\beta\,(R_m+1)-1} \left[c\beta\,(n-R_1-\cdots R_{m-1}-m+1) \times \mu_{i,m-1:m-1:n}^{(R_1,\cdots,R_{m-1}+R_m+1)} + \delta \mu_{i,m:m:n}^{(R_1,\cdots,R_m)^{(1,-c+1)}} \right]. \tag{32}$$

4 The Doubly Truncated Weibull Gamma Distribution

In this section, we present recurrence relations for the single and product moments of progressively type-II right censored order statistics from the doubly truncated Weibull gamma distribution .

The probability density function of the doubly truncated Weibull gamma distribution is given by:

$$f_t(x) = \frac{1}{P-Q} \frac{c}{\delta} \beta x^{c-1} \left[1 + \frac{1}{\delta} x^c \right]^{-\beta-1}, \tag{33}$$

$$0 < Q_1 < x < P_1, \cdots,\ \alpha, \beta,\ \delta > 0, \cdots, x \ge 0.$$

Here, $1-P$ is the proportion of right truncation on the Weibull gamma distribution and Q is the proportion of the left truncation. Thus

$$Q = 1 - \left[1 + \frac{1}{\delta} Q_1^c \right]^{-\beta}, \tag{34}$$

and

$$P = 1 - \left[1 + \frac{1}{\delta} P_1^c \right]^{-\beta}. \tag{35}$$

Thus cumulatative distribution function of doubly truncated Weibull gamma distribution is given by:

$$F_t(x) = \frac{1}{P-Q} \left[\left[1 + \frac{1}{\delta} Q_1^c \right]^{-\beta} - \left[1 + \frac{1}{\delta} x^c \right]^{-\beta} \right], \tag{36}$$

so, the characterizing differential equation for this distribution is given by:

$$\left[x+\delta x^{-c+1}\right] f_t(x) = c\beta\left[\frac{1-P}{P-Q}\right] + c\beta\left[1-F_t(x)\right]. \tag{37}$$

Thus, we can conclude new recurrence relations for the single and product moments of progressively type-II right censored order statistics from the doubly truncated Weibull gamma distribution.

Relation 15 *For* $2\leq m\leq n-1, k>-1$ *and* $R_1>-1$,

$$\begin{aligned}\mu_{1:m:n}^{(R_1,\cdots,R_m)^{(k+1)}} &= \frac{c\beta}{(k+1)-c\beta(R_1+1)} \\ &\times\left(\frac{1-P}{P-Q}\right)\left[(n-R_1-1)\,\mu_{1:m-1:n}^{(R_1+R_2+1,R_3,\cdots,R_m)^{(k+1)}}\right. \\ &\left.-\, nQ_1^{k+1} + R_1\frac{n}{n-1}\mu_{1:m:n-1}^{(R_1-1,R_2,\cdots,R_m)^{(k+1)}}\right] \\ &+ (n-R_1-1)\,\mu_{1:m-1:n}^{(R_1+R_2+1,R_3,\cdots,R_m)^{(k+1)}} \\ &-\, nQ_1^{k+1} - \frac{\delta(k+1)}{c\beta}\mu_{1:m:n}^{(R_1,\cdots,R_m)^{(k-c+1)}}\Bigg]. \end{aligned} \tag{38}$$

Relation 16 *For* $2\leq i\leq m-1, m\leq n-1, k>-1$ *and* $R_i>-1$,

$$\begin{aligned}\mu_{i:m:n}^{(R_1,\cdots,R_m)^{(k+1)}} &= \frac{c\beta}{(k+1)-c\beta(R_i+1)} \\ &\times\left[\left(\frac{1-P}{P-Q}\right)\frac{A_{n,R_i}}{A_{n-1,R_{i-1}}}\mu_{i:m-1:n-1}^{(R_1,\cdots,R_i+R_{i+1},\cdots,R_m)^{(k+1)}}\right. \\ &-\frac{A_{n,R_{i-1}}}{A_{n-1,R_{i-2}}}\mu_{i-1:m-1:n-1}^{(R_1,\cdots,R_{i-1}+R_i.,\cdots,R_m)^{(k+1)}} \\ &\left.+R_i\frac{A_{n,R_{i-1}}}{A_{n-1,R_{i-1}}}\mu_{i:m:n-1}^{(R_1,\cdots,R_i-1,\cdots,R_m)^{(k+1)}}\right] \\ &+\left[(n-R_1-\cdots R_i-i)\mu_{i:m-1:n}^{(R_1,\cdots,R_i+R_{i+1}+1,\cdots,R_m)^{(k+1)}}\right. \\ &-(n-R_1-\cdots R_{i-1}-i+1)\mu_{i-1:m-1:n}^{(R_1,\cdots,R_{i-1}+R_i+1,\cdots,R_m)^{(k+1)}} \\ &\left.-\frac{\delta(k+1)}{c\beta}\mu_{i:m:n}^{(R_1,\cdots,R_m)^{(k-c+1)}}\right]. \end{aligned} \tag{39}$$

Relation 17 *For* $2\leq m\leq n-1, k>-1$ *and* $R_m>-1$,

$$\begin{aligned}\mu_{m:m:n}^{(R_1,\cdots,R_m)^{(k+1)}} &= \frac{c\beta}{(k+1)-c\beta(R_m+1)} \\ &\times\left[-\left(\frac{1-P}{P-Q}\right)\frac{A_{n,R_{m-1}}}{A_{n-1,R_{m-2}}}\mu_{m-1:m-1:n-1}^{(R_1,\cdots,R_{m-1}+R_m)^{(k+1)}}\right. \\ &\left.-R_m\frac{A_{n,R_{m-1}}}{A_{n-1,R_{m-1}}}\mu_{m:m:n-1}^{(R_1,\cdots R_m-1)^{(k+1)}}\right] \\ &+\left[-(n-R_1-\cdots R_{m-1}-m+1)\mu_{m-1:m-1:n}^{(R_1,\cdots,R_{m-1}+R_m+1)^{(k+1)}}\right. \\ &\left.-\frac{\delta(k+1)}{c\beta}\mu_{m:m:n}^{(R_1,\cdots,R_m)^{(k-c+1)}}\right]. \end{aligned} \tag{40}$$

Relation 18 *For* $1 \leq i < j \leq m-1, m \leq n$ *and* $R_j > -1$,

$$\mu_{i,j:m:n}^{(R_1,R_2,\cdots R_m)} = \frac{c\beta}{1-c\beta(R_j+1)} \times \left[\frac{(1-P)}{(P-Q)} \frac{A_{n,R_j}}{A_{n-1,R_{j-1}}} \mu_{i,j:m-1:n-1}^{(R_1,\cdots,R_j+R_{j+1},\cdots,R_m)} - \frac{A_{n,R_{j-1}}}{A_{n-1,R_{j-2}}} \mu_{i,j-1:m-1:n}^{(R_1,\cdots,R_{j-1}+R_j,\cdots,R_m)} + R_j \frac{A_{n,R_{j-1}}}{A_{n-1,R_{j-1}}} \mu_{i,j:m:n-1}^{(R_1,\cdots,R_j-1,\cdots,R_m)} \right] + \left[(n-R_1-\cdots R_j-j)\, \mu_{i,j:m-1:n-1}^{(R_1,\cdots,R_j+R_{j+1}+1,\cdots,R_m)} - (n-R_1-\cdots R_{j-1}-j+1)\, \mu_{i,j-1:m-1:n}^{(R_1,\cdots,R_{j-1}+R_j+1,\cdots,R_m)} - \frac{\delta}{c\beta} \mu_{i,j:m:n}^{(R_1,\cdots,R_m)^{(1,-c+1)}} \right]. \tag{41}$$

Relation 19 *For* $1 \leq i \leq m-1, m \leq n$ *and* $R_m > -1$,

$$\mu_{i,m:m:n}^{(R_1,\cdots,R_m)} = \frac{c\beta}{c\beta(R_m+1)-1} \left[\left(\frac{1-P}{P-Q} \right) \frac{A_{n,R_{m-1}}}{A_{n-1,R_{m-2}}} \mu_{i,m-1:m-1:n-1}^{(R_1,\cdots R_{m-1}+R_m)} + R_m \frac{A_{n,R_{m-1}}}{A_{n-1,R_{m-1}}} \mu_{i,m:m:n-1}^{(R_1,\cdots,R_m-1)} \right] - \left[(n-R_1-\cdots-R_{m-1}-m+1)\, \mu_{i,m-1:m-1:n}^{(R_1,\cdots,R_{m-1}+R_m+1)} - \frac{\delta}{c\beta} \mu_{i,m:m:n}^{(R_1,\cdots,R_m)^{(1,-c+1)}} \right]. \tag{42}$$

Remark. Setting $c = 1$ and $\delta = 1$ in the recurrence relations given in section (2) and (3), we deduce the recurrence relations for the single and product moments from the Lomax distribution. [17, Hassan and Sultan (2005)]

Remark. Setting $p = 1, Q = 0$ in Theorems (16), (17), (18), (19) and (20), we obtain Theorems (7), (8), (9), (14), (15).in our paper.

Remark. Setting $R_1 = R_2 = \cdots . = R_m = 0$, so that $m = n$ in which the case of the progressively type-II censored order statistics gives the usual order statistics $X_{1:n}, X_{2:n}, \cdots, X_{n:n}$, the relations established for the Weibull gamma and Lomax distributions reduce to the corresponding recurrence relations based on the usual order statistics. [20, Malik et al. (1998)].

5 Progressively Type-II Right Censored Transformation

By using Equations (3) and (4), we can write the joint density function of $X_{1:m:n}, X_{2:m:n}, \cdots, X_{m:m:n}$ of a progressively type-II right censored sample from the Weibull gamma distribution, with censoring scheme $(R_1, R_2, \cdots, R_m)$, in the form:

$$f_{X_{1:m:n},X_{2:m:n},\cdots X_{m:m:n}}(x_1,x_2,\cdots,x_m) = A_{n,R_{m-1}} \prod_{i=1}^{m} \frac{\alpha\beta}{\delta} x_i^{\alpha-1} \left[1+\frac{1}{\delta}x_i^{\alpha}\right]^{-\beta(R_i+1)-1} \tag{43}$$

$$0 < x_1 < x_2 < \cdots < x_m < \infty,$$

where $A_{n,R_{m-1}}$ is defined by (2). [6, Balakrishnan and Aggarwala(2000)].

Notation 20 *For simplicity, put* $c = \alpha$ *in Equations* (3) *and* (4).

Since, the joint density function is more complicated, so we try to find relationship between the Weibull gamma distribution and uniform distribution.

Let $U_{1:m:n}, U_{2:m:n}, \cdots, U_{m:m:n}$ be the progressively type-II right censored order statistics of size m from the sample of size n with censoring scheme $(R_1, R_2, \cdots, R_m)$ be from the uniform $(0,1)$ distribution.

The exact moments of progressively type-II right censored order statistics from the uniform $(0,1)$ distribution can be written in the form:

$$
\begin{aligned}
E(U_{i:m:n}) &= 1 - \prod_{j=m-i+1}^{m} \alpha_j, i = 1, 2, \cdots, m, \\
Var(U_{i:m:n}) &= \left(\prod_{j=m-i+1}^{m} \alpha_j \right) \left(\prod_{j=m-i+1}^{m} \gamma_j - \prod_{j=m-i+1}^{m} \alpha_j \right), \\
Cov(U_{i:m:n}, U_{k:m:n}) &= \left(\prod_{j=m-i+1}^{m} \alpha_j \right) \left(\prod_{j=m-k+1}^{m} \gamma_j - \prod_{j=m-k+1}^{m} \alpha_j \right), k < i,
\end{aligned}
\tag{44}
$$

where

$$
\begin{aligned}
a_i &= i + \sum_{j=m-i+1}^{m} R_j, i = 1, 2, \cdots, m, \\
\alpha_i &= \frac{a_i}{a_i + 1}, i = 1, 2, \cdots, m, \\
\beta_i &= \frac{1}{(a_i + 1)(a_i + 2)}, i = 1, 2, \cdots, m, \\
\gamma_i &= \alpha_i + \beta_i, i = 1, 2, \cdots, m.
\end{aligned}
\tag{45}
$$

[6, Balakrishnan and Aggarwalla(2000)].

These expressions enable us to derive the approximate means, variances, and covariances for the Weibull gamma distribution using the the following theorems. [18, Lindely (1969, pp.133-134)]

Theorem 21.*If X is a random variable with $E(X) = \mu$, $D^2(X) = \sigma^2$, and $y = \phi(x)$ then, for sufficiently small σ, and well-behaved ϕ*

$$E(Y) \simeq \phi(\mu) + \frac{1}{2}\sigma^2 \phi^{\cdot\cdot}(\mu), \tag{46}$$

and

$$D^2(Y) \simeq (\phi^{\cdot}(\mu))^2 \sigma^2. \tag{47}$$

Theorem 22.*If X and Y are random variables with $E(X) = \mu$, $E(Y) = \upsilon$, $D^2(X) = \sigma^2$, $D^2(Y) = \tau^2$ and $\rho(X,Y) = \rho$, and $Z = \phi(x,y)$ then, for sufficiently small σ and τ, and well behaved ϕ*

$$E(Z) \simeq \phi(\mu, \upsilon) + \frac{1}{2}\sigma^2 \frac{\partial^2 \phi}{\partial x^2} + \rho\sigma\tau \frac{\partial^2 \phi}{\partial x \partial y} + \frac{1}{2}\tau^2 \frac{\partial^2 \phi}{\partial y^2}, \tag{48}$$

and

$$D^2(Z) \simeq \sigma^2 \left(\frac{\partial \phi}{\partial x} \right)^2 + 2\rho\sigma\tau \left(\frac{\partial \phi}{\partial x} \frac{\partial \phi}{\partial y} \right) + \tau^2 \left(\frac{\partial \phi}{\partial y} \right)^2, \tag{49}$$

where all partial derivatives are evaluated at $x = \mu$, $y = \upsilon$.

6 Deriving Moments Using Transformation

Since, the joint density function is more difficult to use it to find the moments, so we get relationship between the Weibull gamma distribution and uniform distribution, as follows:

$$U \simeq 1 - \left(1 + \frac{1}{\delta} x^{\alpha} \right)^{-\beta}. \tag{50}$$

So by using Equations (46), (47) and (48), the approximate moments can be written as follows:

$$E(X_{i:m:n}) \simeq \left[\delta\left([1-\mu]^{\frac{-1}{\beta}}-1\right)\right]^{\frac{1}{\alpha}}$$
$$+\frac{1}{2}\sigma^2\left\{\left[\frac{\delta}{\beta}(1-\mu)^{\frac{-1}{\beta}-1}\right]^2\frac{1}{\alpha}(\frac{1}{\alpha}-1)\left[(\delta\left([1-\mu]^{\frac{-1}{\beta}}-1\right)\right]^{\frac{1}{\alpha}-2}\right.$$
$$\left.+\frac{\delta}{\alpha\beta}\left(\frac{1}{\beta}+1\right)(1-\mu)^{\frac{-1}{\beta}-2}\left[(\delta\left([1-\mu]^{\frac{-1}{\beta}}-1\right)\right]^{\frac{1}{\alpha}-1}\right\}, \tag{51}$$

$$D^2(X_{i:m:n}) \simeq \sigma^2\left(\frac{\delta}{\alpha\beta}\right)^2\left[\left[(\delta\left([1-\mu]^{\frac{-1}{\beta}}-1\right)\right]^{\left(\frac{1}{\alpha}-1\right)}[1-\mu]^{\left(\frac{-1}{\beta}-1\right)}\right]^2, \tag{52}$$

$$Z=\left[\delta\left([1-u_i]^{\frac{-1}{\beta}}-1\right)\right]^{\frac{1}{\alpha}}\left[\delta\left([1-u_j]^{\frac{-1}{\beta}}-1\right)\right]^{\frac{1}{\alpha}},$$

and

$$Cov(X_{i:m:n},X_{j:m:n}) \simeq E(Z)-E(X_{i:m:n})E(X_{j:m:n}). \tag{53}$$

We use these moments to derive the best linear unbiased estimators for the location (μ) and scale (σ) parameters of Weibull gamma distribution.

7 Estimation of Parameters

The best linear unbiased and maximum likelihood methods are used to obtain estimators of the location (μ) and scale (σ) parameters. Let the probability density function is given by :

$$f(x)=\frac{1}{\sigma}\frac{\alpha\beta}{\delta}\left(\frac{x-\mu}{\sigma}\right)^{\alpha-1}\left[1+\frac{1}{\delta}\left(\frac{x-\mu}{\sigma}\right)^{\alpha}\right]^{-\beta-1}, x>\mu, \tag{54}$$

also the distribution function is given by:

$$F(x)=1-\left(1+\frac{1}{\delta}\left(\frac{x-\mu}{\sigma}\right)^{\alpha}\right)^{-\beta}. \tag{55}$$

7.1 Best linear unbiased estimates (BLUEs)

Consider an arbitrary continuous distribution $F(x)$. Suppose that the progressively censored order statistics can be represented by the linear transformation $\mathbf{Y}=\mu\mathbf{1}+\sigma\mathbf{X}$, where the vector $\mathbf{X}$ represent a vector of progressively type-II censored order statistics from the standard distribution $F(x)$, then the best linear unbiased estimators of μ and σ will be minimizing the generalized variance $Q(\theta)=(\mathbf{Y}-A\theta)^T\sum^{-1}(\mathbf{Y}-A\theta)$ with respect to θ where $\theta=(\mu,\sigma)^T$, A is the $p\times p$ matrix, $\mathbf{1}$ is $p\times 1$ vector with components all 1's, μ is the mean vector of $\mathbf{X}$ and $\sum$ is the variance-covariance matrix of $\mathbf{X}$. The minimum occurs when

$$\mu^*=-\mu^T\Gamma Y=\sum_{i=1}^{m}A_iY_{i:m:n}, \tag{56}$$

and

$$\sigma^*=\mathbf{1}^T\Gamma Y=\sum_{i=1}^{m}B_iY_{i:m:n}, \tag{57}$$

where

$$\Gamma=\sum^{-1}\left(\mathbf{1}\mu^T-\mu\mathbf{1}^T\right)\sum^{-1}/\Delta, \tag{58}$$

and

$$\Delta=\left(\mathbf{1}^T\sum^{-1}\mathbf{1}\right)\left(\mu^T\sum^{-1}\mu\right)-\left(\mathbf{1}^T\sum^{-1}\mu\right)^2. \tag{59}$$

[6, Balakrishnan and Aggarwala(2000)].

From these results, we can get the variances and covariances of the estimators in the form:

$$Var(\mu^*) = \left(\sigma^2 \mu^T \sum^{-1} \mu\right) \diagup \Delta, \tag{60}$$

$$Var(\sigma^*) = \left(\sigma^2 \mathbf{1}^T \sum^{-1} \mathbf{1}\right) \diagup \Delta, \tag{61}$$

$$Cov(\mu^*, \sigma^*) = \left(-\sigma^2 \mu^T \sum^{-1} \mathbf{1}\right) \diagup \Delta. \tag{62}$$

The coefficients A_i, B_i, $i = 1, 2, \cdots m$ which represented in Equations (56) and (57) satisfy the relations $\sum_{i=1}^{m} A_i = 1$, $\sum_{i=1}^{m} B_i = 0$.[6, Balakrishnan and Aggarwala(2000)].

7.2 Maximum likelihood estimators

Let $X_{1:m:n}, X_{2:m:n} \cdots X_{m:m:n}$ be the progressively type-II right censored order statistics of size m from the sample of size n with censoring scheme $(R_1, R_2, \cdots, R_m)$ taken from the Weibull gamma distribution whose probability function is given by (54) and the cummulatative distribution function is given by (55), then likelihood function can be written in the form:

$$L(\mu, \sigma) = A_{n.R_{m-1}} \prod_{i=1}^{m} f\left(X_{i:m:n}\right)\left[1 - F\left(X_{i:m:n}\right)\right]^{R_i}, \tag{63}$$

where c is normalizing constant, see [6, Balakrishnan and Aggarwala(2000)].

The likelihood function to be maximized for estimators of μ and σ is given by:

$$L(\mu, \sigma) = (\text{constant})\,(\alpha\beta)^m \sigma^{n\beta\alpha} \delta^{n\beta} \prod_{i=1}^{m} (x_i - \mu)^{\alpha-1} \left[\delta\sigma^\alpha + (x_i - \mu)^\alpha\right]^{-\beta(R_i+1)-1}. \tag{64}$$

The log-likelihood function can be written in form:

$$\begin{aligned} \ln L(\mu, \sigma) &= \ln \text{constant} + m \ln \alpha\beta + n\alpha\beta \ln \sigma + n\beta \ln \delta + (\alpha - 1) \sum_{i=1}^{m} \ln (x_i - \mu) \\ &- \sum_{i=1}^{m} \left[\beta(R_i + 1) + 1\right] \ln\left(\delta\sigma^\alpha + (x_i - \mu)^\alpha\right), \end{aligned} \tag{65}$$

by differentiating the log-likelihood function given by (65) with respect to μ and σ. The resulting equations to be solved for maximum likelihood estimators μ and σ are given by:

$$\sum_{i=1}^{m} \frac{\alpha\left[\beta(R_i + 1) + 1\right]\left(x_i - \widehat{\mu}\right)^{\alpha-1}}{\delta\widehat{\sigma}^\alpha + (x_i - \widehat{\mu})^\alpha} - (\alpha - 1) \sum_{i=1}^{m} \frac{1}{x_i - \widehat{\mu}} = 0. \tag{66}$$

$$\frac{n\beta\alpha}{\widehat{\sigma}} - \sum_{i=1}^{m} \left[\beta(R_i + 1) + 1\right] \frac{c\delta\widehat{\sigma}^{\alpha-1}}{\delta\widehat{\sigma}^\alpha + (x_i - \widehat{\mu})^\alpha} = 0. \tag{67}$$

Since Equations (66) and (67) cannot be solved analytically, so we can use MATLAB program to solve these equations.

8 Simulation Study

Let us consider the following table represented different schemes of progressively censored data:

So, by using Equations (56) and (57), the coefficients of the $BLUE's$ of μ and σ from the Weibull gamma distribution using different schemes represented in table (1) about $\mu = 0$ and $\sigma = 1$ are obtained in the following tables (2,3,4 and 5):

Table 1: sample sizes and censoring schemes from the Weibull gamma distribution.

m	n	*scheme*
5	15	$R_1 = [2\ 0\ 4\ 0\ 4]$
6	20	$R_2 = [4\ 0\ 4\ 0\ 2\ 4]$
7	25	$R_3 = [4\ 0\ 4\ 2\ 2\ 4\ 2]$
8	30	$R_4 = [2\ 2\ 4\ 4\ 4\ 2\ 0\ 4]$

Table 2: Coefficients of Blues of σ and μ from the Weibull gamma distribution using the first scheme.

*sch*1		$\delta = 0.5, \beta = 0.25, \alpha = .5$		$\delta = 1.5, \beta = 1, \alpha = 1.5$		$\delta = 2, \beta = 1.5, \alpha = 2$	
m	n	A_i	B_i	A_i	B_i	A_i	B_i
5	15	0.3357	−0.3444	0.5232	−0.6892	0.6828	−1.0293
		0.4657	−0.2104	0.5773	−0.4841	0.6739	−0.6327
		0.3717	−0.0381	0.3891	−0.0893	0.4058	−0.1178
		0.0236	0.2561	−0.1156	0.5870	−0.2340	0.7659
		−0.1967	0.3368	−0.3740	0.7757	−0.5286	1.0139
		$\simeq 1$	$\simeq 0$	$\simeq 1$	$\simeq 0$	$\simeq 1$	$\simeq 0$

Table 3: Coefficients of Blues of σ and μ from the Weibull gamma distribution using the Second scheme.

*sch*2		$\delta = 0.5, \beta = 0.25, \alpha = 0.5$		$\delta = 1.5, \beta = 1, \alpha = 1.5$		$\delta = 2, \beta = 1.5, \alpha = 2$	
m	n	A_i	B_i	A_i	B_i	A_i	B_i
6	20	0.1621	−0.4013	0.3480	−0.7713	0.4836	−0.9421
		0.4056	−0.2514	0.5351	−0.5295	0.6408	−0.6806
		0.4024	−0.0648	0.4477	−0.1789	0.4926	−0.2545
		0.2116	0.2005	0.1257	0.3608	−0.0661	0.4306
		0.0210	0.2912	−0.1191	0.5783	−0.2279	0.7255
		−0.2028	0.2259	−0.3374	0.5406	−0.4552	0.7211
		$\simeq 1$	$\simeq 0$	$\simeq 1$	$\simeq 0$	$\simeq 1$	$\simeq 0$

Table 4: Coefficients of Blues of σ and μ from the Weibull gamma distribution using the Third scheme.

*sch*3		$\delta = 0.5, \beta = 0.25, \alpha = 0.5$		$\delta = 1.5, \beta = 1, \alpha = 1.5$		$\delta = 2, \beta = 1.5, \alpha = 2$	
m	n	A_i	B_i	A_i	B_i	A_i	B_i
7	25	−0.8200	−0.9465	−0.1782	−0.8276	−0.0459	−0.8444
		−0.1277	−0.5728	0.2998	−0.6130	0.4399	−0.7098
		0.3581	−0.1292	0.4952	−0.2548	0.5866	−0.3616
		0.7466	0.4003	0.5135	0.2398	0.5103	0.1742
		0.8161	0.7097	0.3466	0.5866	0.2572	0.5897
		0.5380	0.6851	0.0491	0.6689	−0.0908	0.7486
		−0.5109	−0.1466	−0.5290	0.2002	−0.6573	0.4034
		$\simeq 1$	$\simeq 0$	$\simeq 1$	$\simeq 0$	$\simeq 1$	$\simeq 0$

Table 5: Coefficients of Blues of σ and μ from the Weibull gamma distribution using the fourth scheme.

$sch4$		$\delta=0.5,\beta=0.25,\alpha=0.5$		$\delta=1.5,\beta=1,\alpha=1.5$		$\delta=2,\beta=1.5,\alpha=2$	
m	n	A_i	B_i	A_i	B_i	A_i	B_i
8	30	-0.4775	-1.8120	-0.2157	-0.7244	-0.2412	-0.5281
		-0.0232	-1.2434	0.2359	-0.6744	0.3212	-0.6851
		0.3349	-0.3368	0.4922	-0.3771	0.6228	-0.5290
		0.5642	0.6512	0.5648	0.0490	0.6849	-0.1731
		0.5871	1.3873	0.4258	0.4674	0.4798	0.2743
		0.3377	1.4786	0.0708	0.7100	0.0094	0.6701
		-0.0547	0.5671	-0.2618	0.5060	-0.4119	0.6652
		-0.2685	-0.6919	-0.3120	0.0435	-0.4651	0.3046
		$\simeq 1$	$\simeq 0$	$\simeq 1$	$\simeq 0$	$\simeq 1$	$\simeq 0$

Also, the variances and covariances of the estimators μ and σ can be represented in the following table:

Table 6: the variances and co variances of the estimators μ^* and σ^* from the weibull gamma distribution .

δ	β	α	m	n	sch	$Var(\mu^*)$	$Var(\sigma^*)$	$Cov(\mu^*,\sigma^*)$
0.5	0.25	0.5	5	15	1	5.5359	0.0110	0.0453
			6	20	2	6.0932	0.2626	-0.0206
			7	25	3	11.5396	1.1215	0.4186
			8	30	4	11.6510	3.2271	-0.1334
1.5	1	1.5	5	15	1	0.0010	0.0040	0.0062
			6	20	2	1.8969	0.0781	-0.0220
			7	25	3	2.3443	0.7210	-0.0600
			8	30	4	2.5058	0.7511	-0.1529
2	1.5	2	5	15	1	1.8108	0.0013	0.0014
			6	20	2	1.8515	0.0252	-0.0094
			7	25	3	2.0298	0.0588	-0.0293
			8	30	4	2.2291	0.1259	-0.0651

MSE of μ and σ from the Weibull gamma distribution with $\mu=0$ and $\sigma=1$ can be represented in the following table:

Table 7: MSE of μ and σ from the Weibull gamma distribution using different scheme.

δ	β	α	m	n	sch	$MSE(\mu^*)$	$MSE(\sigma^*)$	$MSE(\widehat{\mu})$	$MSE(\widehat{\sigma})$
0.5	0.25	0.5	5	15	1	0.3516	1.3527	$1.5246e-012$	$1.8314e-012$
			6	20	2	0.1407	0.2454	$1.3985e-012$	$1.0118e-012$
			7	25	3	0.0688	0.2354	$1.2511e-013$	$3.4817e-013$
			8	30	4	0.0317	0.0468	$1.1433e-013$	$1.1211e-013$
1.5	1	1.5	5	15	1	0.5138	0.1682	$1.7668e-007$	$1.7662e-008$
			6	20	2	0.2831	0.1184	$1.5681e-009$	$1.9753e-010$
			7	25	3	0.0542	0.1049	$1.1102e-009$	$1.5039e-010$
			8	30	4	0.0533	0.0951	$1.6474e-010$	$1.4631e-010$
2	1.5	2	5	15	1	0.8050	0.3606	$1.1579e-009$	$1.0979e-009$
			6	20	2	0.6980	0.1316	$1.0739e-009$	$3.2984e-010$
			7	25	3	0.5014	0.0831	$5.9680e-010$	$1.4333e-010$
			8	30	4	0.1533	0.0047	$1.5537e-010$	$8.0946e-012$

From the numerical results presented in tables 2, 3, 4, 5 and 7, we can conclude the following:

1. As a check of the entries of tables 2, 3, 4 and 5, we see that $\sum_{i=1}^{n} A_i \simeq 1$, $\sum_{i=1}^{n} B_i \simeq 0$.

2.From Table (7), we see that as n increases, the mean square error $MSE(\mu^*)$ and $MSE(\sigma^*)$ decrease for all censoring schemes and all values of α, δ and β.
3.From Table (7), we see that as n increases, the mean square error $MSE(\widehat{\mu})$ and $MSE(\widehat{\sigma})$ decrease for all censoring schemes and all values of α, δ and $\beta \cdots$

9 Numerical Examples

Example 1*A progressively type-II censored sample of size $m = 5$ from a sample of size $n = 15$ from the Weibull gamma distribution with $\mu = 0$, $\sigma = 1$, $\delta = 1.5, \beta = 1$, $\alpha = 1.5$ with scheme $R_i = (2\ 0\ 4\ 0\ 4)$, was simulated using MATLAB program. The simulated progressively type-II right censored sample is given by:*

Table 8: Progreesively Type-II right censored sample generated from the weibull gamma distribution.

$X_{i:5:15}$	0.0503	0.2537	0.2705	0.2935	0.6190
R_i	2	0	4	0	4

By making use of equation (56) and (57), and using the coefficients A_i and B_i given in table (2) for $n = 15$ and $m = 5$, we get the BLUE's of the μ and σ as follows:

$$\begin{aligned}\mu^* &= (0.5232 \times 0.0503) + (0.5773 \times 0.2537) + (0.3891 \times 0.2705) \\ &\quad + (-0.1156 \times 0.2935) + (-0.3740 \times 0.6190) \\ &= 0.01259492\end{aligned}$$

and

$$\begin{aligned}\sigma^* &= (-0.7892 \times 0.0503) + (-0.4841 \times 0.2537) + (-0.0893 \times 0.2705) \\ &\quad + (0.5870 \times 0.2935) + (0.7757 \times 0.6190) \\ &= 0.46577422\end{aligned}$$

The standard error of the estimates μ^* and σ^* are

$$SE(\mu^*) = \sigma^* (Var(\mu^*))^{\frac{1}{2}} = 0.465774220 \times (0.0010)^{\frac{1}{2}} = 0.01472907$$

$$SE(\sigma^*) = \sigma^* (Var(\sigma^*))^{\frac{1}{2}} = 0.465774220 \times (0.0040)^{\frac{1}{2}} = 0.02945148$$

Using the same data,we can get by simulation

$$\widehat{\mu} = 0.000505005$$

$$\widehat{\sigma} = 0.686868111$$

Then, the best linear unbiased prediction for the failure following $Y_{5:5:15}^{(2\ 0\ 4\ 0\ 4)}$ may be determined simply by equating μ^* or σ^* based on the sample of size $m = 5$ to μ^* and σ^* based on the sample of size $n = 15$ with progressive censoring $(2\ 0\ 4\ 0\ 0\ 3)$ whose coefficients are given in the form:

Table 9: coefficients of the BLUEs of μ and σ from the Weibull gamma distribution using different scheme.

A_i	0.3347	0.5248	0.4522	0.0886	-0.1521	-0.2481
B_i	-0.5139	-0.4075	-0.1814	0.2888	0.4517	0.3623

then

$$\begin{aligned}\mu^* &= (0.3347 \times 0.0503) + (0.5248 \times 0.2537) + (0.4522 \times 0.2705) \\ &\quad + (0.0886 \times 0.2935) + (-0.1521 \times 0.6190) + (-0.2481 \times y^*_{6:6:15})\end{aligned}$$

Upon equating this with $\mu^* = 0.012594920$ and solving, we get the first-order approximation to the BLUE of $y^*_{6:6:15}$ as

$$y^*_{6:6:15} = \frac{0.20415147 - 0.01259492}{0.2481} = 0.772094115$$

*Remark.*One can see that the values of μ^*close to 0 and σ^* close to 1.

Example 2*We consider the following set of data reported by Nelson (1982). Nelson presents the results of a life-test experiment in which specimens of a type of electrical insulating fluid were subject to a constant voltage stress (34kv/minutes). In analyzing the complete data, Nelson assumed a Weibull distribution for the times to breakdown. The 19 log-times to breakdown are:*

-1.66073, -0.248461, -0.040822, 0.270027, 1.02245, 1.15057, 1.42311, 1.54116, 1.57898, 1.8718, 1.9947, 2.08069, 2.11263, 2.48989, 3.45789, 3.48186, 3.52371, 3.60305, 4.28895.

We consider the following progressive type-II censored sample generated by Viveros and Balakrishnan (1994). In this sample the vector of log-times to breakdown and the progressive censoring scheme are presented in the table 10: In order

Table 10: Progressively type-II right censored sample from the real data set.

$X(i)$	-1.66073	-0.248461	-0.040822	0.270027	1.02245	1.57898	1.8718	1.9947
$R(i)$	0	0	3	0	3	0	0	5

to find relationship between the previous data and data generated by Weibull gamma distribution presented in table 11. We have to compute correlation coefficient. Since, the correlation coefficient between two sets of two data is very high

Table 11: Progressively type-II right censored sample generated from the real data set.

$X(j)$	0.8977	0.9548	0.9759	0.9876	1.0684	1.1541	1.4584	1.4714
$R(j)$	0	0	3	0	3	0	0	5

then, Weibull gamma distribution is appropriate for the data (set 1).

From that, we can get:

$$\mu^* = 0.2580, \qquad \sigma^* = 0.4454$$
$$\hat{\mu} = 0.2021 \qquad \hat{\sigma} = 0.3211$$

and the standard error of $\mu^*$and σ^* are written:

$$SE(\mu^*) = \sigma^*(var(\mu^*)\hat{}(1/2)) = 0.4454((1.5667)\hat{}(1/2)) = 0.5574.$$

$$SE(\sigma^*) = \sigma^*(var(\sigma^*)\hat{}(1/2)) = 0.4454((0.0090)\hat{}(1/2)) = 0.13362.$$

References

[1] Abd El-Baset A. Ahmed and Mohammed A. Fawzy. Recurrence relations for single moments of generalized order statistics from doubly truncated distribution, *Journal of Statistical Planning and Inference*, **117**, 241-249 (2003).

[2] Arnold, B. C., Balakrishnan, N. and Nagaraja, H. N., *A First Course in Order Statistics*, John Wiley & Sons, New York, (1992).

[3] Asgharzadeh, A., Point and interval estimation for a generalized logistic distribution under progressive Type-II censoring, *Communications in Statistics- Theory and Methods*, **35**, 1682-1702 (2006).

[4] Balakrishnan, N., Progressive Censoring Methodology: An Appraisal, Test., **16**, 211-296 (2007) (with discussion).

[5] Balakrishnan, N. and Aggarwala, R., Recurrence relations for single and product moments of order statistics from a generalized logistic distribution with applications to inference and generalizations to double truncation, Handbook of Statistics, **17**, 85-126 (1998).

[6] Balakrishnan, N. and Aggrawala, R. *Progressive Censoring-Theory, Methods and Applications*, Birkh$\bar{}$auser, Boston, (2000).

[7] Balakrishnan, N. and Asgharzadeh, A., Inference for scaled half-logistic distribution based on progressively Type-II censored samples, *Communications in Statistics-Theory and Methods*, **34**, 73-87 (2005).

[8] Balakrishnan, N. and Basak, I., Robust estimation under progressive censoring, *Computational Statistics* & *Data Analysis*, **44**, 1-2, 349-376 (2003).

[9] Balakrishnan, N. and Rao, C. R., Some efficiency properties of best linear unbiased estimators, *Journal of Statistical Planning and Inference*, **113**, 551-555 (2003).

[10] Balakrishnan, N. and Sandhu, R. A., A simple simulational algorithm for for generating progressive Type-II censored samples, *The American Statistician*, **49**, 229-230 (1995).

[11] Balakrishnan, N., AL-Hussaini, E. K. and Saleh, H. M., Recurrence relations for moments of progressively censored order statistics from logistic distribution with applications to inference, *Journal of Statistical Planning and Inference,* **141**, 17-30 (2011).
[12] Balakrishnan, N., Childs, A. and Chandraseka, B., An efficient computational method for moments of order statistics under progressive censoring, Statistics, *Probability Letters*, **60**, 359-365 (2002).
[13] Balakrishnan, N., Kannan, N., Lin, C. T., and Wu, S. J. S., Inference for the extreme value distribution under under progressive type-II censoring, J*ournal of Statistical Computation* & *Simulation*, **74**, 25-45 (2004).
[14] David, H. A. and Nagaraja, H. N., *Order Statistics*, Third Edition, John Wiley & Sons, New York, (2003).
[15] Fernandez, A. J., On estimating exponential parameters with general type-II progressively censoring, *Journal of Statistical Planning and Inference*, **121**, 135-147 (2004).
[16] Gibbons, J. D. and Chakraborti, S. *Nonparametric Statistical Inference*, Fourth edition, Marcel Dekker, New York, (2003).
[17] Hassan, M. S. and Sultan, K. S., On progressive censoring samples and their applications, (2005).
[18] Lindely, D. V., *Introduction To Probability and Statistics From a Bayesian Viewpoint*, Camberidge, (1969).
[19] Mahmoud M. A. W. and Mohie El-Din, M.,). The Inverted Linear Exponential Distribution as a Life Time Distribution, *The Egyptian Statistical Journal,* **50**, 13-20 (2006).
[20] Malik, H. J., Balakrishnan, N. and Ahmed, S. E., Recurrence relations and identities for moments of order statistics,I:Arbitrary continous distribution, *Communications in Statistics-Theory and Methods*, **17**, 2623-2655 (1998).
[21] Raqab, M. Z., Asgharzadeh, A. and Valiollahi, R., Prediction for Pareto distribution based on progressively Type-II censored samples, *Computational Statistics* & *Data Analysis,* **54**, 1732-1743 (2010).