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Homotopy Analytical Solution of MHD Fluid Flow and Heat Transfer Problem

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The objective of this paper is to derive, based on the Homotopy Analysis Method (HAM), an exact analytic solution for the boundary value problem of the coupled nonlinear system of ordinary differential equations

$$f''' + ff'' - f'^2 + A\theta + Ef' = 0,$$

$$\theta'' + Kf\theta' = 0,$$

$$f(0) = 0, \quad f'(0) = 1, \quad \theta(0) = 1, \quad f'(\infty) = 0, \quad \theta(\infty) = 0.$$

This system arises in the study of the steady magnetohydrodynamic (MHD), viscous, incompressible and electrically conducting fluid flow over a linearly stretching surface and heat transfer problem in presence of a transverse constant magnetic field and a uniform free stream of constant velocity and temperature. The obtained analytical uniformly valid solution is verified graphically and numerically and compared with the numerical results reported previously. The solution agrees with the previous reported results.

Keywords: Homotopy analysis method, nonlinear partial differential equations, MHD fluid, flow and heat transfer.

1 Introduction

The MHD flow and heat transfer over a stretching sheet is one of the very important problems in fluid mechanics. It had been discussed for the first time by Sakiadis [17]. In last decades the applications of this problem has been widely spreaded in metallurgical industry, polymer processing, and paper production [2-4]. Accordingly, this problem gained more attention and many scientists [5-13] discussed this problem but from a numerical calculation point of view only.

The flow of a Newtonian fluid with heat transfer with or without Hall effect have been studied by many authors [2, 14, 18, 19, 21, 22, 26] and the results are very important in the design of the duct wall and the cooling arrangements [20]. The related problems of flow over a linearly stretching plate with different thermal boundary conditions are investigated by many researchers such as Crane [7], Gupta [8], and Rajagopal [15].

The magnetohydrodynamic (MHD) problems of flow of an electrically conducting fluid over a stretching porous plate in a porous medium with an external transverse uniform magnetic field has many applications in petroleum industry, purification of crude oil and fluid droplets sprays wire and fiber coating and polymer technology, production of plastic sheets and foils, and cold drawing of plastic sheets. All these processes depend on the physical/rheological properties of the fluid around the sheet. Many studies to understand the features of the flow over a stretching sheet had been done traditionally for Newtonian fluids, although the fluids used in industrial purposes are non-Newtonian.

Recently, great attention has been focused on the boundary layer problem of flow of non-Newtonian fluids over a stretching sheet. Zhang *et al.* [31] investigated the steady laminar flow of a non-Newtonian fluid obeying the power-law model over a stretching surface. They concluded that the problem has a unique normal solution for $0 < n < 1$ and has a unique generalized normal solution for $n > 1$. Rafael [5] proved the existence of solution of the boundary layer problem and in [6] discussed the steady boundary layer of a second grade fluid in a porous medium over a stretching sheet with chemical reaction using similarity transformation and the Rung-Kutta numerical methods. Asif *et al.* [1] introduced the homotopy analysis solution of unsteady boundary layer over a permeable stretching sheet and neglected the effect of the magnetic field. Wang and Pop [27] derived the solution of the flow of a power-law fluid film on unsteady stretching sheet without porosity and neglected the magnetic field effects using the homotopy analysis method. Liao [11, 12] introduced the analytic solution of the steady state non-Newtonian MHD fluid flow over a stretching sheet by means of HAM.

Hang Xu and Liao [28] introduced the homotopy analysis solution for the unsteady power-law fluid flows on an impulsively stretching sheet taking into consideration the Hall effects. Hayat *et al.* [9] analyzed the MHD boundary layer flow of an upper convected Maxwell fluid over a porous stretching sheet by means of homotopy analysis method. Andersson et al. [3] derived the series solution for the steady flow of a power law fluid over a stretching sheet the neglected the porosity and the magnetic field effects. Vajravelu and Cannon [24] reported the solution of steady state of viscous Newtonian fluid over a non-linearly stretching sheet. Vajravelu [23], and Vajravelu and Nayfeh [25], studied the steady boundary layer flow of a newtonian fluid over a stretching sheet. Hang and Liao [29] introduced a dual solution of boundary layer flow over upstream moving plate by using the homotopy analysis method.

Mahmoud *et al.* [13] used the method of successive approximations and the numerical shooting method to investigate the magnetic field effects on the steady state boundary layer flow of a non-Newtonian power-law fluid without porosity effects. Mahmmet [30] used the similarity transformation and the numerical variable step Rung-Kutta method to discuss the unsteady boundary layer flow of a power-law fluid over a stretching surface in the absence of the porosity and magnetic field effects. Sajid and Hayat [16] applied the homotopy analysis method to study the nonsimilar series solution for boundary layer flow of a third order non-Newtonian fluid over a stretching surface.

In this paper we study the steady boundary layer non-Newtonian power-law fluid flows over a porous stretching sheet in a porous medium with a transverse magnetic field. We use the homotopy analysis method by Liao [10] to derive the analytical solution of the velocities component of the fluid. The numerical verification of the obtained analytical solution is given, graphical representation of the results of the study of the effects of the magnetic field and variable porosity on the velocity of the fluid flow is presented and discussed.

2 Problem Formulation

Consider the steady MHD viscous, incompressible, electrically conducting fluid flow over a linearly stretching surface and heat transfer, in the presence of a transverse constant magnetic field and a uniform free stream of constant velocity and temperature governed by the continuity equations, the momentum equation the energy equation and the boundary conditions read, respectively as follows:

The continuity equation

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0. \tag{2.1}$$

The momentum equation

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2}+g\beta T-\frac{\sigma B_0^2}{\rho}u. \tag{2.2}$$

The energy equation

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\frac{k}{\rho C_p}\frac{\partial^2 T}{\partial y^2}. \tag{2.3}$$

The boundary conditions:

$$u(x,0)=ax,\quad v(x,0)=0,\quad T(x,0)=T_w=\text{const},$$
$$u(x,\infty)=0\quad\text{and}\quad T(x,\infty)=0. \tag{2.4}$$

where a is the stretching rate constant, u, v are the velocity components of the fluid, ν is the kinematic viscosity, g is the acceleration due to gravity, β is the coefficient of thermal expansion, T is the temperature, σ is the electric conductivity, B_0 is the imposed magnetic field, ρ is the density, and k is the thermal conductivity. In terms of the stream function the velocity components are

$$u = \frac{\partial \psi}{\partial y}, \qquad v = -\frac{\partial \psi}{\partial x}. \tag{2.5}$$

Using the transformations

$$\psi = x\sqrt{a\nu} f(\eta), \qquad \eta = y\sqrt{\frac{a}{\nu}}, \qquad \theta = \frac{T}{T_w}, \tag{2.6}$$

the system of equations (2.1)-(2.3) can be transformed to

$$f''' + ff'' - f'^2 + Gr\theta - M^2 f' = 0, \tag{2.7}$$

$$\theta'' + Prf\theta' = 0, \tag{2.8}$$

and the continuity equation (2.1) is satisfied identically, and the boundary conditions become

$$f(0) = 0, \quad f'(0) = 1, \quad \theta(0) = 1, \quad f'(\infty) = 0 \quad \text{and} \quad \theta(\infty) = 0, \tag{2.9}$$

where the prime denotes differentiation with respect to η, $Gr = g\beta T_w/(a^2 x)$ is the local Grashof number, $M^2 = \sigma B_0^2/(a\rho)$ is the magnetic parameter, and $Pr = \rho\nu C_p/k$ is the Prandtl number.

The system of equations (2.7)-(2.9) is a coupled system of nonlinear ordinary differential equations and it is difficult to solve by the common methods of solution of the system of ordinary differential equation.

3 Homotopy Analytic Solution

In this section, the homotopy analysis method (HAM) is applied to obtain analytic solutions for the equations (2.7)-(2.9). For this purpose we choose the set of bases functions $\{e^{-n\eta}; n \geq 0 \text{is an integer}\}$ to approximate the unknown functions $f(\eta)$ and $\theta(\eta)$ respectively, as

$$f(\eta) = f_0(\eta) + \sum_{m=1}^{\infty} f_m(\eta), \quad \theta(\eta) = \theta_0(\eta) + \sum_{m=1}^{\infty} \theta_m(\eta), \tag{3.1}$$

where

$$f_0(\eta) = 1 - e^{-\eta}, \qquad \theta_0(\eta) = e^{-\eta} \tag{3.2}$$

are taken to be the initial guess approximations. Using the rules of solution expression

$$f(\eta) = \sum_{m=0}^{\infty} f_m(\eta) = \sum_{m=0}^{\infty} \sum_{k=0}^{2m+1} f_{m,k} e^{-k\eta}, \tag{3.3}$$

$$\theta(\eta) = \sum_{m=0}^{\infty} \theta_m(\eta) = \sum_{m=0}^{\infty} \sum_{k=0}^{2m+1} \theta_{m,k} e^{-k\eta}, \tag{3.4}$$

where the coefficients $f_{m,k}$ and $\theta_{m,k}$ are constants and have to be determined. And $\forall m \geq 1$ the mth order aproximation of the solution functions are

$$f_m(\eta) = \sum_{k=0}^{2m+1} f_{m,k} e^{-k\eta}, \tag{3.5}$$

$$\theta_m(\eta) = \sum_{k=0}^{2m+1} \theta_{m,k} e^{-k\eta}. \tag{3.6}$$

According to the above choice of bases functions, the initial guess and the auxiliary linear operators are

$$L_f = \frac{1}{2}\left[\frac{\partial^2}{\partial \eta^2} + \frac{\partial^3}{\partial \eta^3}\right], \qquad L_\theta = e^{\eta}\left[\frac{\partial^2}{\partial \eta^2} + \frac{\partial}{\partial \eta}\right]. \tag{3.7}$$

The Zero-order deformations are

$$(1-q)L_f\left[J_1(\eta,q) - f_0(\eta)\right] = qh_f N_1\left[J_1(\eta,q), J_2(\eta,q)\right], \tag{3.8}$$

$$(1-q)L_\theta\left[J_1(\eta,q) - \theta_0(\eta)\right] = qh_\theta N_2\left[J_1(\eta,q), J_2(\eta,q)\right], \tag{3.9}$$

where $J_1(\eta, q)$ and $J_2(\eta, q)$ are the auxiliary functions, h_f and h_θ are the auxiliary parameters, and $0 \leq q \leq 1$ is the homotopy parameter such that

$$J_1(\eta,0) = f_0(\eta), \quad J_1(\eta,1) = f(\eta), \tag{3.10}$$

$$J_2(\eta,0) = \theta_0(\eta), \quad J_2(\eta,1) = \theta(\eta), \tag{3.11}$$

and the boundary conditions are

$$J_1(0,q) = f(0) = 0, \quad J_2(0,q) = \theta(0) = 1,$$

$$J_2(\infty,q) = \theta(\infty) = 0,$$

$$\left.\frac{\partial J_1(\eta,q)}{\partial \eta}\right|_{\eta=0} = 1, \quad \left.\frac{\partial J_1(\eta,q)}{\partial \eta}\right|_{\eta=\infty} = 0. \tag{3.12}$$

The nonlinear operators are

$$N_1\left[J_1(\eta,q), J_2(\eta,q)\right] = \frac{\partial^3 J_1}{\partial \eta^3} + J_1\frac{\partial^2 J_1}{\partial \eta^2} - \left[\frac{\partial J_1}{\partial \eta}\right]^2 + GrJ_2 - M^2\frac{\partial^2 J_1}{\partial \eta^2}, \tag{3.13}$$

$$N_2\left[J_1(\eta,q), J_2(\eta,q)\right] = \frac{\partial^2 J_2}{\partial \eta^2} + PrJ_1\frac{\partial J_2}{\partial \eta}. \tag{3.14}$$

The Taylor's series expansions of the auxilary functions at $q = 0$ reads

$$J_1(\eta,q) = J_1(\eta,0) + \sum_{m=1}^{\infty} \frac{1}{m!} \left. \frac{\partial^m J_1(\eta,q)}{\partial q^m} \right|_{q=0} = f_0(\eta) + \sum_{m=1}^{\infty} f_m(\eta) q^m, \tag{3.15}$$

$$J_2(\eta,q) = J_2(\eta,0) + \sum_{m=1}^{\infty} \frac{1}{m!} \left. \frac{\partial^m J_1(\eta,q)}{\partial q^m} \right|_{q=0} = \theta_0(\eta) + \sum_{m=1}^{\infty} \theta_m(\eta) q^m. \tag{3.16}$$

By the mth order differentiation of equations (3.8) and (3.9) one gets the mth order deformations

$$L_f\left[f_m(\eta) - \chi_m f_{m-1}(\eta)\right] = h_f P_m(\eta), \tag{3.17}$$

$$L_\theta\left[\theta_m(\eta) - \chi_m \theta_{m-1}(\eta)\right] = h_\theta Q_m(\eta), \tag{3.18}$$

where

$$\chi_m = \begin{cases} 0 & m = 1, \\ 1 & m > 1, \end{cases} \tag{3.19}$$

and

$$P_m(\eta) = f'''_{m-1}(\eta) + \sum_{n=0}^{m-1} f_n(\eta) f''_{m-1-n}(\eta) - \sum_{n=0}^{m-1} f'_n f'_{m-1-n} + Gr\theta_{m-1} - M^2 f'_{m-1}(\eta), \tag{3.20}$$

$$Q_m(\eta) = \theta''_{m-1}(\eta) + Pr \sum_{n=0}^{m-1} f_n(\eta) \theta'_{m-1-n}(\eta). \tag{3.21}$$

Substituting (3.20) and (3.21) into (3.17) and (3.18), and taking into account of (3.5) and (3.6), one obtains the following recurrence relations for the coefficients $f_{m,k}$ and $\theta_{m,k}$

$$f_{m,k} = \frac{2h_f}{k^2 + k^3} S_{m,k} + \chi_m \lambda_{m-1,k+1} f_{m-1,k} \quad 1 \le k \le 2m+1, \tag{3.22}$$

and

$$f_{m,0} = -\sum_{k=0}^{2m+1} f_{m,k} e^{-k\eta}, \tag{3.23}$$

$$\theta_{m,k} = \frac{2h_\theta}{k^2 - k} T_{m,k-1} + \chi_m \lambda_{m-1,k} \theta_{m-1,k}, \quad 2 \le k \le 2m+2, \tag{3.24}$$

$$\theta_{m,1} = -\sum_{k=0}^{2m+1} \theta_{m,k} e^{-k\eta}, \tag{3.25}$$

where

$$S_{m,k} = \alpha_{m,k} + \beta_{m,k}, \quad 1 \le k \le 2m+1, \tag{3.26}$$

$$T_{m,k} = \pi_{m,k} + \delta_{m,k}, \quad 1 \le k \le 2m+1, \tag{3.27}$$

$$\alpha_{m,k} = [M^2 k - k^3]\lambda_{m-1,k} f_{m-1,k} + Gr\lambda_{m-1,k}\theta_{m-1,k}, \tag{3.28}$$

$$\beta_{m,k} = \sum_{n=0}^{m-1} \sum_{s=max[1,k-2n-1]}^{min[2m-2n-1,k-1]} [s^2+s(s-k)] f_{n,k-s} f_{m-1-n,s}, \quad 1 \le k \le 2m+1, \tag{3.29}$$

(where $\beta_{m,1} = 0$ and $\beta_{m,2m+1} = 0$)

$$\delta_{m,k} = Pr \sum_{n=0}^{m-1} \sum_{s=max[1,k-2n-1]}^{min[2m-2n,k]} [-s] f_{n,k-s} \theta_{m-1-n,s}, \quad 1 \le k \le 2m+1, \tag{3.30}$$

$$\pi_{m,k} = [k^2]\lambda_{m-1,k}\theta_{m-1-n,s}, \quad 1 \le k \le 2m+1, \tag{3.31}$$

$$\lambda_{m,k} = \begin{cases} 1 & 1 \le k \le 2m+2 \\ 0 & \text{otherwise.} \end{cases} \tag{3.32}$$

Using the above recurrence relations together with the coefficients in the initial guess functions

$$f_{0,0} = 1, \quad f_{0,1} = -1, \quad \theta_{0,0} = 0 \quad \text{and} \quad \theta_{0,1} = 1,$$

we get

$$f_{1,2} = f_{1,3} = 0, \quad \theta_{1,0} = 0, \quad \theta_{1,1} = -\theta_{1,2} - \theta_{1,3},$$

$$f_{1,0} = -f_{1,1} = h_f\left(M^2 - Gr\right), \quad \theta_{1,2} = \frac{1}{2}\left(1 - Pr\right)h_\theta, \quad \theta_{1,3} = \frac{1}{6}Prh_\theta,$$

$$f_{2,4} = 0, \quad f_{2,5} = 0, \quad \theta_{2,0} = 0, \theta_{2,5} = 0,$$

$$f_{2,0} = -f_{2,1} - f_{2,2} - f_{2,3}, \quad \theta_{2,1} = -\theta_{2,2} - \theta_{2,3} - \theta_{2,4},$$

$$f_{2,1} = h_f\left(Gr - M^2\right) + h_f\left[M^2 + h_f\left(Gr - M^2\right)\right], \quad f_{2,2} = \frac{1}{12}h_f h_\theta Gr\left(1 - Pr\right),$$

$$\theta_{2,2} = h_\theta\left[\left(-\frac{1}{4} + \frac{5}{12}Pr - \frac{1}{6}Pr^2\right)h_\theta + \frac{1}{2}h_f\left(Gr - M^2\right)\right],$$

$$\theta_{2,3} = h_\theta\left[\left(\frac{1}{3} - \frac{1}{12}Pr - \frac{2}{9}Pr^2\right)h_\theta - h_f\left(Gr - M^2\right)\right],$$

$$f_{2,3} = \frac{1}{108}GrPrh_f h_\theta, \quad \theta_{2,4} = \frac{1}{12}Pr^2 h_\theta^2. \tag{3.33}$$

Therefore, the analytic series solutions of the system of coupled nonlinear differential equations with boundary conditions (2.7), (2.8) and (2.9) for different approximations can be written as

initial approximations:
$$\begin{aligned} f(\eta) &\approx f_0(\eta) = 1 - e^{-\eta}, \\ \theta(\eta) &\approx \theta_0(\eta) = e^{-\eta}; \end{aligned} \tag{3.34}$$

the first approximations:

$$\begin{aligned} f(\eta) &\approx f_0(\eta) + f_1(\eta) = (1 + f_{1,0}) + (f_{1,1} - 1)\, e^{-\eta}, \\ \theta(\eta) &\approx \theta_0(\eta) + \theta_1(\eta) = \theta_{1,1}e^{-\eta} + \theta_{1,2}e^{-2\eta} + \theta_{1,3}e^{-3\eta}; \end{aligned} \tag{3.35}$$

the second approximations:

$$
\begin{aligned}
f(\eta) &\approx f_0(\eta) + f_1(\eta) + f_2(\eta) \\
&= (f_{0,0} + f_{1,0} + f_{2,0}) + (f_{0,1} + f_{1,1} + f_{2,1})\, e^{-\eta} + f_{2,2}e^{-2\eta} + f_{2,3}e^{-3\eta}, \\
\theta(\eta) &\approx \theta_0(\eta) + \theta_1(\eta) + \theta_2(\eta) \\
&= (\theta_{0,1}+\theta_{1,1}+\theta_{2,1})\, e^{-\eta}+(\theta_{1,2}+\theta_{2,2})\, e^{-2\eta}+(\theta_{1,3}+\theta_{2,3})\, e^{-3\eta}+f_{2,4}e^{-4\eta}.
\end{aligned} \tag{3.36}
$$

All approximations can be obtained by using equations (3.19)-(3.21) and (3.7). The analytic series solution of the problem can be written as

$$f(\eta) = \sum_{m=0}^{\infty} f_m(\eta), \quad \theta(\eta) = \sum_{m=0}^{\infty} \theta_m(\eta). \tag{3.37}$$

It is important to note that the convergence of the obtaied series solution of the coupled system is controlable by the axuiliary homotopy parameter h_f and h_θ which gurantee the convergence of the obtained series. The optimum values are choosen $h_f = -1$, for the f function and $h_\theta = -0.3$ function. These results can be obtained graphically from h-Curve for $f''(0)$, and ($\theta'(0)$, $\theta''(0)$), respectively.

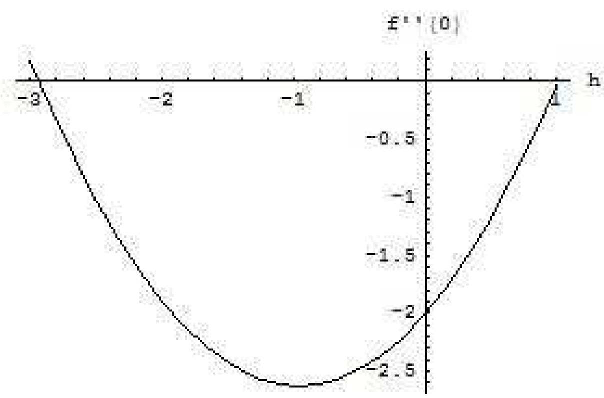

Figure 3.1: The h-curve for $f''(0)$

4 Results and Discussion

The analytical solution of a coupled nonlinear system of differential equation in the form (2.7) and (2.8) subjected to a set of boundary conditions (2.9) is obtained by using the HAM. The obtained analytical solution is verified graphically in figures (3.3) and (3.4) for $M^2 = 0.1$, $Pr = 0.7$, and $Gr = 0.5$ values of the parameters of the problem to show the behaviour of the solution which satisfy the boundary conditions. It can be used to investigate the effects of the physical parameters on the problem an chickout, as a benchmark, the numerical results obtained by the different numerical calculation methods.

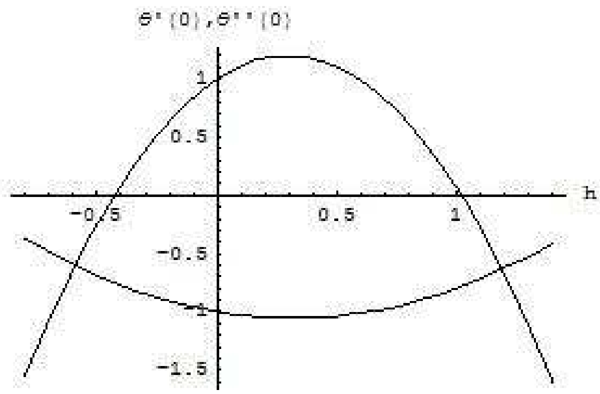

Figure 3.2: The h-curve for $\theta'(0)$ and $\theta''(0)$

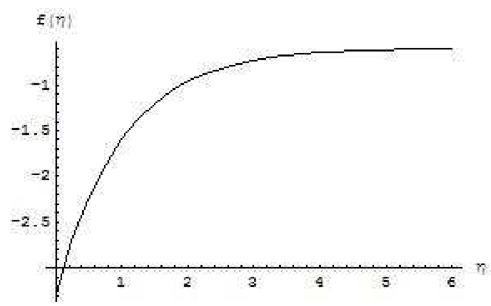

Figure 3.3: The presentation of the solution $f(\eta)$ function

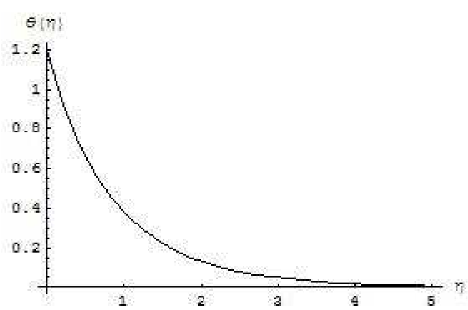

Figure 3.4: The presentation of the solutions $\theta(\eta)$ and $f(\eta)$ functions

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