Journal of Analysis & Number Theory
An International Journal

http://dx.doi.org/10.12785/jant/020211

Common Fixed Point Results in 2-Metric Spaces

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Received: 9 Sep. 2013, Revised: 13 Nov. 2013, Accepted: 12 Mar. 2014
Published online: 1 Jul. 2014

Abstract: In this paper, we prove some fixed point results for the class of mappings satisfying a contractive condition depending on continuous, bijective and sequentially convergent mapping. The main object of this paper is, to obtain fixed point theorems in the setting of 2-metric space using the concept of compatibility. An example is also given in support of our result.

Keywords: Fixed point, compatible pair of maps, 2-metric space, convergent sequence.

1 Introduction and Preliminaries

In recent years, examining the necessary and sufficient conditions for the existence of fixed points is drawing the attention of many researchers. There exist a huge observations on the topic and this is a very functioning field of research at present. The Banach Fixed Point theorem provides an approach for solving various problems in mathematical science and engineering. Many authors have extended, generalized and improved Banach's fixed point theorem in different ways.

In [1,2] Jungck and Rhoades characterize the concepts of δ-compatible and weakly compatible mappings as supplement of the concept of compatible mapping for single valued mappings on metric spaces. Several authors used these concepts to prove some common fixed point theorems (*see* [3,4,5,6]).

The concept of 2-metric space was initially given by Gahler [7] whose abstract properties were suggested by the area of function in Euclidean space. On the way of development, a number of authors like Iseki [8], Rhoades [9], Saha and Dey [10], Kieu Phuong Chi *et al* [11], have studied various aspects of fixed point theory in 2-metric spaces . In [11] Kieu Phuong Chi *et al* proved following theorem,

Theorem 1.[11] *Let (X,ρ) be complete 2-metric space and T be a bijective, continuous and sequentially convergent mapping. Also, suppose that ϕ_1 and ϕ_2 be two self-maps on X such that for every $x,y,a \in X$,*

$$\begin{aligned}
&\rho\left(T\left(\phi_1(x)\right),T\left(\phi_2(y)\right),T(a)\right)\\
&\quad\leqslant a_1\rho\left(T(x),T\left(\phi_1(x)\right),T(a)\right)+a_2\rho\left(T(y),T\left(\phi_2(y)\right),T(a)\right)\\
&\quad+\ a_3\rho\left(T(x),T\left(\phi_2(y)\right),T(a)\right)+a_4\rho\left(T(y),T\left(\phi_1(x)\right),T(a)\right)\\
&\quad+\ a_5\rho\left(T(x),T(y),T(a)\right)
\end{aligned}$$

where a_1,a_2,a_3,a_4 and a_5 are non-negative numbers such that $\sum_{i=1}^{5} a_i < 1$ and $(a_1-a_2)(a_3-a_4) \geq 0$.
Then ϕ_1 and ϕ_2 have a unique common fixed point.

Definition 1.[7] *Let X be a non-empty set. A real valued function d on $X\times X\times X$ is said to be a 2-metric on X if*

1. *for given distinct elements x,y of X, there exists an element z of X such that $d(x,y,z)\neq 0$;*
2. *$d(x,y,z)=0$, when at least two of x,y,z are equal;*
3. *$d(x,y,z)=d(x,z,y)=d(y,z,x)$ for all x,y,z in X;*
4. *$d(x,y,z)\leq d(x,y,w)+d(x,w,z)+d(w,y,z)$ for all x,y,z,w in X.*

When d is 2-metric on X, then ordered pair (X,d) is called a 2-metric space.

Definition 2.[8] *A sequence $\{x_n\}$ in 2-metric space X is said to be a Cauchy sequence if for each $a\in X$,*

$$\lim d\left(x_n,x_m,a\right)=0 \quad as \quad n,m\to\infty.$$

Definition 3.[8] *A sequence $\{x_n\}$ in 2-metric space X is convergent to an element $x\in X$ if for each $a\in X$,*

$$\lim_{n\to\infty} d\left(x_n,x,a\right)=0.$$

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Definition 4.[8] *A complete 2-metric space X is one in which every Cauchy sequence converges to an element of X.*

Definition 5.[13] *A pair $\{S,T\}$ of self map on 2-metric space (X,d) is said to be compatible if*

$$\lim_{n\to\infty} d\left(STx_n, TSx_n, a\right) = 0 \quad \forall \quad a \in X,$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = x.$$

Definition 6.[12] *Let T be a mapping of 2-metric space X into itself. If for all $a \in X$ and $n \to \infty$,*

$$\begin{aligned} d\left(T^n x, u, a\right) &\to 0; \\ \Rightarrow \ d\left(TT^n x, Tu, a\right) &\to 0 \end{aligned}$$

then T is called orbitally continuous.

Definition 7.[11] *A 2-metric space is called sequentially compact if every sequence of X has a convergent subsequence.*

Definition 8.[11] *Let (X,d) be a 2-metric space. A mapping $T : X \to X$ is called sequentially convergent if we have, for every sequence $\{y_n\}$, if $\{Ty_n\}$ is convergence then $\{y_n\}$ is a convergent sequence. T is called subsequentially convergent if we have, for every sequence $\{y_n\}$, if $\{T(y_n)\}$ is convergence then $\{y_n\}$ has a convergent subsequence.*

2 Main Result

We are now going to prove the following main results in the setting of 2-metric space.

Theorem 2. *Let (X,d) be a complete 2-metric space and T be a continuous, bijective and sequentially convergent mapping. Let f_1 and f_2 be a pair of self maps from X to X satisfying the following conditions,*

$$\begin{aligned} & d^2\left(T\left(f_1(x)\right), T\left(f_2(y)\right), T(z)\right) \\ & \leq \alpha d\left(T(x), T\left(f_1(y)\right), T(z)\right) d\left(T(y), T\left(f_2(y)\right), T(z)\right) \\ & + \beta d\left(T(x), T\left(f_2(y)\right), T(z)\right) d\left(T\left(f_1(x)\right), T(y), T(z)\right) \\ & + \gamma d\left(T(x), T(y), T(z)\right) d\left(T\left(f_1(x)\right), T\left(f_2(y)\right), T(z)\right) \end{aligned} \tag{1}$$

$$\begin{aligned} & d^2\left(T\left(f_1(x)\right), T\left(f_2(y)\right), T(z)\right) \\ & \leqslant \alpha d\left[\left(T(x), T\left(f_1(x)\right), T(z)\right)\right] d\left(T(y), T\left(f_2(y)\right), T(z)\right) \\ & \quad + \beta d\left(T(x), T\left(f_2(y)\right), T(z)\right) d\left(T\left(f_1(x)\right), T(y), T(z)\right) \\ & \quad + \gamma d\left(T(x), T(y), T(z)\right) d\left(T\left(f_1(x)\right), T\left(f_2(y)\right), T(z)\right) \end{aligned} \tag{2}$$

and f_1 and f_2 are compatible pair for all $x,y,z \in X$ and for some non-negative constants α,β,γ such that $0 \leq \alpha,\beta,\gamma < 1, \alpha+\gamma < 1$.

Then f_1 and f_2 have a unique common fixed point in X.

Proof. For an arbitrary point $x_0 \in X$, define a sequence $\{x_n\}$ contained in X as follows:

$$f_1\left(x_{2n}\right) = x_{2n+1}, f_2\left(x_{2n+1}\right) = x_{2n+2}, \quad n = 0,1,2,3,\ldots$$

and $\quad T\left(x_n\right) = y_n.$

Now,

$$\begin{aligned} & d^2\left(y_1, y_2, a\right) \\ & = d^2\left(T\left(x_1\right), T\left(x_2\right), T(z)\right) \\ & = d^2\left(T\left(f_1\left(x_0\right)\right), T\left(f_2\left(x_1\right)\right), T(z)\right) \\ & \leqslant \alpha d\left(T\left(x_0\right), T\left(f_1\left(x_0\right)\right), T(z)\right) d\left(T\left(x_1\right), T\left(f_2\left(x_1\right)\right), T(z)\right) \\ & \quad + \beta d\left(T\left(x_0\right), T\left(f_2\left(x_1\right)\right), T(z)\right) d\left(T\left(f_1\left(x_0\right)\right), T\left(x_1\right), T(z)\right) \\ & \quad + \gamma d\left(T\left(x_0\right), T\left(x_1\right), T(z)\right) d\left(T\left(f_1\left(x_0\right)\right), T\left(f_2\left(x_1\right)\right), T(z)\right) \\ & = (\alpha+\gamma) d\left(y_0, y_1, a\right) d\left(y_1, y_2, a\right) \\ & = cd\left(y_0, y_1, a\right), \text{where} \quad c = \alpha+\gamma. \end{aligned} \tag{3}$$

In the same way,

$$d^2\left(y_2, y_3, a\right) \leq cd\left(y_1, y_2, a\right) \tag{4}$$

Using (3) and (4), we have

$$d\left(y_2, y_3, a\right) \leqslant cd\left(y_1, y_2, a\right) \leqslant c^2 d\left(y_0, y_1, a\right).$$

Thus, in general for any positive integer n,

$$d\left(y_n, y_{n+1}, a\right) \leqslant c^n d\left(y_0, y_1, a\right).$$

Now for fixed $n,m > 0,\ m > n$,

$$\begin{aligned} d\left(y_n, y_m, a\right) & \leqslant d\left(y_n, y_{n+1}, a\right) + d\left(y_{n+1}, y_{n+2}, a\right) + \cdots \\ & \quad + d\left(y_{m-1}, y_m, a\right) \\ & \leqslant \left(c^n + c^{n+1} + c^{n+2} + \cdots + c^{m-1}\right) d\left(y_0, y_1, a\right) \\ & \leqslant \frac{c^n\left(1-c^{m-n}\right)}{1-c} d\left(y_0, y_1, a\right). \end{aligned}$$

Letting $m,n \to \infty$, we have $\lim d\left(y_n, y_m, a\right) \to 0$. Which implies $\{y_n\}$ is a Cauchy sequence and it converges to some $y \in X$. Therefore $\lim_{n\to\infty} y_n = \lim_{n\to\infty} T\left(x_n\right) = y$. Since T is sequentially convergent, we infer that $\{x_n\}$ converges to $x \in X$. Due to continuity of T, we can deduce that $T(x) = y$. As f_1 and f_2 are continuous and compatible pair, $\{f_1(x_n)\}$ and $\{f_2(x_n)\}$ converges to same limit say z, then

$$d\left(f_1 f_2 x_n, f_2 f_1 x_n, a\right) \to 0 \quad \text{as } n \to \infty.$$

Therefore,

$$\begin{aligned} \lim_{n\to\infty} f_1 f_2 x_n &= \lim_{n\to\infty} f_2 f_1 x_n \\ f_1\left(\lim_{n\to\infty} f_2\left(x_n\right)\right) &= f_2\left(\lim_{n\to\infty} f_1\left(x_n\right)\right) \\ f_1 z &= f_2 z. \end{aligned}$$

We shall show that z is a unique common fixed point of f_1 and f_2. To do this let, $f_1 z = f_2 z = u$ for some $u \in X$.

We claim that $u = z$. Therefore from (2), we have

$$\begin{aligned} & d^2\left(T\left(f_1(z)\right), T\left(f_2(u)\right), T(z)\right) \\ & \quad \leqslant \alpha d\left(T(z), T\left(f_1(z)\right), T(z)\right) d\left(T(u), T\left(f_2(u)\right), T(z)\right) + \\ & \qquad \beta d\left(T(z), T\left(f_2(u)\right), T(z)\right) d\left(T\left(f_1(z)\right), T(u), T(z)\right) + \\ & \qquad \gamma d\left(T(z), T(u), T(z)\right) d\left(T\left(f_1(z)\right), T\left(f_2(u)\right), T(z)\right) \\ & \quad = 0. \end{aligned}$$

Thus z is the common fixed point of f_1 and f_2. For uniqueness take u_1 be any other fixed point then take $u = u_1$ in the above inequality, we can conclude that $z = u_1$.

Hence z is the unique fixed point of f_1 and f_2.

Theorem 3. *Let X be a bounded complete 2-metric space and T be a continuous, bijective and sequentially convergent mapping and $S : X \mapsto X$ be a continuous self map, defined from X into X such that*

$$\begin{aligned} & d^2\left(T(Sx), T(Sy), T(z)\right) \\ & \quad \leqslant \phi\left[\max\left\{ad^2\left(Tx, Ty, Tz\right), bd\left(Tx, T(Sy), T(z)\right)\right.\right. \\ & \qquad \left.\left. d\left(T(Sx), T(y), T(z)\right)\right\}\right] \end{aligned} \tag{5}$$

for all $x, y, z \in X$ and a, b are positive constants with $0 \leq b < a < 1$ and $\phi : [0,\infty) \to [0,\infty)$ is non decreasing upper semi-continuous and $\phi(t) < t$ for all $t > 0$. Then S has a unique fixed point.

Proof. Define $\{x_n\}$ in X as follows,

$$x_{2n+1} = Sx_{2n}, \quad x_{2n+2} = Rx_{2n+1}, \quad n = 0, 1, 2, \ldots$$

and $\quad T(x_n) = y_n$.

Now,

$$\begin{aligned} & d(y_1, y_2, b) \\ & \quad = d\left(T(x_1), T(x_2), T(z)\right) \\ & \quad = d\left(T(S(x_0)), T(S(x_1)), T(z)\right) \\ & \quad \leqslant \left[\phi\left\{\max\left(ad^2\left(T(x_0), T(x_1), T(z)\right),\right.\right.\right. \\ & \quad \leqslant \left[\phi\left\{\max\left(ad^2(y_0, y_1, b), bd(y_0, y_2, b)\, d(y_1, y_1, b)\right)\right\}\right]^{\frac{1}{2}} \\ & \quad \leqslant \left[\phi\left\{\max\left(ad^2(y_0, y_1, b)\right)\right\}\right]^{\frac{1}{2}} \\ & \quad < a^{\frac{1}{2}} d(y_0, y_1, b). \end{aligned}$$

Thus $d(y_1, y_2, b) < cd(y_0, y_1, b) \quad$ if $0 \leq c = \sqrt{a} < 1$.

In the same way,

$$d(y_2, y_3, b) < cd(y_1, y_2, b) < c^2 d(y_0, y_1, b).$$

Thus in general, we have

$$d(y_n, y_{n+1}, b) < c^n d(y_0, y_1, b).$$

Now for $m, n \in N,\ m > n$

$$\begin{aligned} & d(y_n, y_m, b) \\ & \quad < d(y_n, y_{n+1}, y_m) + d(y_{n+1}, y_{n+2}, y_m) + \cdots \\ & \qquad + d(y_{m-2}, y_{m-1}, y_m) + d(y_n, y_{n+1}, b) \\ & \qquad + d(y_{n+1}, y_{n+2}, b) + \cdots + d(y_{m-1}, y_m, b) \\ & \quad < \left(c^n + c^{n+1} + c^{n+2} + \cdots + c^{m-2}\right) d(y_0, y_1, y_m) \\ & \qquad + \left(c^n + c^{n+1} + \cdots + c^{m-1}\right) d(y_0, y_1, b) \\ & \quad < 2\left(c^n + c^{n+1} + \cdots + c^{m-1}\right) M, \end{aligned}$$

where M is a constant and such M exists, as X is bounded. Letting $n, m \to \infty$ in the inequality,

$$\lim_{n\to\infty} d(y_n, y_m, b) = 0.$$

i.e. $\{y_n\}$ is a Cauchy sequence which is convergent in X say $y \in X$.

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} T(x_n) = y$$

Since T is sequentially convergent, we infer that $\{x_n\}$ converges to $x \in X$. By the hypothesis that T is continuous, we can deduce that $T(x) = y$.

Now, we shall show that x is the unique fixed point of S. Since,

$$x = \lim_{n\to\infty} x_{2n+1} = \lim_{n\to\infty} Sx_{2n} = S\left(\lim_{n\to\infty} x_{2n}\right) = Sx,$$

this shows that $x \in X$ is a fixed point of S. To prove the uniqueness, assume that $x_1 \neq x \in X$ be another fixed point. By using (5), we get

$$\begin{aligned} & d\left(Tx, T(x_1), T(z)\right) \\ & \quad = d\left(T(Sx), T(Sx_1), T(z)\right) \\ & \quad \leqslant \left[\phi\left\{\max\left(ad^2\left(Tx, Tx_1, Tz\right), bd\left(Tx, T(Sx_1), T(z)\right),\right.\right.\right. \\ & \qquad \left.\left.\left. d\left(T(S(x)), T(x_1), T(z)\right)\right)\right\}\right]^{\frac{1}{2}} \\ & \quad = \left[\phi\left\{\max\left(ad^2\left(Tx, Tx_1, Tz\right), bd\left(Tx, Tx_1, Tz\right),\right.\right.\right. \\ & \qquad \left.\left.\left. d\left(Tx, Tx_1, Tz\right)\right)\right\}\right]^{\frac{1}{2}} \\ & \quad \leqslant \left[\phi\left\{ad^2\left(Tx, Tx_1, Tz\right)\right\}\right]^{\frac{1}{2}} < d\left(Tx, Tx_1, Tz\right). \end{aligned}$$

this is a contradiction. Hence $Tx = Tx_1$ or $x = x_1$. This proves that x is a unique fixed point of S.

Theorem 4. *Let S and R be two self maps of a 2-metric space (X, d) into itself. Let T be a continuous, bijective and sequentially convergent mapping such that*

$$\begin{aligned} & d^2\left(T(Sx), T(Ry), T(z)\right) \\ & \quad \leqslant c\phi\left[ad^2\left(Tx, Ty, Tz\right) + (1-a)\max\left\{d^2\left(Tx, T(Sx), Tz\right),\right.\right. \\ & \qquad \left.\left. d^2\left(Ty, T(Ry), T(z)\right)\right\}\right] \end{aligned} \tag{6}$$

$\forall \quad x, y, z \in X, \quad 0 \leq c < 1, \quad a \in (0,1]$ where $\phi : [0,\infty) \to [0,\infty)$ is a non-decreasing upper semi continuous and $\phi(t) < t, \forall \quad t > 0$. If X is x_0 orbitally continuous then S and R have a unique common fixed point in X.

Proof. Define $\{x_n\}$ in X as follows,

$$x_{2n+1} = Sx_{2n}, \quad x_{2n+2} = Rx_{2n+1}, \quad n = 0,1,2,\ldots$$

and $T(x_n) = y_n$.

Now,

$$\begin{aligned}
&d(T(Sx_0), T(Rx_1), T(z)) \\
&\leqslant \left(c\phi\left[ad^2(Tx_0, Tx_1, Tz) + (1-a)\right.\right. \\
&\quad \left.\left.\max\left\{d^2(Tx_0, T(Sx_0), Tz), d^2(T(x_1), T(Rx_1), Tz)\right\}\right]\right)^{\frac{1}{2}} \\
&\leqslant \left(c\phi\left[ad^2(y_0, y_1, b) + (1-a)\right.\right. \\
&\quad \left.\max\left\{d^2(y_0, y_1, b), d^2(y_1, y_2, b)\right\}\right]^{\frac{1}{2}} \\
&\leqslant \sqrt{c}\left(\phi\left[ad^2(y_0, y_1, b)\right]\right)^{\frac{1}{2}}
\end{aligned}$$

[For $z \in X$, $\exists\, b \in X$ such that $T(z) = b$]

$$< \ ed(y_0, y_1, b), \text{if } \sqrt{ac} = \ e.$$

It follows that,

$$d(y_1, y_2, b) < \ ed(y_0, y_1, b)$$

Thus in general,

$$d(y_n, y_{n+1}, b) < \ ed(y_{n-1}, y_n, b), \quad e < 1.$$

Continuing exactly as in proof of Theorem 2, we can show that there exists a point $x \in X$ such that

$$x = Sx = Rx.$$

Also, uniqueness of the fixed point follows immediately from inequality (6). This completes the proof.

Theorem 5. *Let S and R be two orbitally continuous self maps of a bounded 2-metric space X and T be a continuous, bijective and sequentially convergent mapping satisfying.*

$$\begin{aligned}
&\alpha d(T(Sx), T(Ry), T(z)) + \beta \min\{d(Tx, T(Sx), Tz), \\
&\quad d(Ty, T(Ry), Tz)\} - \gamma \min\{d(Tx, T(Ry), Tz), \\
&\quad d(Ty, T(Sx), Tz)\} \\
&\leqslant q\max\{d(Tx, Ty, Tz), d(Tx, T(Sx), Tz), \\
&\quad d(Ty, T(Ry), Tz)\} \qquad (7)
\end{aligned}$$

$\forall \quad x, y, z \in X,\ \alpha, \gamma \geq 0,\ \beta, q > 0$ *with* $q < \alpha + \beta,\ q > \alpha$. *Then S and R have a unique common fixed point in X.*

Proof. Define a sequence $\{x_n\}$ by

$$x_{2n-1} = Sx_{2n-2}, \quad x_{2n} = Rx_{2n-1} \quad \text{for } n = 1,2,3,\ldots, \qquad (8)$$

and $T(x_n) = y_n$.

Now

$$\begin{aligned}
&\alpha d(T(Sx_{2n-2}), T(Rx_{2n-1}), Tz) \\
&+ \beta \min\{d(Tx_{2n-2}, T(Sx_{2n-2}), Tz), \\
&d(Tx_{2n-1}, T(Rx_{2n-1}), Tz)\} \\
&- \gamma \min\{d(Tx_{2n-2}, T(Rx_{2n-1}), Tz), \\
&d(Tx_{2n-1}, T(Sx_{2n-2}), Tz)\} \\
&\leq q\max\{d(Tx_{2n-2}, T(x_{2n-1}), Tz), \\
&d(Tx_{2n-1}, T(Sx_{2n-2}), Tz)\} \\
&d(Tx_{2n-1}, T(Rx_{2n-1}), Tz)\}
\end{aligned}$$

$$\begin{aligned}
&\leqslant q\max\{d(y_{2n-2}, y_{2n-1}, b), d(y_{2n-2}, y_{2n-1}, b), \\
&\quad d(y_{2n-1}, y_{2n}, b)\}.
\end{aligned}$$

Thus

$$\alpha d_{2n-1} + \beta \min\{d_{2n-2}, d_{2n-1}\} \leqslant q\max\{d_{2n-2}, d_{2n-1}\}. \qquad (9)$$

where

$$d_{2n} = d(y_{2n}, y_{2n+1}, b).$$

If $d_{2n-1} \leq d_{2n-2}$ then (7) implies,

$$\begin{aligned}
&(\alpha + \beta) d_{2n-1} \leqslant qd_{2n-2} \\
\implies \quad & d_{2n-1} \leqslant \frac{q}{\alpha + \beta} d_{2n-2} \leqslant cd_{2n-2},
\end{aligned}$$

where $c = \frac{q}{\alpha+\beta} < 1$. Thus we can show that

$$d_{2n-1} \leqslant cd_{2n-2} \leqslant c^2 d_{2n-3} \cdots < c^{2n-1} d_0 \to 0 \text{ as } n \to \infty$$

In he same way, if $d_{2n-2} \leq d_{2n-1}$, we have,

$$\begin{aligned}
\alpha d_{2n-1} + \beta d_{2n-2} &\leqslant qd_{2n-1} \\
\beta d_{2n-2} &\leqslant (q - \alpha) d_{2n-1} \\
d_{2n-2} &\leqslant \frac{q-\alpha}{\beta} d_{2n-1} \\
&\leqslant \delta d_{2n-1} \qquad \text{where } \delta = \frac{q-\alpha}{\beta} < 1.
\end{aligned}$$

Thus, we have

$$d_{2n-2} \leqslant \delta d_{2n-1} \leqslant \delta^2 d_{2n} \cdots < \delta^{2n-2} d_0 \to 0 \text{ as } n \to \infty.$$

Hence the sequence $\{y_n\}$ is a Cauchy sequence in X and it converges to some $y \in X$, that is

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} T(x_n) = y.$$

Since T is sequentially convergent, we infer that $\{x_n\}$ converges to $x \in X$. By the hypothesis that T is continuous, we can deduce that,

$$T(x) = y.$$

Next, we shall show that x is the unique fixed point of S and R.

$$x = \lim_{n\to\infty} x_{2n-1} = \lim_{n\to\infty} x_{2n-2} = S\left(\lim_{n\to\infty} x_{2n-2}\right) = Sx.$$

It remains to show that show that $Sx = Rx$. For this, from inequality (7)

$$\begin{aligned}&\alpha d\left(T\left(Sx\right),T\left(Rx\right),Tz\right)+\beta \min\left\{d\left(Tx,T\left(Sx\right),Tz\right),\right.\\ &\quad d\left(Tx,T\left(Rx\right),Tz\right)\right\}-\gamma \min\left\{d\left(Tx,T\left(Rx\right),Tz\right),\right.\\ &\quad \left.d\left(Tx,T\left(Rx\right),Tz\right)\right\}\\ &\leqslant q\max\left\{d\left(Tx,Tx,Tz\right),d\left(Tx,T\left(Sx\right),Tz\right),\right.\\ &\quad \left.d\left(Tx,T\left(Rx\right),Tz\right)\right\}.\end{aligned}$$

Which implies that $Sx = Rx$. Hence x is the common fixed point of S and R, the uniqueness follows immediately from (7). This completes the proof.

Setting $S = T$ in the above theorem, we have the following corollary.

Corollary 1. *Let S be an orbitally continuous self map of a bounded 2-metric space X and T be a continuous, bijective and sequentially convergent mapping satisfying,*

$$\begin{aligned}&\alpha d\left(T\left(Sx\right),T\left(Sy\right),Tz\right)+\beta \min\left\{d\left(Tx,T\left(Sx\right),Tz\right),\right.\\ &\quad d\left(Ty,T\left(Sy\right),Tz\right)\right\}-\gamma \min\left\{d\left(Tx,T\left(Sy\right),Tz\right),\right.\\ &\quad \left.d\left(Ty,T\left(Sx\right),Tz\right)\right\}\\ &\leqslant q\max\left\{d\left(Tx,Ty,Tz\right),d\left(Tx,T\left(Sx\right),Tz\right),\right.\\ &\quad \left.d\left(Ty,T\left(Sy\right),Tz\right)\right\}\end{aligned}$$

$\forall$ $x,y,z\in X$, $\alpha,\gamma \geqslant 0$, $\beta,q>0$ with $q<\alpha+\beta$ and $q+\gamma>\alpha$. Then S has a unique fixed point in X.

3 Applications

Here we give some applications related to our results. For this, we use a Lebesgue integrable function as a summable for each compact R^+.
Let us define $\psi : [0,\infty) \to [0,\infty)$ as $\psi(t) = \int_0^t \varphi(t)$ $\forall t > 0$ be a non-decreasing and continuous function. Moreover for each $\varepsilon > 0$, $\varphi(\varepsilon) > 0$. Also it implies that $\varphi(t) = 0$ iff $t = 0$.

Theorem 6. *Let (X,d) be a complete 2-metric space and T be a continuous, bijective and sequentially convergent mapping. Let f_1 and f_2 be a pair of self maps from X into X satisfying the following condition,*

$$\begin{aligned}&\int_0^{d^2(T(f_1(x)),T(f_2(y)),Tz)}\varphi(t)\,dt \leq\\ &\alpha\int_0^{d(Tx,T(f_1(x)),Tz).d(Ty,T(f_2(y)),Tz)}\varphi(t)\,dt+\\ &\beta\int_0^{d(Tx,T(f_2(y)),Tz).d(T(f_1(x)),Ty,Tz)}\varphi(t)\,dt+\\ &\gamma\int_0^{d(Tx,Ty,Tz).d(T(f_1(x)),T(f_2(y)),Tz)}\varphi(t)\,dt\end{aligned}$$

where $\varphi\in\psi$ and f_1 and f_2 are compatible pair for all $x,y,z\in X$ and for some non-negative constants α,β,γ with $0\le\alpha,\beta,\gamma<1,\beta+\gamma<1$.
Then f_1 and f_2 have a unique common fixed point in X.

Proof. Taking $\varphi(t) = 1$ and using Theorem 2, we get the desired result.

Theorem 7. *Let X be a bounded complete 2-metric space and T be a continuous, bijective and sequentially convergent mapping and $S : X \to X$ be a continuous self map defined from X into X such that*

$$\int_0^{d(T(Sx),T(Sy),Tz)}\varphi(t)\,dt \le \phi\left(\int_0^{\lambda(x,y,z)}\varphi(t)\,dt\right) \quad (10)$$

where

$$\lambda(x,y,z) = \left[max\left\{ad^2\left(Tx,Ty,Tz\right),bd\left(Tx,T\left(Sy\right),Tz\right).d\left(T\left(Sx\right),Ty,Tz\right)\right\}\right]^{1/2}$$

for all $x,y,z\in X$ and a,b are positive constants with $0\le b<a<1$ and $\phi : [0,\infty)\to[0,\infty)$ is non-decreasing upper semi continuous and $\phi(t)<t$ for all $t>0$. Then S has a unique fixed point.

Proof. Since $\phi(t) < t$, therefore from (10), we have

$$\begin{aligned}&\int_0^{d(T(Sx),T(Sy),Tz)}\varphi(t)\,dt \le\\ &\phi\left(\int_0^{\lambda(x,y,z)}\varphi(t)\,dt\right) < \int_0^{\lambda(x,y,z)}\varphi(t)\,dt \quad (11)\end{aligned}$$

On taking $\varphi(t) = 1$, the result follows from Theorem 3.

Theorem 8. *Let S and R be two orbitally continuous self maps of a bounded 2-metric space X and T be a continuous, bijective and sequentially convergent mapping satisfying:*

$$\begin{aligned}&\alpha\int_0^{d(T(Sx),T(Ry),Tz)}\varphi(t)\,dt+\beta\int_0^{\lambda_1(x,y,z)}\varphi(t)\,dt\\ &-\gamma\int_0^{\lambda_2(x,y,z)}\varphi(t)\,dt \le q\int_0^{\lambda_3(x,y,z)}\varphi(t)\,dt \quad (12)\end{aligned}$$

where

$$\begin{aligned}&\lambda_1(x,y,z)\\ &\quad = min\left\{d\left(Tx,T\left(Sx\right),Tz\right),d\left(Ty,T\left(Ry\right),Tz\right)\right\}\\ &\lambda_2(x,y,z)\\ &\quad = min\left\{d\left(Tx,T\left(Ry\right),Tz\right),d\left(Ty,T\left(Sx\right),Tz\right)\right\}\\ &\lambda_3(x,y,z)\\ &\quad = \left\{d\left(Tx,Ty,Tz\right),d\left(Tx,T(Sx),Tz\right),\right.\\ &\quad\quad \left.d\left(Tx,T(Sx),Tz\right)\right\}\end{aligned}$$

$\forall x,y,z\in X,\alpha,\gamma\ge0,\beta,q>0$ with $q<\alpha+\beta,q>\alpha$.
Then S and R have a unique common fixed point in X.

Proof. Define a sequence $\{x_n\}$ by

$$x_{2n-1} = Sx_{2n-2},\ x_{2n} = Tx_{2n-1} \quad \text{for} \quad n = 1,2,3....$$

and $T\{x_n\} = y_n$.

Now

$$\begin{aligned}\alpha \int_0^{d(T(Sx_{2n-2}),T(Rx_{2n-1}),Tz)} &\varphi(t)\,dt \\ +\beta \int_0^{\lambda_1(x_{2n-2},x_{2n-1},z)} &\varphi(t)\,dt \\ -\gamma \int_0^{\lambda_2(x_{2n-2},x_{2n-1},z)} &\varphi(t)\,dt \\ \leq q \int_0^{\lambda_3(x_{2n-2},x_{2n-1},z)} &\varphi(t)\,dt,\end{aligned}$$

where

$$\begin{aligned}\lambda_1(x,y,z) &= \{d(Tx_{2n-2},T(Sx_{2n-2},Tz),\\ &\quad d(Tx_{2n-1},T(Rx_{2n-1}),Tz)\}\\ \lambda_2(x,y,z) &= \{d(Tx_{2n-2},T(Rx_{2n-1},Tz),\\ &\quad d(Tx_{2n-1},T(Sx_{2n-2}),Tz)\}\\ \lambda_3(x,y,z) &= \{d(Tx_{2n-2},Tx_{2n-1},Tz),\\ &\quad d(Tx_{2n-2},T(Sx_{2n-1}),Tz)\}\\ &\quad d(Tx_{2n-1},T(Rx_{2n-1}),Tz)\}\end{aligned}$$

Equation (12) is

$$\begin{aligned}\alpha \int_0^{d_{2n-1}} \varphi(t)\,dt + \beta \int_0^{min\{d_{2n-1},d_{2n-1}\}} \varphi(t)\,dt \leq \\ q \int_0^{max\{d_{2n-1},d_{2n-1}\}} \varphi(t)\,dt,\end{aligned} \tag{13}$$

where $d_{2n} = d(y_{2n}, y_{2n+1}, b)$.

If $d_{2n-1} \leq d_{2n-2}$, then we have

$$\begin{aligned}(\alpha+\beta)\int_0^{d_{2n-1}} \varphi(t)\,dt &\leq q\int_0^{d_{2n-2}} \varphi(t)\,dt\\ \int_0^{d_{2n-1}} \varphi(t)\,dt &\leq \frac{q}{\alpha+\beta}\int_0^{d_{2n-2}} \varphi(t)\,dt\\ &\leq \int_0^{d_{2n-2}} \varphi(t)\,dt\end{aligned}$$

where $c = \frac{q}{\alpha+\beta} < 1$. Analogously we can show that

$$\int_0^{d_{2n-1}} \varphi(t)\,dt \leq c\int_0^{d_{2n-2}} \varphi(t)\,dt \leq .... \leq c^{2n-1}\int_0^{d_0} \varphi(t)\,dt.$$

Thus $\int_0^{d_{2n-1}} \varphi(t)\,dt \to 0$ as $n \to \infty$. Again by definition of $\varphi(t)$, we obtain

$$d_{2n-1} \to 0 \qquad \text{as } n \to \infty.$$

Similarly if $d_{2n-2} \leq d_{2n-1}$, we have

$$\alpha \int_0^{d_{2n-1}} \varphi(t)\,dt + \beta \int_0^{d_{2n-2}} \varphi(t)\,dt \leq q\int_0^{d_{2n-1}} \varphi(t)\,dt$$

$$\int_0^{d_{2n-2}} \varphi(t)\,dt \leq \left(\frac{q-\alpha}{\beta}\right)\int_0^{d_{2n-1}} \varphi(t)\,dt \leq \delta\int_0^{d_{2n-1}} \varphi(t)\,dt$$

where $\delta = \frac{q-\alpha}{\beta} < 1$.

Thus we have

$$\int_0^{d_{2n-2}} \varphi(t)\,dt \leq \delta\int_0^{d_{2n-1}} \varphi(t)\,dt \leq ..... \leq \delta^{2n-2}\int_0^{d_0} \varphi(t)\,dt.$$

Hence we conclude that, $\int_0^{d_{2n-2}} \varphi(t)\,dt \to 0$ as $n \to \infty$.

By definition of $\varphi(t)$ we have $d_{2n-2} \to 0$ as $n \to \infty$.

Thus the sequence $\{y_n\}$ is a Cauchy sequence in X and it converges to some $y \in X$, that is, $\lim_{n\to\infty} y_n = \lim_{n\to\infty} T(x_n) = y$.

Since T is sequentially convergent, we infer that x_n converges to $x \in X$. By the hypothesis that T is continuous, we can deduce that $T(x) = y$.

We shall show that x is the unique fixed point of S and R. Now $x = \lim_{n\to\infty} x_{2n-1} = \lim_{n\to\infty} x_{2n-2} = S(\lim_{n\to\infty} x_{2n-2}) = Sx$.

Now, we show that $Sx = Rx$.

From equation (12),

$$\alpha \int_0^{d(T(Sx),T(Rx),Tz)} \varphi(t)\,dt + \beta\int_0^{\lambda_1(x,x_1,z)} \varphi(t)\,dt - \gamma\int_0^{\lambda_2(x,x_1,z)} \varphi(t)\,dt \leq q\int_0^{\lambda_3(x,x_1,z)} \varphi(t)\,dt$$

$$\Rightarrow (\alpha - q)\int_0^{d(T(Sx),T(Rx),Tz)} \varphi(t)\,dt \leq 0$$

$$\text{or } \int_0^{d(T(Sx),T(Rx),Tz)} \varphi(t)\,dt = 0, \qquad \text{as } q > \alpha.$$

By definition of $\varphi(t)$, we have $d(T(Sx), T(Rx), Tz) = 0$. This implies $T(Sx) = T(Rx)$ or $Sx = Rx$.

Hence x is the common fixed point of S and R, and the uniqueness follows from equation (12). This completes the proof.

Example 1. Let $X = \{a,b,c,d\}$ and we define $d : X \times X \times X \to R_+$ as follows:-
$d(a,b,c) = d(b,c,d) = 0; d(a,b,d) = d(a,c,d) = 3$ and $d(x,y,z) = 0$ if $x,y,z \in X$ such that two of them are equal.
Clearly (X,d) is a complete 2-metric space. Define $f_1, f_2 : X \to X$ as follows:-

$$f_1(x) = \begin{cases} a, & \text{x=a} \\ c, & \text{x=b} \\ d, & \text{x=c} \\ b, & \text{x=d} \end{cases} \qquad f_2(x) = \begin{cases} a, & \text{x=a} \\ d, & \text{x=b} \\ b, & \text{x=c} \\ c, & \text{x=d} \end{cases}$$

Clearly f_1 and f_2 are compatible mappings.

Also we define a bijective, continuous and sequentially convergent mapping $T : X \to X$ as

$$T(x) = \begin{cases} b, & \text{x=a} \\ c, & \text{x=b} \\ d, & \text{x=c} \\ a, & \text{x=d} \end{cases}$$

Take $x = a, y = b$ and $z = c$ in the (2), we have

$$\begin{aligned} d^2 (T(f_1(a)), T(f_2(b)), Tc) \leq \\ \alpha d(Ta, T(f_1(a)), Tc)\, d(Tb, T(f_2(b)), Tc) + \\ \beta d(Ta, T(f_2(b)), Tc)\, d(T(f_1(a)), Tb, Tc) + \\ \gamma d(Ta, Tb, Tc)\, d(T(f_1(a)), T(f_2(b)), Tc) \\ \Rightarrow d^2(b,a,c) \leq \alpha d(b,b,b)\, d(c,a,d) + \\ \beta d(b,a,d)\, d(b,c,d) + \gamma d(b,c,d)\, d(b,d,d) \end{aligned}$$

Hence the condition of Theorem 2 is satisfied. The mapping f_1 and f_2 have a unique fixed point a.

Example 2. Let $X = [0,1]$ be a metric space with the usual metric $d(x,y) = |x-y|$. Define $f_1(x) = x^3$ and $f_2(x) = 2 - x\ \forall x \in X$, then f_1 and f_2 are compatible mapping.
Now we define 2-metric space on X by,
$d(x,y,z) = min\{|x-y|, |y-z|, |z-x| : \forall x,y,z \in X\}$.
Also we define $T = 1 + logx$ be a bijective, continuous and sequentially convergent mapping.
Taking $x = 1, y = \frac{1}{4}$ and $z = \frac{1}{2}$ in the condition (2), we get

$$\begin{aligned} d^2\left(T(f_1(1)), T\left(f_2\left(\frac{1}{4}\right)\right), T\left(\frac{1}{2}\right)\right) \leq \\ \alpha d\left(T(1), T\left(f_1\left(\frac{1}{4}\right)\right), T\left(\frac{1}{2}\right)\right) \\ d\left(T\left(\frac{1}{4}\right), T\left(f_2\left(\frac{1}{4}\right)\right), T\left(\frac{1}{2}\right)\right) + \\ \beta d\left(T(1), T\left(f_2\left(\frac{1}{4}\right)\right), T\left(\frac{1}{2}\right)\right) \\ d\left(T(f_1(1)), T\left(\frac{1}{4}\right), T\left(\frac{1}{2}\right)\right) \\ + \gamma d\left(T(1), T\left(\frac{1}{4}\right), T\left(\frac{1}{2}\right)\right) \\ d\left(T(f_1(1)), T\left(f_2\left(\frac{1}{4}\right)\right), T\left(\frac{1}{2}\right)\right) \end{aligned}$$

or $d\left(1 + log1, 1 + log\frac{7}{4}, 1 + log\frac{1}{2}\right) \leq (\beta + \gamma)\, d\left(1 + log1, 1 + log\frac{1}{4}, 1 + log\frac{1}{2}\right)$

$0.5596 \leq 0.693$ as $\beta + \gamma < 1$, which is true. Also f_1 and f_2 have fixed point 1.

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