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Exact Solutions for the Nonlinear PDE Describing the Model of DNA Double Helices using the Improved (G'/G)-Expansion Method

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Abstract: The objective of this article is to apply the improved (G'/G)- expansion method for constructing many exact solutions with parameters of the nonlinear partial differential equation (PDE) describing the model of DNA double helices. The stretch of the hydrogen bonds is considered as a nonlinear chain with cubic and quadratic potential. When the parameters take special values, many solitary wave solutions and periodic wave solutions can be found. Comparison between some of our new results and the well-known results are given.

Keywords: DNA double helices, Exact solutions, Solitary solutions, Periodic solutions, Improved (G'/G)- expansion method

1 Introduction

Nonlinear phenomena play crucial roles in applied mathematics and physics. Exact solutions for nonlinear partial differential equations (PDEs) play an important role in many phenomena in such as fluid mechanics, hydrodynamics, optics, plasma physics and so on. With the development of solitary theory many powerful methods for obtaining the exact solutions of the nonlinear evolution equations are presented and can be found in Refs. [1–20]. An attractive nonlinear model of the nonlinear science is the deoxynbonucleic acid (DNA). The DNA molecule encodes the information that organisms need to live and reproduce themselves. The DNA structure has been studied last decades (see for example [22–32]). The discovery of the double-helix structure of the DNA molecule has been established a strong relationship between its structure and function. Zhang et al. [31] have studied the following nonlinear partial differential equation describing the model of DNA double helices using the homogeneous balance method:

$$y_{tt} - c^2 y_{xx} + \omega_0^2 y - \lambda_1^2 y^2 + \delta^2 y^3 + \eta y_t = 0, \tag{1}$$

where $c^2, \omega_0^2, \lambda_1^2, \delta^2$ are well-known constants, which can be found in [31], and η is the damping constant. The objective of this article is to apply the improved (G'/G)- expansion method to find many exact traveling wave solutions of the model (1), namely the hyperbolic, trigonometric and rational function solutions. Comparison between some of our new results and the well-known results obtained in [31] will be given later. The rest of this article is organized as follows: In Sec. 2, the derivation of the model (1) is given. In Sec. 3 the description of the improved (G'/G)- expansion method is obtained. In Sec. 4 some conclusions are given.

2 Derivation of the model of DNA double helices (1)

With reference to [31], the helical axis for the Watson-Crick model of DNA is taken in the x-direction. The base and its complementary base may vibrate each other along the direction of hydrogen bonds in a base pair. Thus, it is assumed that the displacements of the nth base and its complementary base along the direction of H-bonds are given by y_n and y_n' respectively. Since the H-bond interaction has a remarkable nonlinearity due to its charge-transfer interaction, then the H- bond can be approximately described by the formula

$$V(r_n) = V_0 + Ar_n^2 - Br_n^3 + Cr_n^4, \tag{2}$$

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where $r_n = y_n - y'_n$ and A, B, C are constants. The stacking energy between two neighboring base pairs is indicated by

$$\Delta L_n = \frac{1}{2}L(y_n - y_{n-1})^2 + \frac{1}{2}L(y_{n+1} - y_n)^2, \quad (3)$$

where L is a parameter. Let M be the average mass of a base, then the Hamiltonian of the DNA system can be written as:

$$H = \sum_n \left\{ \frac{1}{2}M(\dot{y}_n^2 + \dot{y'}_n^2) + V(y_n - y'_n) \right\} + \sum_n \left\{ \frac{1}{2}L\left[(y_n - y_{n-1})^2 + (y'_n - y'_{n-1})^2 \right] \right\}, \quad (4)$$

The differential equations may be written as:

$$M\ddot{y}_n = -\frac{\partial H}{\partial y_n} = -2A\left(y_n - y'_n\right) + 3B(y_n - y'_n)^2 - 4C(y_n - y'_n)^3 + L(y_{n+1} - 2y_n + y_{n-1}) - \eta\dot{y}_n, \quad (5)$$

$$M\ddot{y}'_n = -\frac{\partial H}{\partial y'_n} = 2A\left(y_n - y'_n\right) - 3B(y_n - y'_n)^2 + 4C(y_n - y'_n)^3 + L(y'_{n+1} - 2y'_n + y'_{n-1}) - \eta\dot{y}'_n, \quad (6)$$

where η is the damping constant. We shall limit us to study the relative motion of the base such that

$$y_n = -y'_n. \quad (7)$$

Consequently, we have the differential equation

$$M\ddot{y}_n = -4Ay_n + 12By_n^2 - 32Cy_n^3 + L(y_{n+1} - 2y_n + y_{n-1}) - \eta\dot{y}_n. \quad (8)$$

The base spacing d equals 3.4 A^0 for B-DNA, it is obtained using the continuum approximation

$$y_n(t) \to y(x,t), \quad \sum_n \to \frac{1}{d}\int dx. \quad (9)$$

Hence, the equation of motion can be reduced to the form of the model (1) where

$$c^2 = \frac{L}{M}d^2, \ \omega_0^2 = \frac{4A}{M}, \ \lambda_1^2 = \frac{12B}{M}, \ \delta^2 = \frac{32c}{M}$$

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3 Description of the improved (G'/G)-expansion method

Suppose that we have the following nonlinear evolution equation:

$$F(u, u_t, u_x, u_{tt}, u_{xx}, \ldots) = 0, \quad (10)$$

where F is a polynomial in $u(x,t)$ and its partial derivatives, in which the highest order derivatives and the nonlinear terms are involved. In the following, we give the main steps of this method [20, 21] as follows:

Step 1. We use the traveling wave transformation

$$u(x,t) = u(\xi), \ \ \xi = \ell(x + \beta t + k), \quad (11)$$

to reduce Eq. (10) to the following ordinary differential equation (ODE):

$$P(u, u', u'', \ldots) = 0, \quad (12)$$

where ℓ, β, k are constants while P is a polynomial in $u(\xi)$ and its total derivatives, while the dashes denote the derivatives with respect to ξ.

Step 2.We assume that Eq. (12) has the formal solution

$$u(\xi) = \sum_{i=-m}^{m} a_i \left(\frac{G'(\xi)}{G(\xi)}\right)^i, \quad (13)$$

where $a_i \ (i = -m, \ldots, m)$ are constants to be determined, and $G(\xi)$ satisfies the following linear ODE:

$$G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0, \quad (14)$$

where λ and μ are constants.

Step 3.The positive integer m in Eq. (13) can be determined by balancing the highest-order derivatives with the nonlinear terms appearing in Eq. (12).

Step 4. We substitute (13) along with Eq. (14) into Eq. (12) to obtain polynomials in $\left(\frac{G'}{G}\right)^i$, $(i = 0, \pm1, \pm2, \ldots)$. Equating all the coefficients of these polynomials to zero, yields a set of algebraic equations, which can be solved using the Maple to find a_i, β, ℓ.

Step 5. Since the solutions of Eq.(14) are well-known for us, then we have the following ratios:

(i) If $\lambda^2 - 4\mu > 0$, we have

$$\frac{G'(\xi)}{G(\xi)} = -\frac{\lambda}{2} + \frac{\sqrt{\lambda^2 - 4\mu}}{2} \times \left[\frac{c_1 \sinh\left(\frac{\xi}{2}\sqrt{\lambda^2 - 4\mu}\right) + c_2 \cosh\left(\frac{\xi}{2}\sqrt{\lambda^2 - 4\mu}\right)}{c_1 \cosh\left(\frac{\xi}{2}\sqrt{\lambda^2 - 4\mu}\right) + c_2 \sinh\left(\frac{\xi}{2}\sqrt{\lambda^2 - 4\mu}\right)} \right], \quad (15)$$

(ii) If $\lambda^2 - 4\mu < 0$, we have

$$\frac{G'(\xi)}{G(\xi)} = -\frac{\lambda}{2} + \frac{\sqrt{4\mu - \lambda^2}}{2} \times \left[\frac{-c_1 \sin\left(\frac{\xi}{2}\sqrt{4\mu - \lambda^2}\right) + c_2 \cos\left(\frac{\xi}{2}\sqrt{4\mu - \lambda^2}\right)}{c_1 \cos\left(\frac{\xi}{2}\sqrt{4\mu - \lambda^2}\right) + c_2 \sin\left(\frac{\xi}{2}\sqrt{4\mu - \lambda^2}\right)} \right] \quad (16)$$

(iii) If $\lambda^2 - 4\mu = 0$, we have

$$\frac{G'(\xi)}{G(\xi)} = -\frac{\lambda}{2} + \frac{c_1}{c_1 + c_2\xi}, \tag{17}$$

where c_1 and c_2 are arbitrary constants.

Step 6. We substitute the values of a_i, β, ℓ as well as the ratios (15)-(17) into (13) to obtain many exact solutions of Eq.(10).

4 Exact solutions of the DNA model (1)

In this section, we apply the proposed method of Sec. 3, to construct the exact solutions of the DNA double helices modeling (1). To this end, we use the wave transformation (11) to reduce Eq. (1) to the following ODE:

$$\ell^2(\beta^2 - c^2)y'' + \omega_0^2 y - \lambda_1^2 y^2 + \delta^2 y^3 + \eta\ell\beta y' = 0, \tag{18}$$

where $\beta^2 - c^2 \neq 0$. Balancing y'' with y^3 we get $m = 1$. Consequently, we have

$$y(\xi) = a_1\left(\frac{G'(\xi)}{G(\xi)}\right) + a_0 + a_{-1}\left(\frac{G'(\xi)}{G(\xi)}\right)^{-1} \tag{19}$$

where a_1, a_0, a_{-1} are constants to be determined , such that $a_1 \neq 0$ or $a_{-1} \neq 0$. Substituting (19) along with Eq. (14) into Eq. (18) and equating all the coefficients of $\left(\frac{G'}{G}\right)^i, (i = 0, \pm1, \pm2, \pm3)$ to zero, we obtain the following algebraic equations:

$\left(\frac{G'}{G}\right)^{-3}$: $2a_{-1}\mu^2\ell^2(\beta^2 - c^2) + \delta^2 a_{-1}^3 = 0$,
$\left(\frac{G'}{G}\right)^{3}$: $2a_1\ell^2(\beta^2 - c^2) + \delta^2 a_1^3 = 0$,
$\left(\frac{G'}{G}\right)^{-2}$: $3a_{-1}\mu\lambda\ell^2(\beta^2 - c^2) - \lambda_1^2 a_1^2 + 3a_0 a_{-1}^2\delta^2 + a_{-1}\mu\eta\ell\beta = 0$,
$\left(\frac{G'}{G}\right)^{2}$: $3a_1\lambda\ell^2(\beta^2 - c^2) - \lambda_1^2 a_1^2 + 3a_0 a_1^2\delta^2 - a_1\eta\ell\beta = 0$,

$\left(\frac{G'}{G}\right)^{-1}$: $(2a_{-1}\mu + a_{-1}\lambda^2)\ell^2(\beta^2 - c^2) + \omega_0^2 a_{-1} - 2a_0 a_{-1}\lambda_1^2 + \delta^2(3a_1 a_{-1}^2 + 3a_{-1}a_0^2) + a_{-1}\lambda\eta\ell\beta = 0$,
$\left(\frac{G'}{G}\right)$: $(a_1\lambda^2 + 2a_1\mu)\ell^2(\beta^2 - c^2) + \omega_0^2 a_1 - 2a_0 a_1\lambda_1^2 + \delta^2(3a_0^2 a_1 + 3a_{-1}a_1^2) - a_1\lambda\eta\ell\beta = 0$,

$\left(\frac{G'}{G}\right)^{0}$: $(a_1\mu\lambda + a_{-1}\lambda)\ell^2(\beta^2 - c^2) + \omega_0^2 a_0 - \lambda_1^2(a_0^2 + 2a_1 a_{-1}) + \delta^2(a_0^3 + 6a_0 a_{-1}a_1) + (a_{-1} - a_1\mu)\eta\ell\beta = 0$,

On solving the above algebraic equations using the Maple or Mathematica, we obtain the following cases:

Case 1.

$$a_0 = \frac{2\lambda_1^2}{3\delta^2},\ a_1 = \frac{-\lambda(9\omega_0^2 - 2\lambda_1^4)}{12\mu\delta^2\lambda_1^2},\ a_{-1} = \frac{2\mu\lambda_1^2}{3\lambda\delta^2},\ \eta = 0,$$
$$\lambda_1 = \lambda_1,\ c = c,\ \ell = \ell,$$
$$\beta = \frac{1}{18a_{-1}\delta^3\ell}\times \left[\sqrt{324a_{-1}^2\ell^2c^2\delta^6 - 2\lambda_1^8 + 18\lambda_1^2\delta^2\omega_0^2 - \frac{81}{2}\omega_0^4\delta^4}\right], \tag{20}$$

Case 2.

$$a_0 = \frac{(\omega_0 + \lambda\ell\sqrt{c^2 - \beta^2})}{\sqrt{2}\delta},\ a_1 = \frac{\ell\sqrt{2(c^2 - \beta^2)}}{\delta},\ a_{-1} = 0,$$

$$\mu = \frac{\ell^2\lambda^2(\beta^2 - c^2) + \omega_0^2}{4(\beta^2 - c^2)\ell^2}, \eta = 0, \lambda_1^2 = \frac{3}{\sqrt{2}}\omega_0\delta, c = c,\ \beta = \beta, \ell = \ell, \tag{21}$$

Case 3.

$$a_0 = \frac{(\lambda_1^4 + 2\lambda\eta\ell\beta\delta^2)}{2\lambda_1^2\delta^2},\ a_1 = \frac{2\eta\beta\ell}{\lambda_1^2},\ a_{-1} = 0,$$

$$\mu = \frac{(-\lambda_1^8 + 4\delta^2\omega_0^2\lambda_1^4 + 4\delta^4\eta^2\beta^2\lambda^2\ell^2)}{16\eta^2\beta^2\ell^2\delta^4}, \lambda_1 = \lambda_1,\ \beta = \beta,\ \ell = \ell,$$
$$\eta = \eta,\ \lambda = \lambda,\ c^2 = \frac{\beta^2}{\lambda_1^4}(\lambda_1^4 + 2\eta^2\delta^2). \tag{22}$$

4.1 Exact solutions of the DNA model (1) for case 1

Substituting (20) into (19) and using (15) - (17), we have the following exact solutions for the model (1):
(i)If $\lambda^2 - 4\mu > 0$ (Hyperbolic function solutions), we have the exact solution

$$y(\xi) = \frac{2\lambda_1^2}{3\delta^2} - \frac{\lambda(9\omega_0^2\delta^2 - 2\lambda_1^4)}{12\mu\delta^2\lambda_1^2}\times \left\{-\frac{\lambda}{2} + \frac{\sqrt{\lambda^2-4\mu}}{2}\left[\frac{c_1\sinh\left(\frac{\xi}{2}\sqrt{\lambda^2-4\mu}\right) + c_2\cosh\left(\frac{\xi}{2}\sqrt{\lambda^2-4\mu}\right)}{c_1\cosh\left(\frac{\xi}{2}\sqrt{\lambda^2-4\mu}\right) + c_2\sinh\left(\frac{\xi}{2}\sqrt{\lambda^2-4\mu}\right)}\right]\right\}$$
$$+ \frac{2\mu\lambda_1^2}{3\lambda\delta^2}\left\{-\frac{\lambda}{2} + \frac{\sqrt{\lambda^2-4\mu}}{2}\left[\frac{c_1\sinh\left(\frac{\xi}{2}\sqrt{\lambda^2-4\mu}\right) + c_2\cosh\left(\frac{\xi}{2}\sqrt{\lambda^2-4\mu}\right)}{c_1\cosh\left(\frac{\xi}{2}\sqrt{\lambda^2-4\mu}\right) + c_2\sinh\left(\frac{\xi}{2}\sqrt{\lambda^2-4\mu}\right)}\right]\right\}^{-1}. \tag{23}$$

Substituting the formulas (8), (10), (12) and (14) obtained in [33] into (23) we have respectively the following kink-type traveling wave solutions:
(1) If $|c_1| > |c_2|$, then

$$y_1(\xi) = \frac{2\lambda_1^2}{3\delta^2} - \frac{\lambda(9\omega_0^2\delta^2 - 2\lambda_1^4)}{12\mu\delta^2\lambda_1^2}\times \left\{-\frac{\lambda}{2} + \frac{\sqrt{\lambda^2-4\mu}}{2}\tanh\left(\frac{\xi}{2}\sqrt{\lambda^2-4\mu} + sgn(c_1c_2)\psi_1\right)\right\}$$
$$+ \frac{2\mu\lambda_1^2}{3\lambda\delta^2}\times \left\{-\frac{\lambda}{2} + \frac{\sqrt{\lambda^2-4\mu}}{2}\tanh\left(\frac{\xi}{2}\sqrt{\lambda^2-4\mu} + sgn(c_1c_2)\psi_1\right)\right\}^{-1}. \tag{24}$$

(2) If $|c_2| > |c_1| \neq 0$, then

$$y_2(\xi) = \frac{2\lambda_1^2}{3\delta^2} - \frac{\lambda(9\omega_0^2\delta^2 - 2\lambda_1^4)}{12\mu\delta^2\lambda_1^2} \times \left\{ -\frac{\lambda}{2} + \frac{\sqrt{\lambda^2-4\mu}}{2} \coth\left(\frac{\xi}{2}\sqrt{\lambda^2-4\mu} + sgn(c_1c_2)\psi_2\right) \right\} + \frac{2\mu\lambda_1^2}{3\lambda\delta^2} \times \left\{ -\frac{\lambda}{2} + \frac{\sqrt{\lambda^2-4\mu}}{2} \coth\left(\frac{\xi}{2}\sqrt{\lambda^2-4\mu} + sgn(c_1c_2)\psi_2\right) \right\}^{-1}. \tag{25}$$

(3) If $|c_2| > |c_1| = 0$, then

$$y_3(\xi) = \frac{2\lambda_1^2}{3\delta^2} - \frac{\lambda(9\omega_0^2\delta^2 - 2\lambda_1^4)}{12\mu\delta^2\lambda_1^2} \times \left\{ -\frac{\lambda}{2} + \frac{\sqrt{\lambda^2-4\mu}}{2} \coth\left(\frac{\xi}{2}\sqrt{\lambda^2-4\mu}\right) \right\} + \frac{2\mu\lambda_1^2}{3\lambda\delta^2} \left\{ -\frac{\lambda}{2} + \frac{\sqrt{\lambda^2-4\mu}}{2} \coth\left(\frac{\xi}{2}\sqrt{\lambda^2-4\mu}\right) \right\}^{-1}. \tag{26}$$

(4) If $|c_1| = |c_2|$, then

$$y_4(\xi) = \frac{2\lambda_1^2}{3\delta^2} - \frac{\lambda(9\omega_0^2\delta^2 - 2\lambda_1^4)}{12\mu\delta^2\lambda_1^2} \left\{ -\frac{\lambda}{2} + \frac{\sqrt{\lambda^2-4\mu}}{2} \right\} + \frac{2\mu\lambda_1^2}{3\lambda\delta^2} \left\{ -\frac{\lambda}{2} + \frac{\sqrt{\lambda^2-4\mu}}{2} \right\}^{-1}, \tag{27}$$

where $\psi_1 = \tanh^{-1}(\frac{|c_2|}{|c_1|})$, $\psi_2 = \coth^{-1}(\frac{|c_2|}{|c_1|})$ and $sgn(c_1c_2)$ is the sign function.

(ii)If $\lambda^2 - 4\mu < 0$ (Trigonometric function solutions), we have the exact solution

$$y(\xi) = \frac{2\lambda_1^2}{3\delta^2} - \frac{\lambda(9\omega_0^2\delta^2 - 2\lambda_1^4)}{12\mu\delta^2\lambda_1^2} \times \left\{ -\frac{\lambda}{2} + \frac{\sqrt{4\mu-\lambda^2}}{2} \left[\frac{-c_1\sin\left(\frac{\xi}{2}\sqrt{4\mu-\lambda^2}\right) + c_2\cos\left(\frac{\xi}{2}\sqrt{4\mu-\lambda^2}\right)}{c_1\cos\left(\frac{\xi}{2}\sqrt{4\mu-\lambda^2}\right) + c_2\sin\left(\frac{\xi}{2}\sqrt{4\mu-\lambda^2}\right)} \right] \right\} + \frac{2\mu\lambda_1^2}{3\lambda\delta^2} \times \left\{ -\frac{\lambda}{2} + \frac{\sqrt{4\mu-\lambda^2}}{2} \left[\frac{-c_1\sin\left(\frac{\xi}{2}\sqrt{4\mu-\lambda^2}\right) + c_2\cos\left(\frac{\xi}{2}\sqrt{4\mu-\lambda^2}\right)}{c_1\cos\left(\frac{\xi}{2}\sqrt{4\mu-\lambda^2}\right) + c_2\sin\left(\frac{\xi}{2}\sqrt{4\mu-\lambda^2}\right)} \right] \right\}^{-1}, \tag{28}$$

Now, simplify (28) to get the following periodic solutions:

$$y_1(\xi) = \frac{2\lambda_1^2}{3\delta^2} - \frac{\lambda(9\omega_0^2\delta^2 - 2\lambda_1^4)}{12\mu\delta^2\lambda_1^2} \times \left\{ -\frac{\lambda}{2} + \frac{\sqrt{4\mu-\lambda^2}}{2} \tan\left(\xi_0 - \frac{\xi}{2}\sqrt{4\mu-\lambda^2}\right) \right\} + \frac{2\mu\lambda_1^2}{3\lambda\delta^2} \left\{ -\frac{\lambda}{2} + \frac{\sqrt{4\mu-\lambda^2}}{2} \tan\left(\xi_1 - \frac{\xi}{2}\sqrt{4\mu-\lambda^2}\right) \right\}^{-1}, \tag{29}$$

and

$$y_2(\xi) = \frac{2\lambda_1^2}{3\delta^2} - \frac{\lambda(9\omega_0^2\delta^2 - 2\lambda_1^4)}{12\mu\delta^2\lambda_1^2} \times \left\{ -\frac{\lambda}{2} + \frac{\sqrt{4\mu-\lambda^2}}{2} \cot\left(\xi_0 - \frac{\xi}{2}\sqrt{4\mu-\lambda^2}\right) \right\} + \frac{2\mu\lambda_1^2}{3\lambda\delta^2} \left\{ -\frac{\lambda}{2} + \frac{\sqrt{4\mu-\lambda^2}}{2} \cot\left(\xi_2 + \frac{\xi}{2}\sqrt{4\mu-\lambda^2}\right) \right\}^{-1}, \tag{30}$$

where $\xi_1 = \tan^{-1}\left(\frac{c_2}{c_1}\right)$, $\xi_2 = \cot^{-1}\left(\frac{c_2}{c_1}\right)$ and $c_1^2 + c_2^2 \neq 0$.

(iii) If $\lambda^2 - 4\mu = 0$ (Rational function solutions),then we have

$$y(\xi) = \frac{2\lambda_1^2}{3\delta^2} - \frac{\lambda(9\omega_0^2\delta^2 - 2\lambda_1^4)}{12\mu\delta^2\lambda_1^2} \left\{ -\frac{\lambda}{2} + \frac{c_1}{c_1+c_2\xi} \right\} + \frac{2\mu\lambda_1^2}{3\lambda\delta^2} \left\{ -\frac{\lambda}{2} + \frac{c_1}{c_1+c_2\xi} \right\}^{-1}. \tag{31}$$

4.2 Exact solutions of the DNA model (1) for case 2

Substituting (21) into (19) and using (15) - (17), we have the following exact solutions for the model (1):

(i)If $\lambda^2 - 4\mu > 0$ (Hyperbolic function solutions),then we have the exact solution

$$y(\xi) = \frac{1}{\sqrt{2}\delta}(\omega_0 + \lambda\ell\sqrt{c^2-\beta^2}) + \frac{\ell\sqrt{2(c^2-\beta^2)}}{\delta} \times \left\{ -\frac{\lambda}{2} + \frac{\sqrt{\lambda^2-4\mu}}{2} \left[\frac{c_1\sinh\left(\frac{\xi}{2}\sqrt{\lambda^2-4\mu}\right) + c_2\cosh\left(\frac{\xi}{2}\sqrt{\lambda^2-4\mu}\right)}{c_1\cosh\left(\frac{\xi}{2}\sqrt{\lambda^2-4\mu}\right) + c_2\sinh\left(\frac{\xi}{2}\sqrt{\lambda^2-4\mu}\right)} \right] \right\} \tag{32}$$

Substituting the formulas (8), (10), (12) and (14) obtained in [33] into (32) we have respectively the following kink-type traveling wave solutions:

(1) If $|c_1| > |c_2|$, then

$$y_1(\xi) = \frac{1}{\sqrt{2}\delta}(\omega_0 + \lambda\ell\sqrt{c^2-\beta^2}) + \frac{\ell\sqrt{2(c^2-\beta^2)}}{\delta} \times \left\{ -\frac{\lambda}{2} + \frac{\sqrt{\lambda^2-4\mu}}{2} \tanh\left(\frac{\xi}{2}\sqrt{\lambda^2-4\mu} + sgn(c_1c_2)\psi_1\right) \right\}, \tag{33}$$

(2) If $|c_2| > |c_1| \neq 0$, then

$$y_2(\xi) = \frac{1}{\sqrt{2}\delta}(\omega_0 + \lambda\ell\sqrt{c^2-\beta^2}) + \frac{\ell\sqrt{2(c^2-\beta^2)}}{\delta} \times \left\{ -\frac{\lambda}{2} + \frac{\sqrt{\lambda^2-4\mu}}{2} \coth\left(\frac{\xi}{2}\sqrt{\lambda^2-4\mu} + sgn(c_1c_2)\psi_2\right) \right\}. \tag{34}$$

(3) If $|c_2| > |c_1| = 0$, then

$$y_3(\xi) = \frac{1}{\sqrt{2}\delta}(\omega_0 + \lambda\ell\sqrt{c^2-\beta^2}) + \frac{\ell\sqrt{2(c^2-\beta^2)}}{\delta} \times \left\{ -\frac{\lambda}{2} + \frac{\sqrt{\lambda^2-4\mu}}{2} \coth\left(\frac{\xi}{2}\sqrt{\lambda^2-4\mu}\right) \right\}. \tag{35}$$

(4) If $|c_1| = |c_2|$, then

$$y_4(\xi) = \frac{1}{\sqrt{2}\delta}(\omega_0 + \lambda\ell\sqrt{c^2-\beta^2}) + \frac{\ell\sqrt{2(c^2-\beta^2)}}{\delta} \times \left\{ -\frac{\lambda}{2} + \frac{\sqrt{\lambda^2-4\mu}}{2} \right\}. \tag{36}$$

where $\psi_1 = \tanh^{-1}(\frac{|c_2|}{|c_1|})$, $\psi_2 = \coth^{-1}(\frac{|c_2|}{|c_1|})$ and $sgn(c_1c_2)$ is the sign function.

Remark: On comparing our result (33) with the well-known result (18) of Ref. [31] we deduce the following results:

(1) Setting $\omega_0 = \ell\sqrt{c^2-\beta^2}$ and $sgn(c_1c_2) = 0$, the solitary wave solution (33) can be simplified to become

$$y_1(\xi) = \Gamma + \Gamma \tanh\left(\frac{\ell}{2}(x+\beta t+k)\right) \tag{37}$$

where $\Gamma = \frac{\ell}{\sqrt{2\delta}}\sqrt{c^2-\beta^2}$.

(2) If we set $\alpha = 0$ and $\eta = 0$ in the formulas (17)-(19) of Ref. [31] we have the same formula (37) while Γ is given by $\Gamma = \pm\frac{3\ell}{\sqrt{2\delta}}\sqrt{c^2-\beta^2}$. Here Γ is called the amplitude. When $\Gamma > 0$ we get the kink solution, while if $\Gamma < 0$, we get the antikink solution.

(ii)If $\lambda^2 - 4\mu < 0$ (Trigonometric function solutions),then we have the exact solution

$$y(\xi) = \frac{1}{\sqrt{2\delta}}(\omega_0 + \lambda\ell\sqrt{c^2-\beta^2}) + \frac{\ell\sqrt{2(c^2-\beta^2)}}{\delta}\times \left\{-\frac{\lambda}{2} + \frac{\sqrt{4\mu-\lambda^2}}{2}\left[\frac{-c_1\sin\left(\frac{\xi}{2}\sqrt{4\mu-\lambda^2}\right)+c_2\cos\left(\frac{\xi}{2}\sqrt{4\mu-\lambda^2}\right)}{c_1\cos\left(\frac{\xi}{2}\sqrt{4\mu-\lambda^2}\right)+c_2\sin\left(\frac{\xi}{2}\sqrt{4\mu-\lambda^2}\right)}\right]\right\}. \tag{38}$$

Now, simplify (38) to get the following periodic solutions:

$$y_1(\xi) = \frac{1}{\sqrt{2\delta}}(\omega_0 + \lambda\ell\sqrt{c^2-\beta^2}) + \frac{\ell\sqrt{2(c^2-\beta^2)}}{\delta}\times \left\{-\frac{\lambda}{2} + \frac{\sqrt{4\mu-\lambda^2}}{2}\tan\left(\xi_1 - \frac{\xi}{2}\sqrt{4\mu-\lambda^2}\right)\right\}, \tag{39}$$

and

$$y_2(\xi) = \frac{1}{\sqrt{2\delta}}(\omega_0 + \lambda\ell\sqrt{c^2-\beta^2}) + \frac{\ell\sqrt{2(c^2-\beta^2)}}{\delta}\times \left\{-\frac{\lambda}{2} + \frac{\sqrt{4\mu-\lambda^2}}{2}\cot\left(\xi_2 + \frac{\xi}{2}\sqrt{4\mu-\lambda^2}\right)\right\}, \tag{40}$$

where $\xi_1 = \tan^{-1}\left(\frac{c_2}{c_1}\right)$, $\xi_2 = \cot^{-1}\left(\frac{c_2}{c_1}\right)$ and $c_1^2 + c_2^2 \neq 0$.

(iii)If $\lambda^2 - 4\mu = 0$ (Rational function solutions),then we have

$$y(\xi) = \frac{1}{\sqrt{2\delta}}(\omega_0 + \lambda\ell\sqrt{c^2-\beta^2}) + \frac{\ell\sqrt{2(c^2-\beta^2)}}{\delta}\times \left\{-\frac{\lambda}{2} + \frac{c_1}{c_1+c_2\xi}\right\}. \tag{41}$$

4.3 Exact solutions of the DNA model (1) for case 3

Substituting (22) into (19) and using (15) - (17), we have the following exact solutions for the model (1):

(i)If $\lambda^2 - 4\mu > 0$ (Hyperbolic function solutions),then we have the exact solution

$$y(\xi) = \frac{2\eta\beta\ell}{\lambda_1^2}\times \left\{-\frac{\lambda}{2} + \frac{\sqrt{\lambda^2-4\mu}}{2}\left[\frac{c_1\sinh\left(\frac{\xi}{2}\sqrt{\lambda^2-4\mu}\right)+c_2\cosh\left(\frac{\xi}{2}\sqrt{\lambda^2-4\mu}\right)}{c_1\cosh\left(\frac{\xi}{2}\sqrt{\lambda^2-4\mu}\right)+c_2\sinh\left(\frac{\xi}{2}\sqrt{\lambda^2-4\mu}\right)}\right]\right\} + \frac{1}{2\lambda_1^2\delta^2}(\lambda_1^4 + 2\lambda\ell\eta\beta\delta^2) \tag{42}$$

Substituting the formulas (8), (10), (12) and (14) obtained in [33] into (42) we have respectively the following kink-type traveling wave solutions:

(1) If $|c_1| > |c_2|$, then

$$y_1(\xi) = \frac{1}{2\lambda_1^2\delta^2}(\lambda_1^4 + 2\lambda\ell\eta\beta\delta^2) + \frac{2\eta\beta\ell}{\lambda_1^2}\times \left\{-\frac{\lambda}{2} + \frac{\sqrt{\lambda^2-4\mu}}{2}\tanh\left(\frac{\xi}{2}\sqrt{\lambda^2-4\mu} + sgn(c_1c_2)\psi_1\right)\right\}, \tag{43}$$

(2) If $|c_2| > |c_1| \neq 0$, then

$$y_2(\xi) = \frac{1}{2\lambda_1^2\delta^2}(\lambda_1^4 + 2\lambda\ell\eta\beta\delta^2) + \frac{2\eta\beta\ell}{\lambda_1^2}\times \left\{-\frac{\lambda}{2} + \frac{\sqrt{\lambda^2-4\mu}}{2}\coth\left(\frac{\xi}{2}\sqrt{\lambda^2-4\mu} + sgn(c_1c_2)\psi_2\right)\right\}. \tag{44}$$

(3) If $|c_2| > |c_1| = 0$, then

$$y_3(\xi) = \frac{1}{2\lambda_1^2\delta^2}(\lambda_1^4 + 2\lambda\ell\eta\beta\delta^2) + \frac{2\eta\beta\ell}{\lambda_1^2}\times \left\{-\frac{\lambda}{2} + \frac{\sqrt{\lambda^2-4\mu}}{2}\coth\left(\frac{\xi}{2}\sqrt{\lambda^2-4\mu}\right)\right\}. \tag{45}$$

(4) If $|c_2| = |c_1|$, then

$$y_4(\xi) = \frac{1}{2\lambda_1^2\delta^2}(\lambda_1^4 + 2\lambda\ell\eta\beta\delta^2) + \frac{2\eta\beta\ell}{\lambda_1^2}\times \left\{-\frac{\lambda}{2} + \frac{\sqrt{\lambda^2-4\mu}}{2}\right\}. \tag{46}$$

where $\psi_1 = \tanh^{-1}(\frac{|c_2|}{|c_1|})$, $\psi_2 = \coth^{-1}(\frac{|c_2|}{|c_1|})$ and $sgn(c_1c_2)$ is the sign function.

(ii)If $\lambda^2 - 4\mu < 0$ (Trigonometric function solutions),then we have the exact solution

$$y(\xi) = \frac{2\eta\beta\ell}{\lambda_1^2}\times \left\{-\frac{\lambda}{2} + \frac{\sqrt{4\mu-\lambda^2}}{2}\left[\frac{-c_1\sin\left(\frac{\xi}{2}\sqrt{4\mu-\lambda^2}\right)+c_2\cos\left(\frac{\xi}{2}\sqrt{4\mu-\lambda^2}\right)}{c_1\cos\left(\frac{\xi}{2}\sqrt{4\mu-\lambda^2}\right)+c_2\sin\left(\frac{\xi}{2}\sqrt{4\mu-\lambda^2}\right)}\right]\right\} + \frac{1}{2\lambda_1^2\delta^2}(\lambda_1^4 + 2\lambda\ell\eta\beta\delta^2) \tag{47}$$

Now, simplify (47) to get the following periodic solutions:

$$y_1(\xi) = \frac{1}{2\lambda_1^2\delta^2}(\lambda_1^4 + 2\lambda\ell\eta\beta\delta^2) + \frac{2\eta\beta\ell}{\lambda_1^2}\times \left\{-\frac{\lambda}{2} + \frac{\sqrt{4\mu-\lambda^2}}{2}\tan\left(\xi_1 - \frac{\xi}{2}\sqrt{4\mu-\lambda^2}\right)\right\}, \tag{48}$$

and

$$y_2(\xi) = \frac{1}{2\lambda_1^2\delta^2}(\lambda_1^4 + 2\lambda\ell\eta\beta\delta^2) + \frac{2\eta\beta\ell}{\lambda_1^2}\times \left\{-\frac{\lambda}{2} + \frac{\sqrt{4\mu-\lambda^2}}{2}\cot\left(\xi_2 + \frac{\xi}{2}\sqrt{4\mu-\lambda^2}\right)\right\}, \tag{49}$$

where $\xi_1 = \tan^{-1}\left(\frac{c_2}{c_1}\right)$, $\xi_2 = \cot^{-1}\left(\frac{c_2}{c_1}\right)$ and $c_1^2 + c_2^2 \neq 0$.

5 Some conclusions and discussions

In this article, we have employed the improved (G'/G)-expansion method described in Sec. 3, to find many exact solutions as well as many solitary wave solutions and many periodic solutions (23)-(49) of the nonlinear PDE (1) describing the DNA double helices modeling which look new. This model has been discussed in [31] using the homogeneous balance method, where some solitary wave solutions have been found. On comparing our new results (23)-(49) with that obtained in [31] we conclude that the improved (G'/G)- expansion method used in this article is more effective and giving more exact solutions than the homogeneous balance method used in [31]. According to our knowledge, we deduce that the DNA model (1) and its solutions (23)-(49) have not been discussed elsewhere using the improved (G'/G)- expansion method. Finally, all the solutions of Eq. (1) obtained in this article have been checked with the Maple by putting them back into the original Eq. (1).

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