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A Modified Palindromic Symmetry Model for Square Contingency Tables with Ordered Categories

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Abstract: For the analysis of square contingency tables with the same row and column ordinal classifications, McCullagh (1978) considered the palindromic symmetry (PS) model, which has a multiplicative form for cumulative probabilities that an observation will fall in row (column) category i or below and column (row) category j $(> i)$ or above. The present paper proposes a modified PS model which indicates that (1) the symmetric odds for cumulative probabilities with distance 1 from main diagonal of the table are constant and (2) there is the structure of quasi-symmetry for them with distance k (≥ 2). Also the present paper gives the decomposition of the symmetry model using the proposed model. Examples are given.

Keywords: Cumulative quasi-symmetry, decomposition, marginal homogeneity, palindromic symmetry, square contingency tables

1 Introduction

For an $R \times R$ square contingency table with the same row and column ordinal classifications, let p_{ij} denote the probability that an observation will fall in the ith row and jth column of the table $(i = 1,\ldots,R;\ j = 1,\ldots,R)$. The conditional symmetry model is defined by

$$p_{ij} = \begin{cases} \delta\psi_{ij} & (i < j), \\ \psi_{ij} & (i \geq j), \end{cases} \tag{1}$$

where $\psi_{ij} = \psi_{ji}$; see McCullagh (1978). A special case of this model obtained by putting $\delta = 1$ is the symmetry (S) model (Bowker, 1948; Bishop, Fienberg and Holland, 1975, p. 282). Note that model (1) with δ replaced by δ_{j-i} is Goodman's (1979) diagonals-parameter symmetry model. Caussinus (1965) proposed the quasi-symmetry model for cell probabilities, defined by

$$p_{ij} = \alpha_i \beta_j \psi_{ij} \quad (i \neq j),$$

where $\psi_{ij} = \psi_{ji}$. This may be expressed as

$$p_{ij} = \xi_i \phi_{ij} \quad (i \neq j),$$

where $\phi_{ij} = \phi_{ji}$. The marginal homogeneity (Stuart, 1955) model is defined by

$$p_{i\cdot} = p_{\cdot i} \quad (i = 1,\ldots,R),$$

where

$$p_{i\cdot} = \sum_{t=1}^{R} p_{it}, \quad p_{\cdot i} = \sum_{s=1}^{R} p_{si}.$$

Let

$$G_{ij} = \sum_{s=1}^{i}\sum_{t=j}^{R} p_{st} \quad (i < j),$$

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and

$$G_{ij} = \sum_{s=i}^{R} \sum_{t=1}^{j} p_{st} \quad (i > j).$$

The marginal homogeneity model may be expressed as

$$G_{i,i+1} = G_{i+1,i} \quad (i = 1, \ldots, R-1).$$

McCullagh (1978) considered a multiplicative model for cumulative probabilities $\{G_{ij}\}$, $i \neq j$, which was referred to as the palindromic symmetry (PS) model, including the S model and the conditional symmetry model as special cases. The PS model is defined by

$$G_{ij} = \begin{cases} \Delta \alpha_i \Psi_{ij} \ (i < j), \\ \alpha_{i-1} \Psi_{ij} \ (i > j), \end{cases} \quad p_{ii} = \Psi_{ii}, \tag{2}$$

where $\Psi_{ij} = \Psi_{ji}$ and $\alpha_1 = 1$ without loss of generality. Note that (2) with Δ replaced by Δ_i is the generalized PS model (McCullagh, 1978).

The conditional symmetry model for $\{p_{ij}\}$ is also expressed similarly as a multiplicative form for $\{G_{ij}\}$ as

$$G_{ij} = \begin{cases} \Delta \Psi_{ij} \ (i < j), \\ \Psi_{ij} \ (i > j), \end{cases} \quad p_{ii} = \Psi_{ii}, \tag{3}$$

where $\Psi_{ij} = \Psi_{ji}$. A special case obtained by putting $\Delta = 1$ is the S model. Note that Tomizawa (1989) and Tahata, Yamamoto and Tomizawa (2012) gave the decompositions of the conditional symmetry model and the S model, respectively, using the PS model. Tomizawa (1993) proposed the cumulative diagonals-parameter symmetry model, defined by (3) with Δ replaced by Δ_{j-i}. Miyamoto, Ohtsuka and Tomizawa (2004) proposed the cumulative quasi-symmetry model for $\{G_{ij}\}$, defined by

$$G_{ij} = \gamma_i \Psi_{ij} \quad (i \neq j), \quad p_{ii} = \Psi_{ii},$$

where $\Psi_{ij} = \Psi_{ji}$. This model may be expressed as

$$\frac{G_{ij}}{G_{ji}} = \frac{\gamma_i}{\gamma_j} \quad (i < j);$$

or

$$G_{ij} G_{jk} G_{ki} = G_{ji} G_{kj} G_{ik} \quad (1 \leq i < j < k \leq R).$$

See also Yamamoto, Ando and Tomizawa (2011) for this model.

The PS model is also expressed as

$$\frac{G_{i,i+1}}{G_{i+1,i}} = \Delta \quad (i = 1, \ldots, R-1), \tag{4}$$

and

$$\frac{G_{ij}}{G_{ji}} = \Delta \frac{\alpha_i}{\alpha_{j-1}} \quad (i < j; j \neq i+1). \tag{5}$$

Note that (4) is equivalent to the extended marginal homogeneity model in Tomizawa (1984, 1995) (also see Tahata and Tomizawa, 2008), and (5) is different from the structure of the cumulative quasi-symmetry model because (5) depends on both parameters of Δ and $\{\alpha_i\}$. Now we are interested in considering a new model, which has a structure of cumulative quasi-symmetry for $\{G_{ij}, |j-i| \geq 2\}$, in addition to $\{G_{i,i+1}/G_{i+1,i} = \Delta\}$. For various models of symmetry of cell probabilities, cumulative probabilities and marginal probabilities, see also, e.g., Lawel (2003, Chap. 11), Kateri and Agresti (2007), a reference list in Tomizawa and Tahata (2007), and Agresti (2013, Chap. 11).

Section 2 in the present paper proposes a new model which modifies the PS model. Section 3 gives a decomposition of the S model using the proposed model. Section 4 describes the goodness-of-fit test for models. Section 5 gives examples. Section 6 provides some concluding remarks.

2 Modified palindromic symmetry model

Consider a model defined by

$$G_{ij} = \begin{cases} \beta_i \Psi_{ij} & (i < j; j \neq i+1), \\ \Gamma \beta_i \Psi_{ij} & (j = i+1), \\ \beta_{i-1} \Psi_{ij} & (i > j), \end{cases} \qquad p_{ii} = \Psi_{ii},$$

where $\Psi_{ij} = \Psi_{ji}$ and $\beta_1 = 1$ without loss of generality. We shall refer to this model as the modified palindromic symmetry model (MPS) model. A special case of this model obtained by putting $\Gamma = 1$ and $\{\beta_i = 1\}$ is the S model. The MPS model is also expressed as

$$\frac{G_{i,i+1}}{G_{i+1,i}} = \Gamma \quad (i = 1, \ldots, R-1), \tag{6}$$

and

$$\frac{G_{ij}}{G_{ji}} = \frac{\beta_i}{\beta_{j-1}} \quad (i < j; j \neq i+1). \tag{7}$$

The equation (6) indicates that the cumulative probability that an observation will fall in row category i or below and column category $i+1$ or above (i.e., $G_{i,i+1}$) is Γ times higher than the cumulative probability that the observation falls in row category $i+1$ or above and column category i or below (i.e., $G_{i+1,i}$); this is the structure of the extended marginal homogeneity model. The equation (7) states that the cumulative probability that an observation will fall in row category i or below and column category j $(i < j; j \neq i+1)$ or above, is β_i/β_{j-1} times higher than the cumulative probability that the observation falls in row category j or above and column category i or below.

Let $G^*_{i,j-1} = G_{ij}$ and $G^*_{j-1,i} = G_{ji}$ for $j - i \geq 2$. Then (7) is expressed as

$$\frac{G^*_{i,j-1}}{G^*_{j-1,i}} = \frac{\beta_i}{\beta_{j-1}} \quad (i < j; j \neq i+1). \tag{8}$$

Namely, this indicates that there is the structure of cumulative quasi-symmetry for $\{G_{ij}\}$ with $|j - i| \geq 2$. Note that this structure is different from the PS model. The equation (8) also may be expressed as

$$G^*_{kl} G^*_{lm} G^*_{mk} = G^*_{lk} G^*_{ml} G^*_{km} \quad (1 \leq k < l < m \leq R-1). \tag{9}$$

The equation (9) implies

$$\theta^*_{ij;st} = \theta^*_{st;ij} \quad (1 \leq i < j < s < t \leq R-1), \tag{10}$$

where

$$\theta^*_{ij;st} = \frac{G^*_{is} G^*_{jt}}{G^*_{it} G^*_{js}} = \frac{G_{i,s+1} G_{j,t+1}}{G_{i,t+1} G_{j,s+1}}.$$

Namely, from (10) the MPS model implies the symmetry of odds ratios based on $\{G_{ij}\}$ with $|j - i| \geq 2$.

Let X and Y denote the row and the column variables, respectively. Under the MPS model, $\Gamma > 1$ is equivalent to $F^X_i > F^Y_i$ for $i = 1, \ldots, R-1$, where $F^X_i = \sum_{k=1}^{i} p_{k\cdot}$ and $F^Y_i = \sum_{l=1}^{i} p_{\cdot l}$. Therefore the parameter Γ in the MPS model would be useful for making inferences such as that X is stochastically less than Y or vice versa.

Define the odds ratio besed on $\{G_{ij}\}$, $i \neq j$, by $\theta_{ij;st} = (G_{is}G_{jt})/(G_{it}G_{js})$ for $1 \leq i < j < s < t \leq R$. The PS model implies

$$\theta_{ij;st} = \theta_{st;ij} \quad (1 \leq i < j < s < t \leq R).$$

The MPS model implies

$$\theta_{ij;st} = \theta_{st;ij} \quad (1 \leq i < j < s < t \leq R; s \neq j+1),$$

and

$$\Gamma \theta_{ij;st} = \theta_{st;ij} \quad (1 \leq i < j < s < t \leq R; s = j+1).$$

Therefore, the PS model implies the symmetry of odds ratios based on $\{G_{ij}\}$, $i \neq j$; however, the MPS model implies the symmetry of odds ratios with the asymmetry partially. Note that both the PS and MPS models have the structure of constant of odds $\{G_{i,i+1}/G_{i+1,i}\}$, $i = 1, \ldots, R-1$.

3 Decomposition of symmetry model

Tomizawa, Miyamoto and Ouchi (2006) proposed the cumulative subsymmetry (CSS) model, defined by

$$G_{i,i+2} = G_{i+2,i} \quad (i = 1,\dots,R-2).$$

Consider the model of equality of marginal means,

$$E(X) = E(Y).$$

We shall refer to this model as the ME model.

We obtain the decomposition of the S model as follows:

Theorem 3.1. The S model holds if and only if all the MPS, CSS and ME models hold.

Proof. If the S model holds, then the MPS, CSS and ME models hold. Assuming that all the MPS, CSS and ME models hold, then we shall show that the S model holds. Since the MPS model hold, we see

$$\sum_{i=1}^{R-1} G_{i,i+1} = \Gamma \sum_{i=1}^{R-1} G_{i+1,i}.$$

Also we have

$$F_i^X - F_i^Y = G_{i,i+1} - G_{i+1,i} \quad (i = 1,\dots,R-1),$$

and

$$E(X) = R - \sum_{i=1}^{R-1} F_i^X, \quad E(Y) = R - \sum_{i=1}^{R-1} F_i^Y.$$

Thus we see

$$E(Y) - E(X) = \sum_{i=1}^{R-1} G_{i,i+1} - \sum_{i=1}^{R-1} G_{i+1,i}.$$

Since the ME model holds, we obtain $\Gamma = 1$. From the MPS and CSS models, we can see

$$\frac{G_{i,i+2}}{G_{i+2,i}} = \frac{\beta_i}{\beta_{i+1}} = 1 \quad (i = 1,\dots,R-2).$$

Since $\beta_1 = 1$, thus we see $\{\beta_i = 1\}$. Therefore we obtain $G_{ij} = G_{ji}$ $(i < j)$. Namely, the S model holds. The proof is completed.

4 Goodness-of-fit test

Let n_{ij} denote the observed frequency in the (i,j)th cell of the $R \times R$ table $(i = 1,\dots,R; j = 1,\dots,R)$. Assume that a multinomial distribution applies to the $R \times R$ table. The maximum likelihood estimates of expected frequencies under models could be obtained by using the Newton-Raphson method in the log-likelihood equation. Each model can be tested for goodness-of-fit by the likelihood ratio chi-squared statistic (denoted by G^2) with the corresponding degrees of freedom. The test statistic G^2 is given by

$$G^2 = 2 \sum_{i=1}^{R} \sum_{j=1}^{R} n_{ij} \log\left(\frac{n_{ij}}{\hat{m}_{ij}}\right),$$

where $\hat{m}_{ij}$ is the maximum likelihood estimate of expected frequency m_{ij} under the given model. The number of degrees of freedom for the MPS model is $(R-1)(R-2)/2$, which is equal to that for the PS model.

5 Examples

Example 1. The data in Table 1, taken from Stuart (1955), are constructed from unaided distance vision of 7477 women aged 30-39 employed in Royal Ordnance factories in Britain from 1943 to 1946. [These data have been analyzed by many statisticians, including Stuart (1955), Caussinus (1965), Bishop et al. (1975, p. 284), McCullagh (1978), Goodman (1979), Tomizawa (1989), Tomizawa and Tahata (2007), and Tahata and Tomizawa (2011).]

Table 3 gives the values of likelihood ratio statistic G^2 for testing the goodness-of-fit of each model in Theorem 3.1 and the PS model. [For the values of G^2 for the other models, see the corresponding articles.] The S and ME models fit these data poorly, however, the PS, MPS and CSS models fit these data well. When the MPS and PS models are compared, the G^2 value for the MPS model is less than that for the PS model with the same number of degrees of freedom.

Under the MPS model, the maximum likelihood estimate of Γ is $\hat{\Gamma} = 1.20$. Hence, under this model, the probability that a woman's right eye grade is i or below and her left eye grade is $i+1$ or above is estimated to be $\hat{\Gamma} = 1.20$ times higher than the probability that the woman's right eye grade is $i+1$ or above and her left eye grade is i or below. Since $\hat{\Gamma}$ is greater than 1, the probability that the grade of the right eye is less than k ($k = 2,3,4$) is estimated to be greater than the probability that the grade of the left eye is less than k. Namely, the right eye is estimated to be better than the left eye. Also, the maximum likelihood estimates of $\{\beta_i\}$ are $\hat{\beta}_2 = 0.78$ and $\hat{\beta}_3 = 0.61$, namely, $\hat{G}_{13}/\hat{G}_{31} = \hat{\beta}_1/\hat{\beta}_2 = 1.29$, $\hat{G}_{14}/\hat{G}_{41} = \hat{\beta}_1/\hat{\beta}_3 = 1.64$ and $\hat{G}_{24}/\hat{G}_{42} = \hat{\beta}_2/\hat{\beta}_3 = 1.27$ with $\hat{\beta}_1 = 1$. Therefore, for $i < j$ with $j-i \geq 2$, the probability that a woman's right eye grade is i or below and her left eye grade is j or above is estimated to be $\hat{\beta}_i/\hat{\beta}_{j-1}$ times higher than the probability that the woman's right eye grade is j or above and her left eye grade is i or below.

Also under the PS model, the maximum likelihood estimate of Δ is $\hat{\Delta} = 1.18$ (being close to the value of $\hat{\Gamma}$ of the MPS model). Hence, under the PS model, the probability that a woman's right eye grade is i or below and her left eye grade is $i+1$ or above is estimated to be $\hat{\Delta} = 1.18$ times higher than the probability that the woman's right eye grade is $i+1$ or above and her left eye grade is i or below. Also under the PS model, the maximum likelihood estimates of $\{\alpha_i\}$ are $\hat{\alpha}_2 = 0.93$ and $\hat{\alpha}_3 = 0.88$, namely, $\hat{G}_{13}/\hat{G}_{31} = \hat{\Delta}\hat{\alpha}_1/\hat{\alpha}_2 = 1.27$, $\hat{G}_{14}/\hat{G}_{41} = \hat{\Delta}\hat{\alpha}_1/\hat{\alpha}_3 = 1.35$, and $\hat{G}_{24}/\hat{G}_{42} = \hat{\Delta}\hat{\alpha}_2/\hat{\alpha}_3 = 1.25$ with $\hat{\alpha}_1 = 1$. Therefore, under the PS model, for $i < j$ with $j-i \geq 2$, the probability that a woman's right eye grade is i or below and her left eye grade is j or above is estimated to be $\hat{\Delta}\hat{\alpha}_i/\hat{\alpha}_{j-1}$ times higher than the probability that the woman's right eye grade is j or above and her left eye grade is i or below.

Moreover, under the MPS model, the odds ratio $\theta_{34;12}$ is estimated to be $\hat{\Gamma} = 1.20$ times greater than the odds ratio $\theta_{12;34}$. On the other hand, under the PS model, the odds ratio $\theta_{34;12}$ is estimated to be equal to the odds ratio $\theta_{12;34}$. Therefore, under the MPS model, the ratio of the odds that a woman's left eye grade is 'Best (1)' instead of 'Best (1)' or 'Second (2)' when her right eye grade is 'Third (3)' or 'Worst (4)' to the odds when her right eye grade is 'Worst (4)', is estimated to be $\hat{\Gamma} = 1.20$ times greater than the ratio of the odds that a woman's right eye grade is 'Best (1)' instead of 'Best (1)' or 'Second (2)' when her left eye grade is 'Third (3)' or 'Worst (4)' to the odds when her left eye grade is 'Worst (4)'.

From Theorem 3.1, we see that the poor fit of the S model is caused by the influence of the lack of structure of the ME model rather than the MPS and CSS models.

Example 2. The data in Table 2, taken from Tomizawa (1984, 1985), are constructed from unaided distance vision of 4746 students aged to 18 to about 25 including about 10% women in Faculty of Science and Technology, Science University of Tokyo in Japan examined in April 1982.

We see from Table 3 that the S and ME models fit these data poorly, however, the PS, MPS and CSS models fit these data well. When the MPS and PS models are compared, the G^2 value for the MPS model is somewhat greater than that for the PS model.

Under the MPS model, the maximum likelihood estimate of Γ is $\hat{\Gamma} = 0.85$. Also under the PS model, that of Δ is $\hat{\Delta} = 0.82$. Therefore $\hat{\Gamma}$ is close to $\hat{\Delta}$ for these data. Namely the $\{\hat{G}_{i,i+1}/\hat{G}_{i+1,i}(=0.85)\}$ under the MPS model are close to the $\{\hat{G}_{i,i+1}/\hat{G}_{i+1,i}(=0.82)\}$ under the PS model. In addition, under the MPS model, the maximum likelihood estimates of $\{\beta_i\}$ are $\hat{\beta}_2 = 1.29$ and $\hat{\beta}_3 = 1.17$, namely, $\hat{G}_{13}/\hat{G}_{31} = \hat{\beta}_1/\hat{\beta}_2 = 0.77$, $\hat{G}_{14}/\hat{G}_{41} = \hat{\beta}_1/\hat{\beta}_3 = 0.85$ and $\hat{G}_{24}/\hat{G}_{42} = \hat{\beta}_2/\hat{\beta}_3 = 1.10$ with $\hat{\beta}_1 = 1$. On the other hand, under the PS model, the maximum likelihood estimates of $\{\alpha_i\}$ are $\hat{\alpha}_2 = 1.13$ and $\hat{\alpha}_3 = 0.91$, namely, $\hat{G}_{13}/\hat{G}_{31} = \hat{\Delta}\hat{\alpha}_1/\hat{\alpha}_2 = 0.72$, $\hat{G}_{14}/\hat{G}_{41} = \hat{\Delta}\hat{\alpha}_1/\hat{\alpha}_3 = 0.90$ and $\hat{G}_{24}/\hat{G}_{42} = \hat{\Delta}\hat{\alpha}_2/\hat{\alpha}_3 = 1.01$ with $\hat{\alpha}_1 = 1$. Therefore, for these data, the values of $\{\hat{G}_{ij}/\hat{G}_{ji}\}$, $i < j$, under the MPS model are close to the corresponding values under the PS model.

In a similar manner to Example 1, the interpretations under the MPS and the PS models are obtained although the details are omitted here.

Table 1: Unaided distance vision of 7477 women aged 30-39 employed in Royal Ordnance factories in Britain from 1943 to 1946; from Stuart (1955). (Upper and lower parenthesized values are the maximum likelihood estimates of expected frequencies under the PS and MPS models, respectively.)

Right eye grade	Left eye grade				Total
	Best (1)	Second (2)	Third (3)	Worst (4)	
Best (1)	1520	266	124	66	1976
	(1520.00)	(264.49)	(133.21)	(58.95)	
	(1520.00)	(266.46)	(129.43)	(64.22)	
Second (2)	234	1512	432	78	2256
	(235.59)	(1512.00)	(423.40)	(86.78)	
	(233.52)	(1512.00)	(430.86)	(82.71)	
Third (3)	117	362	1772	205	2456
	(107.30)	(370.90)	(1772.00)	(204.18)	
	(110.99)	(363.15)	(1772.00)	(205.63)	
Worst (4)	36	82	179	492	789
	(43.68)	(72.67)	(179.86)	(492.00)	
	(39.22)	(76.47)	(178.34)	(492.00)	
Total	1907	2222	2507	841	7477

Table 2: Unaided distance vision of 4746 students aged to 18 about 25 including about 10% women in Faculty of Science and Technology, Science University of Tokyo in Japan examined in April 1982; from Tomizawa (1984, 1985). (Upper and lower parenthesized values are the maximum likelihood estimates of expected frequencies under the PS and MPS models, respectively.)

Right eye grade	Left eye grade				Total
	Best (1)	Second (2)	Third (3)	Worst (4)	
Best (1)	1291	130	40	22	1483
	(1291.00)	(129.75)	(41.45)	(19.84)	
	(1291.00)	(131.57)	(44.47)	(18.91)	
Second (2)	149	221	114	23	507
	(149.23)	(221.00)	(107.43)	(25.50)	
	(147.50)	(221.00)	(107.08)	(28.23)	
Third (3)	64	124	660	185	1033
	(62.79)	(130.51)	(660.00)	(190.03)	
	(59.59)	(131.21)	(660.00)	(193.45)	
Worst (4)	20	25	249	1429	1723
	(22.13)	(22.60)	(243.74)	(1429.00)	
	(22.22)	(20.69)	(240.08)	(1429.00)	
Total	1524	500	1063	1659	4746

Table 3: Likelihood ratio chi-squared values G^2 for models applied to the data in Tables 1 and 2.

Applied models	Degree of freedom	G^2	
		Table 1	Table 2
S	6	19.25*	16.96*
PS	3	6.24	1.98
MPS	3	1.55	4.95
CSS	2	5.00	3.86
ME	1	11.98*	9.95*

* means significant at the 0.05 level.

6 Concluding remarks

Compare the PS and the MPS models. From (4) and (6), both of the PS and the MPS models have the structure of extended marginal homogeneity model. Also, from (5) and (7), the MPS model has the structure of cumulative quasi-symmetry for $\{G_{ij}\}$ with $|j-i| \geq 2$, although the PS model does not have the similar structure. From (8), (9) and (10), when we want to see the structure of cumulative quasi-symmetry (i.e., (8) or (9)) including the structure of symmetry of odds ratios based on $\{G_{ij}\}$ with $|j-i| \geq 2$, the MPS model rather than the PS model would be appropriate. In addition, as described in Section 2, the PS model implies the structure of symmetry of odds ratios $\{\theta_{ij;st}\}$ based on $\{G_{ij}\}$, $i \neq j$; however, the MPS model implies the structure of symmetry of odds ratios with the structure of asymmetry partially (being $\Gamma\theta_{ij;st} = \theta_{st;ij}$ for $1 \leq i < j < s < t \leq R$ and $s = j+1$).

The decomposition of the S model into the MPS, CSS and ME models, given by Theorem 3.1, would be useful for seeing the reason for its poor fit when the S model fits the data poorly. Indeed, for the data in Table 1, the poor fit of the S model is caused by the poor fit of the ME model rather than the MPS and CSS models, i.e., by the reason that the mean of grade of the right eye is different from the mean of grade of the left eye (see Example 1).

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