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A New Two-Parameter Lifetime Distribution with Bathtub, Up-Bathtub or Increasing Failure Rate Function

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Abstract: It is a common situation that the failure rate function has a bathtub shape for many mechanical and electronic components. A simple model based on the median of tree /or four identical independent random variables drown from the well known power distribution is presented for modeling this type of data. The failure rate can also be upside-down bathtub shaped or increasing. Most properties of the proposed distribution are investigated. Estimation procedures are introduced as well as a graphical approach via probability plot. An application to a real data set is presented; and a simulation study is provided. Finally, some concluding remarks are presented.

Keywords: Order statistics; Failure Rate; Quantile; Shanon measure of entropy; Simulation

1 Introduction

Families of distributions for the median X of an independent random sample, $Y_1, \cdots, Y_N$, drawn from an arbitrary lifetime distribution with survival function $S_Y(\cdot; \Theta)$, are introduced by Abd-Elrahman [1]. He shows that its survival function, $S_X(\cdot; \Theta)$, has the following form:

$$S_X(x; \Theta) = \sum_{k=0}^{m} C_{m,k} S_Y(x; \Theta)^{2m+1-k}, \qquad m = 0, 1, 2, \cdots \tag{1}$$

where $m = \text{Int}\left(\frac{N-1}{2}\right)$, $\text{Int}(\cdot)$ is the *Elemental Intrinsic Function*, and the $m+1$ coefficients, $C_{m,k}$, are given by

$$C_{m,k} = \frac{(-1)^{m-k} (m+1-k) \binom{m+1}{m+1-k} \binom{2m+1}{m+1}}{2m+1-k}, \qquad k = 0, 1, \cdots, m.$$

Using (1) with $N = 3$ or 4 and $S_Y(\cdot; \Theta)$ is the exponential survival function, he obtained a new distribution. He gave this new lifetime distribution a name, the *Bilal*(θ) distribution.

Unfortunately, the failure rate function related to the *Bilal*(θ) distribution is always monotonically increasing with finite limit. In this paper, a simple model based on the median of tree /or four identical independent random variables drown from the well known power distribution is presented. We show that its failure rate function can have a bathtub shaped. The failure rate can also be upside-down bathtub shaped or increasing. It may be very desirable that our two-parameter model can have such a flexible failure rate function. The layout of this paper is organized as follows:

In Section 2, the proposed distribution and most of its properties are given, which are: the mode, median, mean, the expected value, variance, the rth moments, the coefficient of variation, Kurtosis coefficient, skewness, a closed form of the qth quintile x_q, the Shanon measure of entropy, the Fisher information measure about λ, the lower limit of Cramér–Rao inequality for the parameter λ. In Section 3, some properties of the failure rate function are presented.

Estimation procedures are presented in Section 4. In which, following Balakrishnan *et. al.* [2], we proved the existence and uniqueness of the maximum likelihood estimate of the parameter λ. The $(1-\alpha)100\%$ asymptotic confidence interval for $\hat{\lambda}^{-1}$ is also given. We showed that, the moment estimate of λ, when the parameter β is assumed to be known, is exists in a simple closed form. Its efficiency with respect to (w.r.t.) the *minimum variance unbiased estimate* (MVUE)

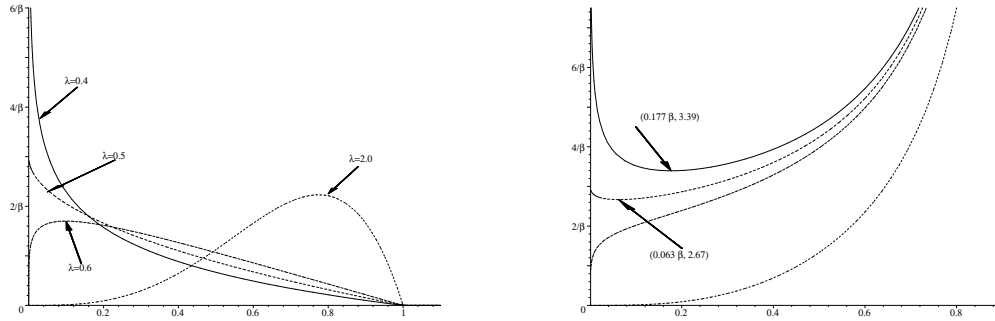* Corresponding author e-mail: ayman275@aun.edu.eg

Fig. 1: Profile of the pdf of the proposed distribution for λ=0.4 (solid), 0.5 (dot), 0.6 (dash), and 2 (dash dot), respectively, w.r.t. $\frac{x}{\beta}$ (I); and its corresponding failure rate function (II).

of λ is equal to 99.9165 %. Estimations of both of the two parameters are studied. Graphical approach is also provided via probability plot.

For illustrative purposes, however, in Section 5, an application to a real data set are considered. Our results are compared with four recent studies to this data.

Section 6 is presented for comparing the performance of the resulting estimators via simulation experiments. Finally we gave some concluding remarks.

2 The proposed model and some properties of the density function

It is well known that, for $\beta, \lambda > 0$, the survival distribution function related to the power distribution has the following form

$$S_Y(y; \beta, \lambda) = \begin{cases} 1 - \left(\frac{y}{\beta}\right)^{\lambda} & \text{if } 0 < y < \beta, \\ 1 & \text{if } x > \beta. \end{cases} \tag{2}$$

In view of (1), let $N = 3$ or 4, *i. e. m* = 1, and replacing $S_Y(\cdot; \Theta)$ by the power survival function, given by (2), we then obtained a new distribution. This distribution is refer to as the *MMHB*(β, λ) distribution. The name *MMHB* is refer to as the initials of my four sons' names. The probability density (pdf), cumulative distribution (cdf) and the failure rate functions of the *MMHB*(β, λ) distribution are, respectively, can be written as

$$f_X(x; \beta, \lambda) = \frac{6\lambda}{\beta} \left(\frac{x}{\beta}\right)^{2\lambda-1} \left(1 - \left(\frac{x}{\beta}\right)^{\lambda}\right), \qquad 0 < x < \beta, (\beta, \lambda > 0), \tag{3}$$

$$F_X(x; \beta, \lambda) = \begin{cases} 1 - \left(1 - \left(\frac{x}{\beta}\right)^{\lambda}\right)^2 \left(1 + 2\left(\frac{x}{\beta}\right)^{\lambda}\right) = \left(\frac{x}{\beta}\right)^{2\lambda} \left(3 - 2\left(\frac{x}{\beta}\right)^{\lambda}\right) & \text{if } 0 < x < \beta, \\ 1 & \text{if } x > \beta. \end{cases} \tag{4}$$

$$H_X(x; \beta, \lambda) = \frac{6\lambda \left(\frac{x}{\beta}\right)^{2\lambda-1} [1 - (\frac{x}{\beta})^{\lambda}]}{\beta [1 - (\frac{x}{\beta})^{2\lambda}(3 - 2(\frac{x}{\beta})^{\lambda})]}, \qquad 0 < x < \beta. \tag{5}$$

Figure 1 depicts profile of the pdf of the proposed distribution; and its corresponding failure rate function. The shape properties of (3) follow from the following theorem.

Theorem 2.1. The pdf of the *MMHB*(β, λ) distribution is a decreasing function for $0 < \lambda \leq 1/2$; and unimodal for $\lambda > 1/2$ with mode at $x_0 = \beta \left(\frac{2\lambda-1}{3\lambda-1}\right)^{\frac{1}{\lambda}}$.

Proof. See the Appendix.

In view of (3), the rth moments of a random variable X, $X \sim MMHB(\beta, \lambda)$ exist for all $\beta > 0$, $\lambda > 0$; and it is then given by:

$$E(X^r) = 6\lambda\beta^r \int_0^1 y^{r+2\lambda-1}(1-y^\lambda)dy = 6\lambda\beta^r \left(\frac{1}{r+2\lambda} - \frac{1}{r+3\lambda}\right) = \frac{6\lambda^2\beta^r}{(r+2\lambda)(r+3\lambda)}. \tag{6}$$

It may be clear that, the expected value, variance; and the variation (CV), Kurtosis (KU) and Skewness (SK) coefficients, can then be easily obtained using (6); and they are respectively given by:

$$E[X] = \frac{6\beta\lambda^2}{(2\lambda+1)(1+3\lambda)}, \qquad Var(X) = \frac{3\beta^2\lambda^2(1+10\lambda+13\lambda^2)}{(1+\lambda)(2+3\lambda)(1+2\lambda)^2(1+3\lambda)^2}, \tag{7}$$

$$CV = \sqrt{\frac{Var(X)}{(E[X])^2}} = \frac{1}{6\lambda}\sqrt{\frac{3(1+10\lambda+13\lambda^2)}{(1+\lambda)(2+3\lambda)}},$$

$$KU = \frac{E\left[(X-E(X)^4)\right]}{E[(X-E(X))^2]}$$

$$= \frac{(1+\lambda)(2+3\lambda)(6+139\lambda+1187\lambda^2+4308\lambda^3+5852\lambda^4+7963\lambda^5+12891\lambda^6+6534\lambda^7)}{3\lambda^2(2+\lambda)(4+3\lambda)(3+2\lambda)(1+10\lambda+13\lambda^2)^2},$$

$$SK = \frac{E\left[(X-E(X)^3)\right]}{E[(X-E(X))^{\frac{3}{2}}]} = (1-\lambda)Q(\lambda)$$

$$= 2(1-\lambda)\frac{[2+35\lambda+185\lambda^2+210\lambda^3][3(1+\lambda)(2+3\lambda)]^{\frac{1}{2}}}{9\lambda(3+2\lambda)(1+10\lambda+13\lambda^2)^{\frac{3}{2}}}\begin{cases} +ve & \text{if } 0 < \lambda < 1, \\ 0 & \text{if } \lambda = 1, \\ -ve & \text{if } \lambda > 1, \end{cases}$$

where for $\lambda > 0$, $Q(\lambda) > 0$. The coefficients CV, KU and SK depends only on the parameter λ.

If $Y = \ln(X)$, $X \sim MMHB(\beta, \lambda)$, then we have

$$E(Y) = \ln(\beta) - \frac{5}{6\lambda}. \tag{8}$$

Therefore, it may be easy to show that, for $r=2,3$ and 4, the rth moments of $Y = \ln(X)$ around $E(Y)$, μ'_r, are given by:

$$\mu'_2 = Var(Y) = \frac{13}{36\lambda^2}, \qquad \mu'_3 = -\frac{35}{108\lambda^3}, \qquad \mu'_4 = \frac{121}{144\lambda^4}. \tag{9}$$

Hence, the KU of Y is equal to $\frac{1089}{1696} = 0.4438$; while the Skewness of Y is equal to $-\frac{70}{169\sqrt{13}} = -1.4934$.

The qth quintile, x_q, is an important quantity, specially for generating random varieties using the inverse transformation method. Let $a_q = \frac{1}{3}\arctan(\frac{2\sqrt{q(1-q)}}{2q-1})$, then the qth quintile can be obtained from (4) as

$$x_q = \beta(\gamma(q))^{\frac{1}{\lambda}}, \tag{10}$$

where

$$\gamma(q) = \begin{cases} \frac{1}{2} - sin(a_q + \frac{\pi}{6}) & \text{if } 0 < q < \frac{1}{2}, \\ \frac{1}{2} & \text{if } q = \frac{1}{2}, \\ \frac{1}{2} + cos(a_q + \frac{\pi}{3}) & \text{if } \frac{1}{2} < q < 1. \end{cases}$$

Hence, the median of a random variable X, $X \sim MMHB(\beta, \lambda)$, is given by $X_{\text{median}} = \beta 2^{\frac{-1}{\lambda}}$.

Using (3), the Shanon measure of entropy, it may be easy to show that it has the following form:

$$E[-\ln(f_X)] = \frac{5}{2} - \ln\left(\frac{6\lambda}{\beta}\right) - \frac{5}{6\lambda}.$$
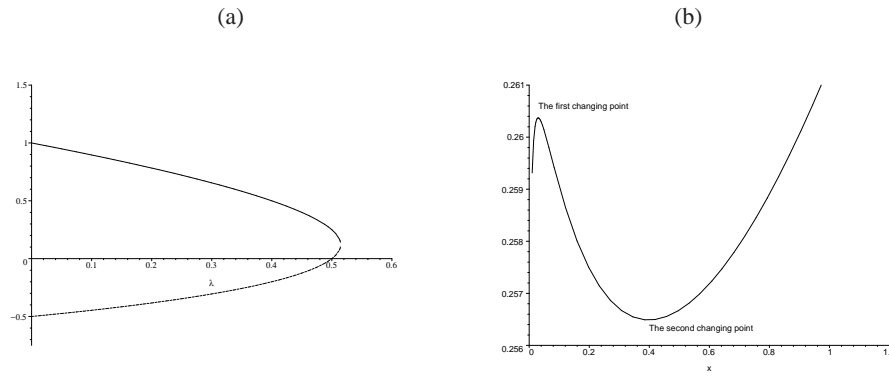
(a) (b)

Fig. 2: (a) Graphs of $\xi_1^\star$ (solid) and $\xi_0^\star$ (dash), w.r.t. the parameter λ. (b) A graph of the failure rate function with $\beta = 10$ and $\lambda = 0.51$.

Let $x_1, x_2, \cdots x_n$ be an independent random sample of size n, taken from $MMHB(\beta, \lambda)$, then the Fisher information measure about λ is given by

$$J(\lambda; n) = -nE\left[\frac{\partial^2 ln(f_X)}{\partial \lambda^2}\right] = \frac{n(12\,Zeta(3) - 12.5)}{\lambda^2} = \frac{0.519566n}{\lambda^2}, \qquad (Zeta(3) = \sum_{i=1}^{\infty} i^{-3}). \tag{11}$$

3 Some properties of the failure rate function

In view of (5), it may be easy to show that

$$t_1 = \lim_{x \to 0^+} H_X(x; \beta, \lambda) = \frac{6\lambda}{\beta} \lim_{x \to 0^+} \left(\frac{x}{\beta}\right)^{2\lambda - 1} = \begin{cases} \infty & \text{if } 0 < \lambda < \frac{1}{2}, \\ \frac{3}{\beta} & \text{if } \lambda = \frac{1}{2}, \\ 0 & \text{if } \lambda > \frac{1}{2}, \end{cases} \tag{12}$$

$$t_2 = \lim_{x \to \beta^-} H_X(x; \beta, \lambda) = \infty \; \forall \lambda > 0, \beta > 0. \tag{13}$$

The shape properties of (5) follow from the following theorem.

Theorem 3.1. The $MMHB(\beta, \lambda)$ distribution has a failure rate function with three different shapes i) bathtub shaped for $0 < \lambda \leq \frac{1}{2}$ with a changing point at $x_1^\star = \beta \left[\frac{(1-\lambda) + \sqrt{9 - 18\lambda + !\lambda^2}}{4}\right]^{\frac{1}{\lambda}}$; ii) upside-down bathtub shaped for $\frac{1}{2} < \lambda < \lambda^\diamond$, $\lambda^\diamond = 9 - 6\sqrt{2} = 0.514718628$, with two changing points at $x_0^\star$ and $x_1^\star$, where $x_{0,1}^\star = \beta \left[\frac{(1-\lambda) \mp \sqrt{9 - 18\lambda + !\lambda^2}}{4}\right]^{\frac{1}{\lambda}}$; and iii) an increasing function for $\lambda \geq \lambda^\diamond$.

Proof. The proof is provided in the Appendix.

In order to give another check for this theorem, one may compere Figure 2-(a), where a profile of the functions $\xi_{0,1}^\star$, which are as the as the definition of $x_{0,1}^\star$, but without both of the power $\frac{1}{\lambda}$ and the coefficient β each, are depicted. In this figure, negative values of $\xi_0^\star$ corresponds to non-real values for $x_0^\star$. Furthermore, Figure **??**-(b) depicts graph of the failure rate function with $\beta = 10$ and $\lambda = 0.51$, i. e., $\frac{1}{2} < \lambda < \lambda^\diamond$.

4 Estimation

Let $X_1, X_2, \cdots, X_n$ be independent random sample with observed value $\mathbf{x} = (x_1, x_2, \cdots, x_n)$ from a $MMHB(\beta, \lambda)$ distribution.

4.1 When β is known

–Moment estimate: The unbiased moment estimate of the parameter λ^{-1}, $\hat{\lambda}_{\beta M}^{-1}$, is given by

$$\hat{\lambda}_{\beta M}^{-1} = \frac{6}{5n} \sum_{i=1}^{n} \ln(\frac{\beta}{x_i}). \tag{14}$$

We find that $\hat{\lambda}_{\beta M}^{-1}$ is very closed to the MVUE for λ^{-1}, this is due to: i) It is an unbiased estimate for λ^{-1}, ii) $Var(\hat{\lambda}_M^{-1}) = \frac{0.52}{n} \lambda^{-2}$; and using (11), its efficiency w.r.t. the lower bound of, *Cramer-Roa Inequality*, the MVUE of λ^{-1} is 99.9165621%.

–Maximum likelihood estimate: Let $y_i = \ln(\frac{\beta}{x_i}), i = 1, 2, \cdots, n$, it follows that, the normal equation of λ is given by

$$l_n(\lambda) = \frac{n}{\lambda} + \sum_{i=1}^{n} \left\{ -2y_i + \frac{y_i e^{-\lambda y_i}}{1 - e^{-\lambda y_i}} \right\} = 0. \tag{15}$$

The maximum likelihood (ML) estimate for λ, $\hat{\lambda}_{\beta ML}$, exists and it is unique (see the Appendix). The solution of (15) gives $\hat{\lambda}_{\beta ML}$ which can be obtained numerically using, e.g., the bisection method. It follows from (11) that, the asymptotic $(1-\lambda)100\%$ confidence interval, ACI, of $\hat{\lambda}_{\beta ML}$ is given by

$$\hat{\lambda}_{\beta ML}^{-1} \pm Z_{\frac{\alpha}{2}} \frac{0.7208 \hat{\lambda}_{\beta ML}^{-1}}{\sqrt{n}}, \tag{16}$$

where $Z_{\frac{\alpha}{2}}$ is the quantile $(1-\frac{\alpha}{2})$ of the standard normal distribution.

4.2 When λ is known

In the following, we introduce three unbiased estimators for the parameter β with their corresponding variances. The first estimator related to the moment method. In view of (7), the unbiased moment estimate of the parameter β, $\hat{\beta}_{\lambda M}$; and its corresponding variance are given by

$$\breve{\beta}_\lambda = \frac{(1+3\lambda)(1+2\lambda)}{6n\lambda^2} \sum_{i=1}^{n} x_i, \qquad Var(\breve{\beta}_\lambda) = \frac{\beta^2 (1+10\lambda+13\lambda^2)}{12n\lambda^2 (\lambda+1)(2+3\lambda)}. \tag{17}$$

The other two unbiased estimators are obtained together with their variances as follows:

Recall the indicator function which is define as $I_{\{x_{(n)}<\beta\}} = 1$ or 0 according as $x_{(n)} < \beta$ or $x_{(n)} \geq \beta$, where $x_{(n)}$ is the largest sample observation, it follows from (3) that, the likelihood function of the parameter β is given by

$$L_n(\beta) = \frac{6^n \lambda^n}{\beta^n} \prod_{i=1}^{n} \left\{ (\frac{x_i}{\beta})^{2\lambda-1} (1-(\frac{x_i}{\beta})^\lambda) \right\} I_{\{x_{(n)}<\beta\}}. \tag{18}$$

Following Gibbons [3], see Johnson *et al.* [4] Page 289 for reference, we investigate three estimators for the parameter β. The first one is biased while the other two estimators are not.

1.The maximum likelihood estimator $\tilde{\beta}_\lambda = X_{(n)}$ with moments

$$E[\tilde{\beta}_\lambda] = \eta_1(n, \lambda)\beta, \qquad Var(\tilde{\beta}_\lambda) = \beta^2 (\eta_2(n, \lambda) - \eta_1^2(n, \lambda)), \tag{19}$$

where

$$\eta_k(n, \lambda) = 1 - k3^n \sum_{i=0}^{n} \frac{(-1)^i (2/3)^i \binom{n}{i}}{k + \lambda(2n+i)}, \qquad k = 1, 2.$$

By (18), since $0 < \eta_1(n, \lambda) < 1$ (compare Figure 3), we may notice that $\tilde{\beta}_{\lambda ML}$ underestimates β.
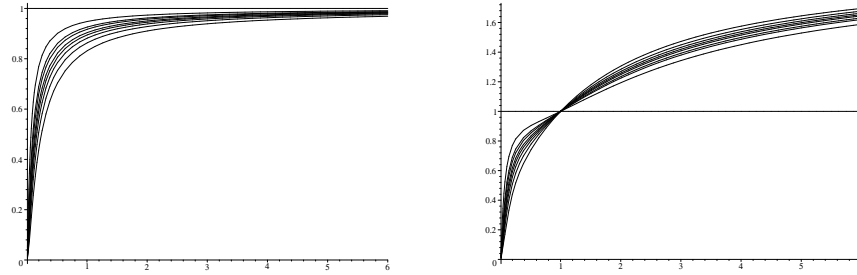
Fig. 3: Plots of the functions $\eta_1(n, \lambda)$ (left) and $\eta_1^\star(n, \lambda)$ (right) for n=10, 15, 20, 25, 30, 40, 50, and 100; and values of the parameter λ in the range (0.01,6.0).
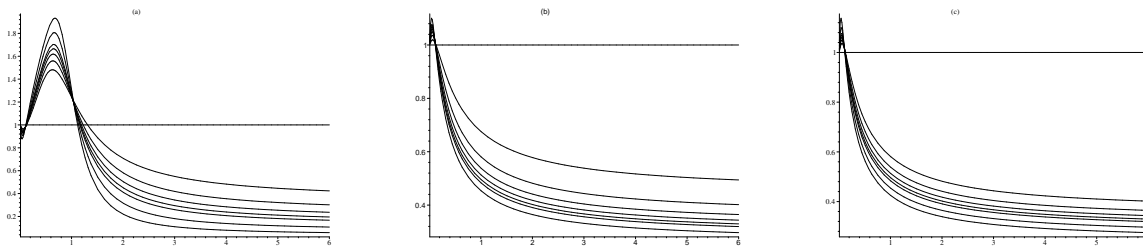

Fig. 4: The relative efficiencies $e_1(n, \lambda)$ (a), $e_2(n, \lambda)$ (b), and $e_3(n, \lambda)$ (c), for n=10, 15, 20, 25, 30, 40, 50, and 100; and $\lambda \in (0.01, 6.0)$.

2.The unbiased estimator $\hat{\beta}_\lambda$; and its corresponding moments are given by

$$\hat{\beta}_\lambda = \frac{x_{(n)}}{\eta_1(n, \lambda)}, \qquad Var(\hat{\beta}_\lambda) = \beta^2 \left[\frac{\eta_2(n, \lambda)}{\eta_1^2(n, \lambda)} - 1 \right]. \tag{20}$$

3.Other unbiased estimator $\beta^\star_\lambda$; and its corresponding moments are given by

$$\beta^\star_\lambda = \frac{x_{(1)} + x_{(n)}}{\eta_1^\star(n, \lambda)}, \qquad Var(\beta^\star_\lambda) = \beta^2 \left[\frac{\eta_2^\star(n, \lambda) - \iota_1^2(n, \lambda) - \eta_1^2(n, \lambda)}{[\eta_1^\star(n, \lambda)]^2} \right]. \tag{21}$$

where $x_{(1)}$ is the smallest sample observation; and for $k = 1, 2, \eta_k(n, \lambda)$ is as given in (19),

$$\iota_k(n, \lambda) = 1 + k \sum_{j=1}^n \binom{n}{j} \left\{ \sum_{i=0}^j \frac{(-1)^{j+i} 2^i 3^{j-i} \binom{j}{i}}{k + \lambda (2j + i)} \right\}$$

and

$$\eta_k^\star(n, \lambda) = \iota_k(n, \lambda) + \eta_k(n, \lambda).$$

Figure 3 depicts graphes of the function $\eta_1(n, \lambda)$ (left) and $\eta_1^\star(n, \lambda)$ (right), for n= 10,15, 20, 25, 30, 40, 50, 100 and values of the parameter λ in the range (0.01, 6). Thesis figure may indicate that, $\eta_1(n, \lambda)$ and $\iota_1(n, \lambda)$ are bounded functions for $n>1$ and $\lambda > 0$; and

$$0 < \eta_1(n, \lambda), \iota_1(n, \lambda) < 1.$$

For $\lambda \in (0.01, 6)$; and n= 10, 15, 20, 25, 30, 40, 50 and 100, Figure 4 (a), (b) and (c) depict the relative efficiency $e_1(n, \lambda)$, $e_2(n, \lambda)$ and $e_3(n, \lambda)$, respectively, where

$$e_1(n, \lambda) = \frac{Var(\breve{\beta}_\lambda)}{Var(\hat{\beta}_\lambda)}, \qquad e_2(n, \lambda) = \frac{Var(\breve{\beta}_\lambda)}{Var(\beta^\star_\lambda)} \quad \text{and} \quad e_3(n, \lambda) = \frac{Var(\hat{\beta}_\lambda)}{Var(\beta^\star_\lambda)}.$$
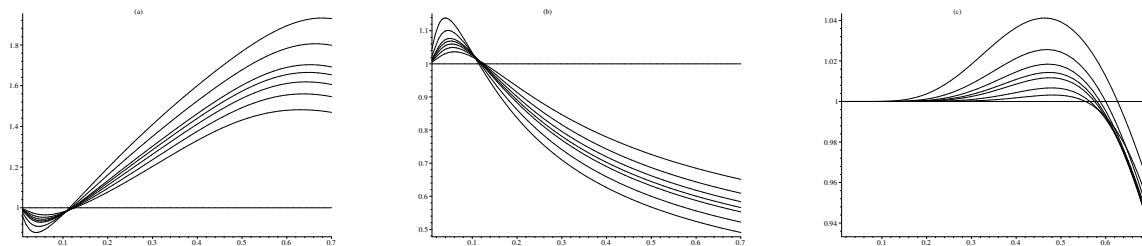
Fig. 5: Same as in Figure 4 but for $\lambda \in (0.01, 0.7)$.

Figure 5 is as the same as in Figure 4 but for $\lambda \in (0.01, 0.7)$.

In view of Figures 4 and 5, we conclude that the value of the parameter λ determines which estimator is more efficient than the other two estimators. However, these figures may indicate that, the estimator which has the lower variance is: $\breve{\beta}_\lambda$ for $\lambda \in (0.01, 0.12)$, $\beta_\lambda^\star$ for $\lambda \in (0.12, 0.61)$ and $\hat{\beta}_\lambda$ for $\lambda \in (0.61, 6)$.

4.3 When both β and λ are unknown

In this case, however, in order to find estimates for both β and λ, initial value for each parameter is needed. We try two different approaches.

Graphical approach:

$MMHB(\beta, \lambda)$ probability plotting (MMHBPP) can be achieved simply by plotting $\ln(x)$ on $z = -\ln[\gamma(F_n(x))]$, where $F_n(\cdot)$ is the empirical distribution function; and $\gamma(\cdot)$ is as given in (10). This results in a least square line $\delta_1 + \delta_2 z = \ln(x)$. The corresponding estimates for β and λ are then respectively given by $\hat{\beta}_G = e^{\delta_1}$ and $\hat{\lambda}_G = \frac{1}{\delta_2}$, which can be used as initial values for much better estimators.

Iterative methods:

Denote V_1 the sample variance of the transformed sample $(\ln(x_1), \ln(x_1), \cdots \ln(x_n))$. Since the variance of a random variable $\ln(X)$, $X \sim MMHB(\beta, \lambda)$, which is given in (9), does not depend on the parameter β then an initial value for an estimator of the parameter λ can be calculated as

$$\lambda_{(0)} = \frac{\sqrt{13}}{6\sqrt{nV_1}}. \tag{22}$$

Once we obtain $\lambda_{(0)}$, three iterative methods can be used to obtain estimators for both of the unknown parameters β and λ. Let $\breve{\lambda}_{(0)} = \hat{\lambda}_{(0)} = \lambda_{(0)}^\star = \lambda_{(0)}$. Then, for $s = 0, 1, 2, \cdots, N$, N is the maximum number of iteration, say 1000, it follows from (14), (17), (20) and (21) that, these proposed methods are as follow:

– Method I:

$$\breve{\beta}_{(s)} = \frac{(1 + 2\breve{\lambda}_{(s)})(1 + 3\breve{\lambda}_{(s)})}{6n\breve{\lambda}_{(s)}^2} \sum_{i=1}^n x_i, \qquad \breve{\lambda}_{(s+1)} = \left\{ \frac{6}{5n} \sum_{i=1}^n \ln(\frac{\breve{\beta}_{(s)}}{x_i}) \right\}^{-1}. \tag{23}$$

– Method II:

$$\hat{\beta}_{(s)} = \frac{x_{(n)}}{\eta_1(n, \hat{\lambda}_{(s)})}, \quad \hat{\lambda}_{(s+1)} = \left\{ \frac{6}{5n} \sum_{i=1}^n \ln(\frac{\hat{\beta}_{(s)}}{x_i}) \right\}^{-1}. \tag{24}$$
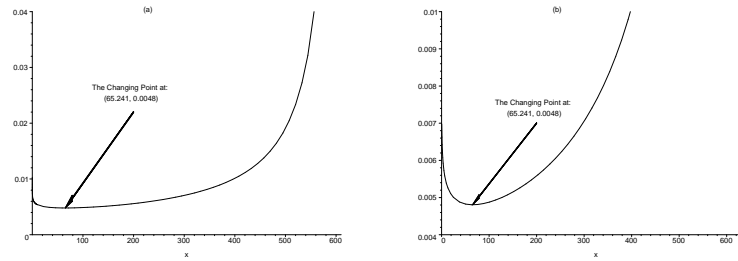
Fig. 6: Estimated failure rate function corresponds to the data set .

– Method III:

$$\beta^{\star}_{(s)} = \frac{x_{(1)} + x_{(n)}}{[\iota_1(n, \lambda^{\star}_{(s)}) + \eta_1(n, \lambda^{\star}_{(s)})]}, \qquad \lambda^{\star}_{(s+1)} = \left\{ \frac{6}{5n} \sum_{i=1}^{n} \ln(\frac{\beta^{\star}_{(s)}}{x_i}) \right\}^{-1}. \tag{25}$$

Each of these iteration methods will be refined until its accuracy is reached according to the stoping rule $|\lambda^{\iota}_{(s+1)} - \lambda^{\iota}_{(s)}| \leq 1.2 \times 10^{-6}$, $s = 1, 2, 3, \cdots$, where, $\lambda^{\iota}_{(\cdot)}$ stands for $\breve{\lambda}_{(\cdot)}$, $\hat{\lambda}_{(\cdot)}$ or $\lambda^{\star}_{(\cdot)}$.

5 Data analysis

For illustration and comparison reasons, we use a real data set that is used in many recent studies which consists of 18 lifetime failure observations of an electronic device. This data set was first analyzed by Wang [5], Xie *et. al.* [6] and then by Rezaei *et. al.* [7]. This data are as follows [Wang [5], Page 309]: 5, 11, 21, 31, 46, 75, 98, 122, 145, 165, 196, 224, 245, 293, 321, 330, 350, 420.

For this data set, estimates of the parameters of the proposed model according to MMHBPP, Methods I, II and III are obtained as described in Section 4.3 and depicted in Table 1 together with some corresponding measures of goodness of fit, namely, the Kolmogorov-Smirnov (K-S) statistics with their corresponding *p*-values, Log Likelihood and the Akaike Information Criterion (AIC). The graphical approach, MMHBPP, results in $\hat{\lambda}_G$=0.4525 which may be used as an initial value for the parameter λ instead of $\lambda_{(0)}$. For this data $\lambda_{(0)}$ is equal to 0.4663. We use each of $\lambda_{(0)}$ and $\hat{\lambda}_G$ as an initial value each, but the corresponding estimators remain the same. Note that, according to the results of Theorem 3.1., since the estimated values of the parameter λ, using any of Method I, II, III or MMHBPP are inside the interval $(0, \frac{1}{2}]$, this data set may then has a bathtub shaped failure rate function.

We compare our results with the four recent studies:

Xie *et. al.* [6] presented a modified Weibull extension (MWE) model with the survival function

$$S(x) = exp\left[\alpha \lambda (1 - e^{(x/\alpha)^{\beta}})\right], \ x > 0, (\alpha, \beta, \lambda > 0).$$

Bebbington *et. al.* [8] presented a flexible Weibull extension (FWE) model , having aging property, base on a generalization of Weibull model, with the survival function

$$S(x) = \exp[-\exp(\alpha x - \beta/x)], \qquad x > 0, (\alpha, \beta > 0).$$

Gupta *et. al.* [9] presented a complete Bayesian analysis of the Weibull extension (BWE) model, using Markov chain Monte Carlo simulation. The survival function for this model is

$$S(x) = \exp\left[-\lambda \alpha^{-1/\beta}(e^{\alpha x^{\beta}} - 1)\right], \ x > 0, (\alpha, \beta, \lambda > 0).$$

Rezaei *et. al.* [7] presented the Exponential Truncated Poisson Maximum (ETPM) model. The ETPM model is obtained by mixing exponential and truncated Poisson maximum distribution, with the survival function

$$S(x) = 1 - \frac{\exp(-\lambda e^{-\beta x}) - e^{-\lambda}}{(1 - e^{-\lambda})}, \ x > 0, (\beta, \lambda > 0).$$

Table 1: Parameter estimators and some measures of goodness of fit for the data set.

Model		$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	K-S	p-value	Log Likelihood	AIC
Other	MWE	287.51	0.93	0.0035	0.1088	0.9765	-109.311	222.6225
	FWE	0.003	24.72	-	0.3159	0.0428	-112.328	228.6559
	BWE	0.76	0.32	0.007	0.1060	0.9821	-110.666	227.3315
	ETPM	-	0.0074	1.1251	0.1035	0.9860	-110.322	224.6449
Propoed	MMHB (M I)	-	607.0871	0.47	0.0768	0.9999	-109.298	222.5961
	MMHB (M II)	-	615.5529	0.4664	0.0804	0.9998	-109.353	222.7057
	MMHB (M III)	-	590.2820	0.4776	0.0812	0.9998	-109.194	222.3875
	MMHB (PP)	-	636.9531	0.4525	0.0867	0.9993	-109.498	222.9962

The resulting estimators and their corresponding goodness of fit measures related to each of these four models are as in Table 3, on Page 1756 of Rezaei *et al.* [7]. These results are also included in Table 1. Although, Table 1 shows that, there is no significant differences between AICs or K-S's, but this may indicate that the fit of our proposed model to this data set is comparable with the corresponding fits obtained using the other four models. Furthermore, the properties of our proposed model may help in estimating the changing point, $x_1^\star$, of the estimated failure rate function. For this data set, based on Method I, on applying Theorem 3.1., given in Section 5, we may have $x_1^\star = 65.241$. Therefore, the coordinates of the corresponding estimated value of the changing point is (65.241, 0.0048). This changing point occurs at the 0.2824 quintile. On the other hand, it follows from (16) that, the 95 % ACI of $\check{\lambda}$ for this data set is (0.353, 0.7047). Figure 6 (a) depicts graphs of the fitted failure rate function corresponds to this data set using Method I. Figure 6 (b) is as the same as in Figure 6 (a), but in the domain (0.04, 0.01). This is to gain some focus about the changing point $x_1^\star$.

6 Simulation experiments

To evaluate the performance of the estimators of the parameters of the proposed distribution, we designed Monte Carlo experiments. In each experiment, for different values of the population parameters and sample sizes, 1000 pseudo–random samples have been generated according to Equation (10) as: $X_j = \beta \left[\gamma(U_j) \right]^{1/\lambda}$, $j = 1, 2, \cdots, n$, where $U_1, U_2, \cdots, U_n$ are n identical independent random observations from the standard Uniform distribution, using the *IMSL* [10] routine DRNUN.

Forty five combinations of the parameters were considered: n= 20, 30, 100, β = 1, 2, 3, λ = 0.2, 0.4, 0.6, 1 and 2 . For each parameters combination, a generated sample is obtained. Based on this sample, the estimators of the parameter β and λ are computed according to Methods I, II and III as described in Section 4. As an estimated risk of the parameter ξ, $\xi = \beta$ or λ, the squared deviations $\left(\hat{\xi} - \xi \right)^2$, can then be obtained and stored. For 1000 repetitions, as estimated risks of the different estimators the root mean squared errors (RMSEs) are computed as the squared root of the average of their corresponding squared deviations. Table 2 displays the estimated risks. For Method I and II, non of the 45,000 generated samples, of course, plus the real data sample has been failed. While Method III fails N_f times. This is because the estimated value of β is less than $x_{(n)}$ for some few cases of the generated samples. We notice that, this number, N_f, increases specially when $\lambda > 1$. This is expected, since, compare Figure 3, $\eta_1^\star(n, \lambda) \geq 1$ for all $\lambda \geq 1$ and $n > 1$, then $\beta^\star$ under estimate the parameter β. Therefore, in Table 3, only the estimated risks related to Methods I and II are displayed.

7 Concluding remarks

We have introduced a new two-parameters life time distribution. We show that its failure rate function can have a bathtub shaped. The failure rate can also be upside-down bathtub shaped or increasing.

This proposed distribution is easy to be involved in statistical software libraries since its reliability function is based on the well known power function. We provide different estimators for the two unknown parameters. Comparing the resulting estimators in Tables 2 and 3 we may notice that, the parameter λ is independent of the parameter β. The parameter λ provides three different shapes for the failure rate function. For the bathtub (or upside-down bathtub) shape failure rate, the changing point(s) of the failure rate function can be easily calculated. The parameter β controls the died line of the hole random phenomena under studying. We derived most of the important properties of the new distribution and closed-form expressions for its moments are also obtained. In view of Table 1, the proposed model fit of the real data set was comparable to that obtained using the other four recent models which appear in the literatures. Due to the flexibility of the failure rate function of our proposed model, we hope that our proposed model can be applied for different practical random phenomenons.

Table 2: RMSE's of the resulting estimators when both β and λ are unknown.

β	λ	n	$\breve{\beta}$	$\hat{\beta}$	$\beta^\star$	N_f
1	0.2	20	0.152806	0.129871	0.129949	-
		30	0.106661	0.081168	0.081187	-
		120	0.025057	0.018581	0.018581	-
	0.4	20	0.056232	0.036956	0.037763	-
		30	0.039202	0.022625	0.022948	-
		120	0.009165	0.004859	0.004876	-
	0.6	20	0.031186	0.017154	0.019669	3
		30	0.021628	0.010467	0.011629	2
		120	0.005092	0.002195	0.002301	-
2	0.2	20	0.611224	0.519482	0.519795	-
		30	0.426645	0.324674	0.324750	-
		120	0.100229	0.074324	0.074324	-
	0.4	20	0.224928	0.147824	0.151054	-
		30	0.156807	0.090498	0.091793	-
		120	0.036660	0.019437	0.019504	-
	0.6	20	0.124744	0.068615	0.078675	3
		30	0.086513	0.041868	0.046516	2
		120	0.020366	0.008779	0.009202	-
3	0.2	20	1.375253	1.168835	1.169538	-
		30	0.959951	0.730516	0.730687	-
		120	0.225514	0.167229	0.167228	-
	0.4	20	0.506088	0.332604	0.339871	-
		30	0.352816	0.203621	0.206534	-
		120	0.082486	0.043734	0.043883	-
	0.6	20	0.280674	0.154383	0.177019	3
		30	0.194655	0.094204	0.104661	2
		120	0.045824	0.019753	0.020706	-
β	λ	n	$\breve{\lambda}$	$\hat{\lambda}$	$\lambda^\star$	N_f
1,2,	0.2	20	0.001937	0.001841	0.001847	-
		30	0.001284	0.001143	0.001144	-
		120	0.000249	0.000240	0.000240	-
or	0.4	20	0.008895	0.007214	0.007600	-
		30	0.005911	0.004504	0.004614	-
		120	0.001118	0.000957	0.000958	-
3	0.6	20	0.021934	0.015672	0.018659	-
		30	0.014571	0.009799	0.010705	-
		120	0.002741	0.002151	0.002182	-

Acknowledgments

Appendix

Proof of Theorem 2.1.
The first derivative of (3) w.r.t. x is given by

$$f_X'(x;\beta,\lambda) = g_p(x;\lambda)\frac{f_X(x;\beta,\lambda)}{x\left(1-\left(\frac{x}{\beta}\right)^\lambda\right)}, \qquad g_p(x;\lambda) = (2\lambda-1)-(3\lambda-1)\left(\frac{x}{\beta}\right)^\lambda. \tag{A1}$$

The function $g_p(x;\lambda)$ has exactly one root at $x_0 = \beta \left(\frac{2\lambda-1}{3\lambda-1} \right)^{\frac{1}{\lambda}}$. Table 4 may give some analysis about this function. Hence, the following three cases arise:

i) If $0 < \lambda < \frac{1}{3}$, then x_0 is an unbounded increasing function of λ. Since, $\lim_{\lambda \to 0^+} x_0 = e\beta$ and $\lim_{\lambda \to \frac{1}{3}^-} x_0 = \infty$, then the minimum value of x_0 tends to $e\beta$ which is grater than β. This implies that x_0 is outside the interval $(0, \beta)$. *e.g.* the value of x_0 at $\lambda = 0.2$ equal to $4.954\,\beta > \beta$. Therefore, $f_X(x;\beta, \lambda)$ is a decreasing function for $0 < \lambda < \frac{1}{3}$.

ii) For $\frac{1}{3} \leq \lambda \leq \frac{1}{2}$, the functions $g_p(x;\lambda)$ and $f'_X(x;\beta, \lambda)$ are negative for all $0 < x < \beta$. This implies that $f_X(x;\beta, \lambda)$ is a decreasing function of λ in the interval $[1/3, 1/2]$.

iii) Suppose now that $\lambda > \frac{1}{2}$, then $0 < \frac{2\lambda-1}{3\lambda-1} < 1$. Hence, $0 < x_0 < \beta$, which implies that, x_0 is a real root for $g_p(x)$ in the interval $0 < x < \beta$, where $g_p(x) > 0$ for $0 < x < x_0$ and $g_p(x) > 0$ for $x_0 < x < \beta$. Hence, $f_X(x;\beta, \lambda)$ has a unique mode at $x = x_0$.

Proof of Theorem 3.1.

The first derivative of the logarithm of (5) w.r.t. x is

$$(\ln[H_X(x;\beta, \lambda)])' = \frac{g_h(x;\lambda)}{x \left(1 - \left(\frac{x}{\beta} \right)^\lambda \right) \left[\left(\frac{x}{\beta} \right)^\lambda \right]}, \qquad g_h(x;\lambda) = (2\lambda - 1) + (\lambda - 1) \left(\frac{x}{\beta} \right)^\lambda + 2 \left(\frac{x}{\beta} \right)^{2\lambda}.$$

It may be clear that the function $g_h(x;\lambda)$ has exactly two roots at $x_0^\star$ and $x_1^\star$, which are given by

$$x_{0,1}^\star = \beta \left[\frac{(1-\lambda) \mp \sqrt{\lambda^2 - 18\lambda + 9}}{4} \right]^{\frac{1}{\lambda}}.$$

Hence, the following cases arise:

i) If $0 < \lambda < \frac{1}{2}$, since

$$\lambda^2 - 18\lambda + 9 = (\lambda^2 - 2\lambda + 1) - 16\lambda + 8 = (1-\lambda)^2 + 8(1 - 2\lambda) > (1-\lambda)^2.$$

Hence, $x_1^\star$ is a real number, while $x_0^\star$ is not. Note that,

$$\lim_{\lambda \to 0^+} x_0^\star = i\beta \lim_{\lambda \to 0^+} \left(\frac{1}{2} \right)^{\frac{1}{\lambda}} = 0, \quad i = \sqrt{-1}, \qquad \lim_{\lambda \to 0^+} x_1^\star = e^{-1}\beta < \beta, \qquad \lim_{\lambda \to \frac{1}{2}} x_0^\star = 0, \qquad \lim_{\lambda \to \frac{1}{2}} x_1^\star = \frac{\beta}{16} < \beta.$$

Therefore, the function $g_h(x;\lambda)$ is a decreing function of x for $0 < x < x_1^\star$ and it is an increasing function of x for $x_1^\star < x < \beta$. This goes in line with (12) and (13), in which $t_1 = t_2 = \infty$, for $0 < \lambda < \frac{1}{2}$. This implies that $H_X(x;\beta, \lambda)$ has a bathtub shaped failure rate for $0 < \lambda \leq \frac{1}{2}$ with a minimum value at $x = x_1^\star$.

Table 3: RMSE's of the resulting estimators $\check{\beta}$, $\hat{\beta}$, $\check{\lambda}$ and $\hat{\lambda}$.

		$\beta = 1$		$\beta = 2$		$\beta = 3$		$\beta = 1,2$ or 3	
λ	n	$\check{\beta}$	$\hat{\beta}$	$\check{\beta}$	$\hat{\beta}$	$\check{\beta}$	$\hat{\beta}$	$\check{\lambda}$	$\hat{\lambda}$
1	20	0.0143	0.0065	0.0572	0.0261	0.1288	0.0587	0.0686	0.0445
	30	0.0098	0.0039	0.0394	0.0156	0.0886	0.0352	0.0455	0.0279
	120	0.0023	0.0008	0.0093	0.0032	0.0211	0.0072	0.0084	0.0059
2	20	0.0044	0.0016	0.0179	0.0067	0.0404	0.0152	0.3081	0.1776
	30	0.0030	0.0010	0.0123	0.0040	0.0278	0.0090	0.2045	0.1113
	120	0.0007	0.0002	0.0030	0.0008	0.0067	0.0018	0.0384	0.0238
3	20	0.0021	0.0007	0.0086	0.0030	0.0195	0.0068	0.7249	0.3992
	30	0.0015	0.0004	0.0060	0.0018	0.0135	0.0040	0.4821	0.2503
	120	0.0003	< 0.00005	0.0014	0.0003	0.0033	0.0008	0.0911	0.0536

Table 4: Some analysis about the function $g_p(x;\lambda)$, given in $(A1)$.

$\lambda \rightarrow$	$0 < \lambda < \frac{1}{3}$	$\frac{1}{3} < \lambda < \frac{1}{2}$	$\frac{1}{2} < \lambda < \infty$
The sign of $(3\lambda - 1)$	$-ve$	$+ve$	$+ve$
The sign of $(2\lambda - 1)$	$-ve$	$-ve$	$+ve$
The sign of $g_p(x;\lambda)$	$-ve$	$-ve$	$+ve$
Type of x_0	Real	Non-real	Real
The value of x_0	$x_0 > \beta$	-	$0 < x_0 < \beta$
The maximum value of $f_X(x;\beta,\lambda)$	at $x = 0$	at $x = 0$	at $x = x_0$

ii) For $\frac{1}{2} < \lambda < 9 - 6\sqrt{2}$, then both $x_0^\star$ and $x_1^\star$ are real numbers and $0 < x_0^\star < x_1^\star < \beta$. But from (12) and (13), for $\lambda > \frac{1}{2}$, $t_1 = 0$ and $t_2 = \infty$ .This implies that $H_X(x;\beta,\lambda)$ has an upside-down bathtub shaped failure rate.

iii) When $\lambda = 9 - 6\sqrt{2}$, then $x_0^\star = x_1^\star = 0.01661\,\beta$. Therefore, using (12) and (13), $H_X(x;\beta,\lambda)$ is an increasing function of x.

iv) Suppose now that $\lambda > 9 - 6\sqrt{2}$, then $g_h(x;\lambda))$ is negative for all $x \in (0,\beta)$. Hence, $H_X(x;\beta,\lambda)$ is an increasing function of x, $\lambda \geq 9 - 6\sqrt{2}$.

Proof of existence and uniqueness of the ML estimate: It follows from (15) that

$$\frac{\partial l_n(\lambda)}{\partial \lambda} = -\frac{n}{\lambda^2} - \sum_{i=1}^{n} \frac{y_i^2 e^{-y_i \lambda}}{\left(1 - e^{-y_i \lambda}\right)^2} < 0.$$

This implies that the ML estimate, $\hat{\lambda}_{ML}$, for λ is unique. To insure that $\hat{\lambda}_{ML}$ exists, following Balakrishnan *et. al.* [2], we rewrite (15) as $h_1(\lambda) = h_2(\lambda)$, where

$$h_1(\lambda) = \frac{n}{\lambda} \quad \text{and} \quad h_2(\lambda) = \sum_{i=1}^{n} \left\{ 2y_i - \frac{y_i e^{-y_i \lambda}}{1 - e^{-y_i \lambda}} \right\}.$$

Since

$$\frac{\partial h_2(\lambda)}{\partial \lambda} = \sum_{i=1}^{n} \frac{y_i^2 e^{-\lambda y_i}}{\left(1 - e^{-\lambda y_i}\right)^2} > 0,$$

$\lim_{\lambda \to 0^+} h_2(\lambda) = -\infty$ and $\lim_{\lambda \to \infty} h_2(\lambda) = 2n \sum_{i=1}^{n} y_i > 0$, $h_2(\lambda)$ is then a pounded increasing function of λ. But $h_1(\lambda)$ is a positive strictly decreasing function with right limit $+\infty$ at 0. This insure that $h_1(\lambda) = h_2(\lambda)$ holds exactly once at some value $\lambda = \lambda^{\circledast}$.

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