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2-Size Resolvability in Graphs

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Abstract: A vertex u in a graph G resolves a pair of distinct vertices x, y of G if the distance between u and x is different from the distance between u and y. A set W of vertices in G resolves the graph G if every pair of distinct vertices of G is resolved by some vertices in W. The metric dimension of a graph, denoted by $dim(G)$, is the smallest cardinality of a resolving set. A resolving set W for a connected graph G of order $n \geq 3$ is called 2-size resolving set if the size of the subgraph $< W >$ induced by W is two. The minimum cardinality of a 2-size resolving set is called the 2-size metric dimension of G, denoted by $tr(G)$. A 2-size resolving set of cardinality $tr(G)$ is called a tr-set. In this paper, we study 2-size resolving sets in some well-known classes of graphs and give some realizable results.

Keywords: Resolving set, 2-size resolving set, 2-size metric dimension.

1. Introduction

In this paper, we consider finite, simple and connected graphs. The vertex and edge sets of a graph G are denoted by $V(G)$ and $E(G)$, respectively. We write $u \sim v$ if two vertices u and v are adjacent (form an edge) in G and write $u \nsim v$ if they are non-adjacent (do not form an edge). We refer [5] for the general graph theoretic notations and terminology not described in this paper.

A research area in graph theory that has increased in popularity during the past few decades is that of studying various methods that can be used to distinguish all the vertices in a connected graph G. Distance in graphs has also been used to distinguish all the vertices of G. The $distance$, $d(u, v)$, between two vertices u and v of a connected graph G is defined as the length of a shortest $u - v$ path in G. For an ordered set $W = \{w_1, w_2, ..., w_k\} \subseteq V(G)$ and a vertex $v \in G$, the k-vector $c_W(v) = (d(v, w_1), d(v, w_2), ..., d(v, w_k)$ is called the *code* of v with respect to W. The set W is called a *resolving set* for G if for any two distinct vertices $v, u \in G$, $c_W(v) \neq c_W(u)$. A resolving set with minimum cardinality is called a *metric basis*, or simply a *basis* of G and that minimum cardinality is called the *metric dimension* of G, denoted by $dim(G)$[4].

For a vertex v in G, the *eccentricity*, $ecc(v)$, is the maximum distance between v and any other vertex of G. The *diameter* of G, denoted by $diam(G)$, is the maximum eccentricity of a vertex v in G. The join of two graphs G_1 and G_2, denoted by $G_1 + G_2$, is a graph with vertex set $V(G_1) \cup V(G_2)$ and an edge set $E(G_1) \cup E(G_2) \cup \{u \sim v \mid u \in V(G_1) \wedge v \in V(G_2)\}$.

Metric dimension was first introduced in the 1970s, independently by Harary and Melter [8], and by Slater [20]. In recent years, a considerable literature regarding this notion has developed (see [1–4, 6, 9, 11–13, 16, 18, 19]). Slater described the usefulness of this idea into long range aids to navigation [20]. Also, this concept has some applications in chemistry for representing chemical compounds [14, 15] and in problems of pattern recognition and image processing, some of which involve the use of hierarchical data structures [18]. Other applications of this concept to navigation of robots in networks and other areas appear in [4, 10, 16]. The problem of determining whether $dim(G) < K$ is an NP-complete problem [7, 16].

To determine whether a given set $W \subseteq V(G)$ is a resolving set for G, W needs only to be verified for the vertices in $V(G) \setminus W$ since every vertex $w \in W$ is the only vertex of G whose distance from w is 0.

A useful property for finding $dim(G)$ is the following:

Lemma 1.[4] *Let W be a resolving set for a connected graph G and $u, v \in V(G)$. If $d(u, w) = d(v, w)$ for all $w \in V(G) \setminus \{u, v\}$, then u or v is in W.*

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This paper introduces a new parameter in the context of resolvability, called the 2-size resolving set, formally defined in the next section, following the idea of one size resolvability in graphs defined by Kwancharone $et\ al.$ [17]. In the next section, we study 2-size resolving sets and the 2-size metric dimension in nontrivial connected graphs, and make a comparison between the metric dimension, one size metric dimension and 2-size metric dimension. We determine the 2-size metric dimension of some specific families of graphs and characterize all the graphs of order n with 2-size metric dimension n and $n-1$. Also, we provide the necessary and sufficient conditions for a pair (k, n) of positive integers with $k \leq n$ $(n, k \geq 3)$ to be realizable as the 2-size metric dimension and order of some connected graph, respectively.

2. 2-Size Resolvability in Graphs

The following two results were proved by Chartrand $et\ al.$ in [4].

Theorem 1.*Let G be a connected graph of order n. Then (i) $dim(G) = 1$ if and only if G is a path P_n on $n \geq 2$ vertices, and (ii) $dim(G) = n-1$ if and only if G is a complete graph K_n on $n \geq 3$ vertices.*

Theorem 2.[4] *Let G be a connected graph of order $n \geq 4$. Then $dim(G) = n-2$ if and only if G is one of the graphs $K_{r,s}$ $(n = r+s$ and $r, s \geq 1)$, or $K_r + \overline{K}_s$ $(n = r+s$ and $r \geq 1, s \geq 2)$, or $K_r + (K_1 \cup K_s)$ $(n = r+s+1$ and $r, s \geq 1)$.*

Thus, if G is a nontrivial connected graph of order n, then

$$1 \leq dim(G) \leq n-1. \quad (1)$$

WKwancharone $et\ al.$ [17] defined the one size resolvability in graphs as follows:

Definition 1.*A resolving set W for a connected graph G of order $n \geq 2$ is called one size resolving set if the size of the subgraph $< W >$ induced by W is one. The minimum cardinality of a one size resolving set is called the one size metric dimension of G, denoted by $or(G)$. A one size resolving set of cardinality $or(G)$ is called an or-set.*

Since the size of the subgraph induced by an or-set is one, it follows that

$$2 \leq or(G) \leq n. \quad (2)$$

In this section, we study 2-size resolving sets which are defined as follows:

Definition 2.*A resolving set W for a connected graph G of order $n \geq 3$ is called 2-size resolving set if the size of the subgraph $< W >$ induced by W is two. The minimum cardinality of a 2-size resolving set is called the 2-size metric dimension of G, denoted by $tr(G)$. A 2-size resolving set of cardinality $tr(G)$ is called a tr-set.*

Since the size of the subgraph induced by a tr-set is two, it follows that

$$3 \leq tr(G) \leq n. \quad (3)$$

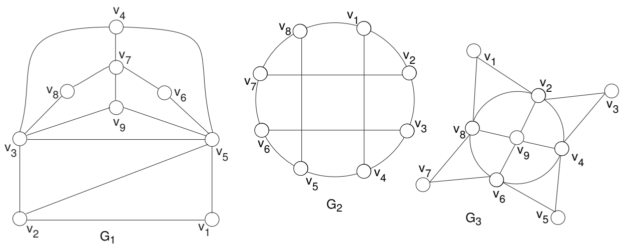

Figure 1 Illustration of 2-size resolving set and comparison between $dim(G)$, $or(G)$ and $tr(G)$

To illustrate the 2-size metric dimension, consider the graph G_1 of Figure 1. One can see that the set $\{v_1, v_3, v_9\}$ is a minimum resolving set for G_1 and is also an or-set for G_1 since the size of the subgraph induced by this set is one. But, there is no resolving set W of cardinality three such that the size of the subgraph induced by W is two. However, if we add the vertex v_8 into the set $\{v_1, v_3, v_9\}$, then the resulting set $W = \{v_1, v_3, v_8, v_9\}$ is a resolving set for G as well as the size of the subgraph $< W >$ induced by W is two. Thus, $dim(G_1) = 3 = or(G_1)$ and $tr(G_1) = 4$.

*Remark.*In a connected graph G, it is not necessary that if $or(G)$ exists, then $tr(G)$ also exists and vice-versa.

*Example 1.*Consider the graph G of Figure 2. The subgraph G' induced by $V(G)\backslash\{y_1, y_2, y_3\}$ has order $p+q+2$. In G', a set $W = \{x_1, x_2\} \cup (V \cup X) \setminus \{z\}$, where $V = \{v_1, v_2, \ldots, v_p\}$ and $X = \{x_3, x_4, \ldots, x_q\}$ $(p, q \geq 4)$, is an or-set for G' but it is easy to see that there is no set W of cardinality at least three in G' such that W is a tr-set for G'. This implies that $or(G') = p+q-1$ and $tr(G')$ does not exist.

*Example 2.*Consider the graph G_2 of Figure 1. The set $W' = \{v_1, v_2, v_3\}$ is a minimum resolving set as well as a tr-set for G_2 since the size of the subgraph induced by this set is two. But, there is no set W of cardinality at least three in G_2 such that W is an or-set for G_2. Because, without loss of generality, if we consider the set $W = \{v_1, v_2\}$, then the third vertex of W will be either v_5 or v_6, or if we consider the set $W = \{v_1, v_4\}$, then the third vertex of W will be either v_6 or v_7, or if we consider the set $W = \{v_1, v_8\}$, then the third vertex of W will be either v_3 or v_6, the induced subgraph $< W >$ have size 1 but, W is not a resolving set for G_2. Similarly, there is no or-set for G_2 of cardinality at least 4. This implies that $tr(G_2) = 3$ and $or(G_2)$ does not exist.

*Remark.*It is possible that $or(G) = tr(G)$ and further it is also possible that $dim(G) = or(G) = tr(G)$ in a nontrivial connected graph G.

*Example 3.*Consider the Petersen graph P shown in Figure 3 and the graph G_3 of Figure 1, respectively. In the Petersen graph P, the set $\{v_1, v_3, v_7\}$is a minimum resolving set for P, but there is no resolving set W of cardinality three such that the size of the subgraph induced by W is one or two. However, it is easy to see that the set $\{v_1, v_3, v_6, v_7\}$ is an or-set and the set $\{v_1, v_3, v_6, v_9\}$ is a tr-set for P (see Theorem 3). Thus, $dim(P) = 3$ and $or(P) = 4 = tr(P)$.

*Example 4.*In the graph G_3 of Figure 1, one can see that the set $\{v_1, v_3, v_5\}$ is a minimum resolving set for G_3, the set $\{v_1, v_2, v_5\}$ is an or-set and the set $\{v_1, v_2, v_3\}$ is a tr-set for G_3, which implies that $dim(G_3) = or(G_3) = tr(G_3) = 3$.

From the definitions and the above discussion, we conclude that, in a nontrivial connected graph G of order n, if both $or(G)$ and $tr(G)$ exist, then $dim(G) \leq or(G)$ and $dim(G) \leq tr(G)$.

Let u be a vertex of a graph G. The *open neighborhood* of u is $N(u) := \{v \in V(G) : v \sim u \text{ in } G\}$, and the *closed neighborhood* of u is $N[u] := N(u) \cup \{u\}$. Two distinct vertices u, v are *adjacent twins* if $N[u] = N[v]$ and *non-adjacent twins* if $N(u) = N(v)$. Observe that if u, v are adjacent twins, then $u \sim v$ in G and if u, v are non-adjacent twins, then $u \not\sim v$ in G. Adjacent twins are called *true twins* and non-adjacent twins are called *false twins*. If u, v are adjacent or non-adjacent twins, then u, v are *twins*. A set $U \subseteq V(G)$ is called a $twin\text{-}set$ of G if u, v are twins in G for every pair of distinct vertices $u, v \in U$. The next lemma follows from the definitions.

Lemma 2.[9] *If u, v are twins in a connected graph G, then $d(u, x) = d(v, x)$ for every vertex $x \in V(G) \setminus \{u, v\}$.*

Corollary 1.[9] *Suppose that u, v are twins in a connected graph G and W resolves G. Then u or v is in W. Moreover, if $u \in W$ and $v \notin W$, then $(W \setminus \{u\}) \cup \{v\}$ also resolves G.*

Thus, we have the following useful remark:

*Remark.*If U is a twin-set in a connected graph G of order n with $|U| = m \geq 2$, then every resolving set for G contains at least $m - 1$ vertices from U.

Lemma 3.*There exists a graph G such that every tr-set for G must contain all the vertices of some twin-set.*

*Proof.*Let G be the graph as shown in Figure 2 obtained from $K_{2,p}$, whose vertex sets are $\{u_1, u_2\}$ and $V = \{v_1, \ldots, v_p\}$ with $p \geq 4$, by adding the vertices $x_1, \ldots, x_q$ with $q \geq 4$ and the vertices y_1, y_2, y_3 such as $x_i \sim u_1$; $1 \leq i \leq q$, $x_1 \sim x_2$, $y_i \sim v_1$; $1 \leq i \leq 3$, $y_1 \sim y_2$ and

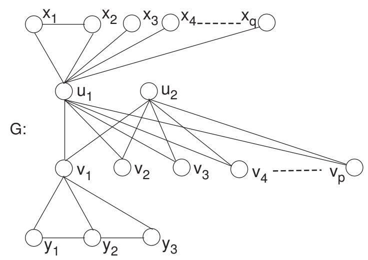

Figure 2 The graph G with tr-set $(V' \cup X \cup X' \cup \{y_1, y_2, y_3\}) \setminus \{u, x\}$ for some $u \in V' \cup X'$ and $x \in X$

$y_2 \sim y_3$. Then G contains four distinct twin-sets of cardinality at least two, namely $V' = \{v_2, \ldots, v_p\}, X = \{x_1, x_2\}, X' = \{x_3, \ldots, x_q\}$ and $Y' = \{y_1, y_3\}$. Note that, every tr-set for G is of the form $(V' \cup X \cup X' \cup \{y_1, y_2, y_3\}) \setminus \{u, x\}$ for some $u \in V' \cup X'$ and $x \in X$. From where the lemma follows.

Let U be a twin set of G, then the subgraph $< U >$ induced by U is either an empty graph or a complete graph on $|U|$ vertices. Thus, we have the following straightforward lemma:

Lemma 4.*Let G be a connected graph and let U be a twin-set of G with $|U| \geq 3$. If the subgraph $< U >$ induced by U is not an empty graph, then $tr(G)$ is not defined.*

3. 2-Size Resolving Sets in Some Well-Known Graphs

Here we determine the 2-size metric dimension of some well-known classes of graphs.

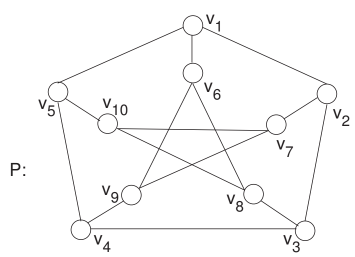

Figure 3 The Petersen graph P with $tr(P) = 4$

Theorem 3.*Let P be the Petersen graph. Then $tr(P) = 4$.*

*Proof.*Let $W = \{v_1, v_3, v_6, v_9\}$ be a set of vertices of P, then the size of the subgraph $< W >$ induced by W is two and all the codes $c_W(v_2) = (1,1,2,2), c_W(v_4) = (2,1,2,1), c_W(v_5) = (1,2,2,2), c_W(v_7) = (2,2,2,1), c_W(v_8) = (2,1,1,2), c_W(v_{10}) = (2,2,2,2)$ of the vertices of $V(P) \setminus W$ are distinct, which implies that $tr(P) \leq 4$.

For the lower bound, assume contrarily that W' is a tr-set for P of cardinality three. Let us call the the vertices $v_1 \ldots, v_5$, the outer vertices, and the vertices $v_6 \ldots, v_{10}$, the inner vertices. Then it is straightforward to see that (a) no three vertices (outer or inner) with consecutive indices form a resolving set for P, and (b) no three outer (inner) vertices form a resolving set set for P. Thus, without loss of generality, we can suppose that $v_1 \in W'$. Then $c_{W'}(v_2) = c_{W'}(v_5)$ when $W' = \{v_1, v_6, v_8\}$, or $W' = \{v_1, v_6, v_9\}$, a contradiction. Since P is 3-regular symmetric graph, considering one case is enough. Thus $tr(P) \geq 4$.

Theorem 4. *Let G be a path on $n \geq 3$ vertices, or a cycle on $n \geq 4$ vertices. Then $tr(G) = 3$.*

Proof. Consider a set $W \subseteq V(G)$ consisting of three consecutive vertices of G. Then it is straightforward to see that (a) W is a resolving set for G, and (b) the size of the subgraph $< W >$ induced by W is two. Thus, W is a 2-size resolving set for G. Therefore, it follows by (3) that $tr(G) = 3$.

Since every subgraph induced by a set of at least three vertices in a complete graph of order $n \geq 3$ is of size at least three. Therefore, 2-size metric dimension is not defined for the complete graphs. However, the removal of any edge e from the complete graphs of order 3 and 4 defines the 2-size metric dimension as we show in the next result.

Theorem 5. *Let G be a complete graph of order $n \geq 3$ and let $G - e$ be the graph obtained by deleting one edge from G. Then $tr(G - e)$ exists and $tr(G - e) = 3$ if and only if $G = K_3$ and K_4.*

Proof. It is not difficult to see that the theorem is true for a complete graph of order 3 and 4. Let us assume that G is a complete graph of order $n \geq 5$. Then $G - e \cong K_r + \overline{K_2}$ for all $r \geq 3$ so, $dim(G-e) = n-2$, by Theorem 2. Since there are two twin-sets, say X and Y, of cardinality r and 2, respectively, in $G - e$ and W must contain at least $r - 1$ vertices from X and at least 1 vertex from Y, by Remark 2. It follows that the size of the subgraph induced by any resolving set for $G - e$ is greater than two. Thus, $tr(G-e)$ does not exist.

Similarly, the removal of any two edges from the complete graphs of order 4 and 5 also defines the 2-size metric dimension. The proof of the following result is same as the proof of the previous result, so we omit it.

Theorem 6. *Let G be a complete graph of order $n \geq 4$ and let $G-2e$ be the graph obtained by deleting two edges from G. Then $tr(G - 2e)$ exists and $tr(G - 2e) = 3$ if and only if $G = K_4$ and K_5.*

Theorem 7. *Let $K_{r,s}$ be a complete bipartite graph with $1 \leq r \leq s$. Then $tr(K_{r,s})$ exists and $tr(K_{r,s}) = 3$ if and only if $1 \leq r < s \leq 3$.*

Proof. It is a routine exercise to see that $tr(K_{r,s})$ exists and $tr(K_{r,s}) = 3$ if and only if $1 \leq r, s \leq 3$ and $r \neq s$ when $r = 3$ or $s = 3$. Now, let $K_{r,s}$ be a complete bipartite graph with $r, s \geq 3$. Let $U = \{u_1, \ldots, u_r\}$ and $V = \{v_1, \ldots, v_s\}$ be partite sets of $K_{r,s}$. Let W be a tr-set for $K_{r,s}$, then it follows from Remark 2 that W contains at least $r - 1$ vertices from U and $s - 1$ vertices from V. However, the size of the subgraph $< W >$ induced by W is greater than two, which is a contradiction.

4. Realizable Results

IAs we have noticed that $3 \leq tr(G) \leq n$ for all connected graphs G of order $n \geq 3$ such that $tr(G)$ exist. We are able to characterize all the nontrivial connected graphs with 2-size metric dimension n and $n - 1$.

Theorem 8. *Let G be a connected graph of order $n \geq 3$. Then $tr(G) = n$ if and only if $G = P_3 = K_{1,2}$.*

Proof. Let $G = P_3 = K_{1,2}$, then, by Theorems 4 and 7, we have $tr(G) = 3 = n$. Conversely, let G be a connected graph of order $n \geq 3$ with $tr(G) = n$ and let W be a tr-set for G of cardinality n. Since the size of the subgraph $< W >$ induced by W is two and G is a connected graph, it follows that $|W| = |V(G)| = 3$. Then $G = P_3$.

It is an immediate consequence of the above theorem that if G is a connected graph of order $n \geq 4$, then $tr(G) \leq n - 1$.

Theorem 9. *Let G be a connected graph of order $n \geq 4$. Then $tr(G) = n - 1$ if and only if $G \in \{P_4, C_4 \cong K_{2,2}, K_{1,3}, K_4 - e, K_4 - 2e\}$.*

Proof. Let $G \in \{P_4, C_4 \cong K_{2,2}, K_{1,3}, K_4 - e, K_4 - 2e\}$. From Theorems 4-7, it follows that $tr(G) = 3 = n-1$. To verify the converse, suppose that G is a connected graph of order $n \geq 4$ with $tr(G) = n - 1$. For $n = 4$, it is straightforward to see that $G \in \{P_4, C_4 \cong K_{2,2}, K_{1,3}, K_4 - e, K_4 - 2e\}$. Thus, we assume that $n \geq 5$. Let W be a tr-set for G of cardinality $n - 1$ and let $V(G) \setminus W = \{x\}$. Then we have the following two cases:

Case 1. If u, v and w are adjacent vertices in the subgraph $< W >$ of G induced by W in such a way that $u \sim v$, $v \sim w$ and $u \not\sim w$. Then x is adjacent to every independent vertex of $W \setminus \{u, v, w\}$ and to at least one vertex of $\{u, v, w\}$, say u. Let $W' = W \setminus \{y\}$, where y is one of $W \setminus \{u, v, w\}$. Since $d(u, x) = 1$ and $d(u, y) = 2$, it follows that $c_{W'}(x) \neq c_{W'}(y)$ and so W' is a tr-set for G with cardinality $n - 2$, a contradiction.

Case 2. If u, v, w and y are adjacent vertices in the subgraph $< W >$ of G induced by W in such a way that $u \sim v$ and $w \sim y$. Then for $n \geq 5$, x is adjacent to at least one vertex of $\{u, v\}$, say u, and to at least one vertex of $\{w, y\}$, say w. Thus, it is easy to see that the set $W' = (W \setminus \{v, y\}) \cup \{x\}$ is tr-set for G of cardinality $n-2$, a contradiction.

From Theorems 8 and 9, we have the following corollary:

Corollary 2.*Let G be a connected graph of order $n \geq 5$. Then*

$3 \leq tr(G) \leq n-2.$

Now, we provide a necessary and sufficient condition for a pair (k, n) of positive integers with $k \leq n$ such that k is realizable as the 2-size metric dimension for a connected graph of order n.

Theorem 10.*For each pair (k, n) of positive integers with $k \leq n$, there exists a connected graph G of order n and $tr(G) = k$ if and only if $n \in \{3, 4\}$ and $k = 3$, or $n \geq 5$ and $3 \leq k \leq n-2$.*

*Proof.*By Theorems 8, 9 and Corollary 2, it remains to show that there exists a connected graph G of order n and $tr(G) = k$ for $n \geq 5$ and $3 \leq k \leq n-2$. By Theorem 4, graph $G = P_n$ with $n \geq 5$ satisfies $tr(G) = 3$. Now, we assume that $n \geq 6$ and $4 \leq k \leq n-2$. Let G be a graph obtained from paths $H_1 : v_1, v_2, \ldots, v_{n-k}$, $H_2 : u_1, u_2, u_3$ and $k-3$ new vertices $w_1, \ldots, w_{k-3}$ with $u_i \sim v_1$ and $w_j \sim v_1$ for $i = 1, 2, 3$ and $1 \leq j \leq k-3$. Thus, G is a connected graph of order n as shown in Figure 4.

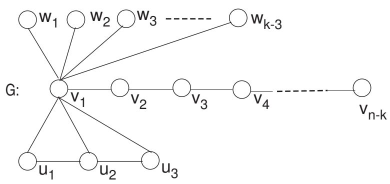

Figure 4 The graph G of order n and $tr(G) = k$

First we show that $tr(G) \leq k$. Let $W = \{u_1, u_2, u_3, w_1, \ldots, w_{k-3}\} \subset V(G)$. Since the size of the subgraph $< W >$ induced by W is two and $c_W(v_i) = (i, i, \ldots, i)$ for each $1 \leq i \leq n-k$, it follows that W is a 2-size resolving set for G. Now, we show that $tr(G) \geq k$. Assume contrarily that $tr(G) \leq k-1$. Let W' be a tr-set for G with $|W'| \leq k-1$. Since $\{u_1, u_3\}$ is a twin-set in G, it follows by Remark 2 that W' contains at least one vertex from $\{u_1, u_3\}$, say u_1. We consider two cases according to the parity of k.

Case 1. $k = 4$, then $|W'| \leq k-1 = 3$. Since the size of the subgraph $< W' >$ induced by W' must be two, it follows that W' is either $V(H_2)$, or $\{u_1, v_1, u_3\}$, or $\{u_1, v_1, v_2\}$, or $\{u_1, v_1, w_1\}$. However, $c_{W'}(v_2) = c_{W'}(w_1)$ if $W' = V(H_2)$ or $\{u_1, v_1, u_3\}$; $c_{W'}(w_1) = c_{W'}(u_3)$ if $W' = \{u_1, v_1, v_2\}$ and $c_{W'}(u_3) = c_{W'}(v_2)$ if $W' = \{u_1, v_1, w_1\}$, a contradiction.

Case 2. $k \geq 5$. Since $\{w_1, \ldots, w_{k-3}\}$ is a twin-set in G, so Remark 2 implies that W' contains at least $k-4$ vertices from $\{w_1, \ldots, w_{k-3}\}$, without loss of generality, say w_i; $1 \leq i \leq k-4$. Since the size of the subgraph $< W' >$ induced by W' must be two, it follows that $W' = V(H_2) \cup \{w_1, \ldots, w_{k-4}\}$. But, $c_{W'}(v_2) = c_{W'}(w_{k-3})$, a contradiction.

Therefore, from the Cases 1 and 2, we have $tr(G) \geq k$.

5. Conclusion

In this paper, we study the notion of 2-size resolvability in graphs and its relationship with metric dimension and one size resolvability. Also, we study 2-size resolving sets in some well-known families of graphs and give some realizable results. Furthermore, from the definitions of $dim(G)$, $or(G)$, $tr(G)$, from the inequalities (2) and (3), and from the Remark 2, we leave to the reader the following conjectures:

*Conjecture 1.*For $k \geq 1$, if $dim_k(G)$ denotes the k-size metric dimension of a connected graph G of order $n > k$ and $n \neq 4$, then $k+1 \leq dim_k(G) \leq n$.

*Conjecture 2.*In a nontrivial connected graph G of order n, if both $or(G)$ and $tr(G)$ exist, then $or(G) \leq tr(G)$.

*Conjecture 3.*If a nontrivial connected graph G of order n has l disjoint twin-sets and $tr(G)$ is defined, then for all $n \geq 5$, $tr(G) \leq n-l$.

However, we show that the upper bound of $tr(G)$ given in the Conjecture 2 is attainable. Let G be a graph of order $n \geq 6$ obtained from the complete graph K_4 with vertex set $\{v_1.v_2, v_3, v_4\}$, by deleting an edge between the vertices v_2 and v_4 and by adding $n-4$ new vertices $w_1, w_2, \ldots, w_{n-4}$ such that $w_i \sim v_1$ for all i; $1 \leq i \leq n-4$. Then, there are two twin-sets in G of cardinality at least two, namely $\{v_2, v_4\}$ and $\{w_1, w_2, \ldots, w_{n-4}\}$, which implies that $l = 2$. One can see the vertices v_2, v_3, v_4 together with any $n-5$ vertices from the set $\{w_1, w_2, \ldots, w_{n-4}\}$ form a tr-set for G of cardinality $n-l$, which implies that $tr(G) = n-l$. Further, in the graph G of Figure 2, $tr(G) < n-l$ (see Lemma 3).

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