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The Modified Interior Point Algorithm for Linear Optimization

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In this paper, we describe a new method for finding search directions for interior point methods (IPMs) in linear optimization (LO). The theoretical complexity of the new algorithms are calculated and we prove that the iteration bound is $O(\log(n/\epsilon))$ in this case too.

Keywords: Linear programming, interior point methods, path-following, Newton's method, central path, search-direction.

1 Introduction

In this paper we discuss interior point methods (IPMs) for solving linear optimization (LO) problems. Linear optimization is an area of mathematical programming which deals with the minimization or maximization of a linear function, subject to linear constraints. These constrains can be expressed by equalities or inequalities. Dantzig proposed the well-known simplex method for solving LO problems in 1947. The simplex method has been continuously improved in the past fifty years. For different variants of the simplex method there are constructed examples illustrating that in the worst case the number of iterations required by the algorithm can be exponential. The first polynomial algorithm for solving LO problems is the ellipsoid method of Khachiyan in 1979. This method is important from a theoretical point of view, but is not so efficient in practice. An alternative variant was defined by Karmakar in 1984. His algorithm uses interior points of the polytope of to approximate the optimal solution. The complexity of this algorithm is smaller than Khachiyan's and the implementation of Karmakar's algorithm proved to be efficient in practice too, especially when the size of the problem is large. Interior point algorithms were studied by Frish in 1955, and Gill exhibited barrier algorithm of Frish algorithm communicate with Karmakar's algorithm perfectly. In the past twenty years, some papers have been written about interior point algorithms. For an overview of results see Todd (1986), Gonzaga (1989), Wright (1996), Ye (1997), and Darvay (2002).

In section 2 we give path-following algorithms and in section 3 we show the Newton's method. We propose a new search-directions in sections 4 and 5. In section 6 we conduct the complexity analysis of the algorithm, and then the conclusion is given in section 7.

2 Path-Following Algorithms

We consider the LP problem in the following standard form

$$\begin{array}{ll} \min & c^T x \\ s.t & Ax = b, \\ & x \geq 0. \end{array} \tag{2.1}$$

where $A \in R^{m \times n}$ with rank$(A) = m,\ b \in R^m$ and $c \in R^n$. The dual of this problem can be writhen in the following form

$$\begin{array}{ll} \max & b^T y \\ s.t & A^T y + s = c, \\ & s \geq 0. \end{array} \tag{2.2}$$

We assume that the interior point condition (IPC) holds for these problems, i.e., there exist (x^o, y^o, s^o) such that

$$\begin{aligned} Ax^o = b, &\quad x^o \geq 0, \\ A^T y^o + s^o = c, &\quad s^o \geq 0. \end{aligned}$$

and

$$\begin{aligned} F_P &= \{x \in R^n_+ :\ Ax = b\}, \\ F_D &= \{(y, s) \in R^m \times R^n_+ :\ A^T y + s = c\}, \end{aligned}$$

where F_P and F_D are solution sets of primal and dual problems, respectively. Their interiors are as the follows:

$$\begin{aligned} F^o_P &= \{x \in R^n_{++} :\ Ax = b\}, \\ F^o_D &= \{(y, s) \in R^m \times R^n_{++} :\ A^T y + s = c\}, \end{aligned}$$

where $R_+ = \{x \in R,\ x \geq 0\}$ and $R_{++} = \{x \in R,\ x > 0\}$.

Using the self-dual embedding method, a larger LO problem can be constructed in such a way that the IPC holds for that problem (Darvay, 2002). Hence, the IPC can be assumed without loss of generality. Finding the optimal solutions of both the primal-dual problems, is equivalent to solving the following system:

$$\begin{aligned} Ax &= b, \quad x \geq 0, \\ A^T y + s &= c, \quad s \geq 0, \\ xs &= 0, \end{aligned} \tag{2.3}$$

where xs denotes the coordinatewise product of the vectors x and s.

The first condition of system (2.3) merely states that the candidate point must be feasible; that is, it must satisfy the constraints of the problem. This is usually referred to as primal feasibility. The second condition is usually referred to as dual feasibility. The last relation is the complementarity condition. Primal-dual IPMs generally replace the complementarity condition by a parameterized equation, for example

$$\begin{aligned} Ax &= b, \quad x \geq 0, \\ A^T y + s &= c, \quad s \geq 0, \\ xs &= \mu e, \end{aligned} \tag{2.4}$$

where $\mu > 0$ and $e = [1, 1, \ldots, 1]^T$. If the IPC is satisfied, then for a fix $\mu > 0$ the system (2.4) has a unique solution that can be obtained from the convex problem as

$$\begin{aligned} \min \quad & c^T x - \mu \sum_{j=0}^{n} \ln\, x_j \\ s.t \quad & Ax = b, \\ & x \geq 0. \end{aligned} \tag{2.5}$$

This solution is called the $\mu-$center, and the set of $\mu-$center for $\mu > 0$ forms the central path. The target-following approach starts from the observation that the system (2.4) can be generalized by replacing the vector μe with an arbitrary positive vector w^2. Thus we have the system

$$\begin{aligned} Ax &= b, \quad x \geq 0, \\ A^T y + s &= c, \quad s \geq 0, \\ xs &= w^2, \end{aligned} \tag{2.6}$$

where $w \geq 0$.

3 The Newton's Method

Let $F : R^n \rightarrow R^n$ be a continuously differentiable function, and let $J(x)$ be the Jacobi matrix attached to F. Consider the system

$$F(x) = 0.$$

Suppose there is the vector x^o, then a sequence of points is obtained by

$$x^{k+1} = x^k - J(x^k)^{-1} F(x^k).$$

If Δx^k is a step direction vector, then

$$x^{k+1} = x^k + \Delta x^k$$

and

$$J(x^k)\Delta x^k = -F(x^k).$$

If x^o is sufficiently near to a solution of F, then this sequence is convergent. The analysis of the Newton's method is very important from the point of view of IPMs.

4 A New Class of Directions

In this section we define a new method for finding search directions for IPMs. We consider the function

$$\varphi \in C^1,\ \varphi : R_+ \to R_+.$$

Furthermore, we suppose that the inverse function φ^{-1} exists. Then, the system (2.6) can be written in the following equivalent form:

$$\begin{array}{rll} Ax & = b, & x \geq 0, \\ A^T y + s & = c, & s \geq 0, \\ \varphi(xs) & = \varphi(w^2). & \end{array} \tag{4.1}$$

And we can apply the Newton's method to the system (4.1) to obtain a new class of search directions. We mention that a direct generalization of the approach defined in (4.1) would be the following variant. The system (2.4) is equivalent to

$$Ax = b, \quad x \geq 0,$$

$$A^T y + s = c, \quad s \geq 0, \tag{4.2}$$

$$\varphi\left(\frac{xs}{\mu}\right) = \varphi(e). \tag{4.3}$$

If $t = 1/\mu$ the above system is equivalent to

$$\begin{array}{rll} Ax & = b, & x \geq 0, \\ A^T y + s & = c, & s \geq 0, \\ \varphi(txs) & = \varphi(e). & \end{array} \tag{4.4}$$

And using Newton's method in the system (4.3) yields new search directions. For our purpose the first approach is more convenient, so in this paper we use the system (4.3). Let us introduce the vector

$$v = \sqrt{\frac{xs}{\mu}} = \sqrt{txs}.$$

Suppose that we have $Ax = b$ and $A^T y + s = c$ for a triplet (x, y, s) such that $x \geq 0$ and $s \geq 0$, then x and s are strictly feasible. Applying Newton's method to the system (4.3) we obtain

$$\begin{aligned} A\Delta x &= 0, \\ A^T \Delta y + \Delta s &= 0, \\ ts\acute{\varphi}(txs)\Delta x + tx\acute{\varphi}(txs)\Delta s &= \varphi(e) - \varphi(txs), \\ s\Delta x + x\Delta s &= \frac{\varphi(e) - \varphi(txs)}{t\acute{\varphi}(txs)}. \end{aligned} \tag{4.5}$$

Furthermore, we denote

$$d_x = \frac{v\Delta x}{x},\ d_s = \frac{v\Delta s}{s},$$

and so we have

$$\mu v(d_x + d_s) = \mu v(\frac{v\Delta x}{x} + \frac{v\Delta s}{s}) = \frac{\mu v^2 \Delta x}{x} + \frac{\mu v^2 \Delta s}{s} = s\Delta x + x\Delta s \tag{4.6}$$

and

$$d_x d_s = \frac{v^2 \Delta x \Delta s}{xs} = \frac{\Delta x \Delta s}{\mu} = t\Delta x \Delta s, \tag{4.7}$$

as $v^2 = xs/\mu$. Hence the system (4.4) can be written in the following equivalent form:

$$\bar{A} d_x = 0, \tag{4.8}$$

$$t\bar{A}\Delta y + d_s = 0, \tag{4.9}$$

$$d_x + d_s = p_v, \tag{4.10}$$

where

$$p_v = \frac{\varphi(e) - \varphi(v^2)}{v\acute{\varphi}(v^2)} \tag{4.11}$$

and $\bar{A} = A \cdot \text{diag}(d)$ with the notation

$$\text{diag}(\xi) = \begin{bmatrix} \xi_1 & 0 & \dots & 0 \\ 0 & \xi_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \xi_n \end{bmatrix}$$

for any vector ξ.

Darvay considered that $\varphi(x) = x$, and $\varphi(x) = \sqrt{x}$ yielded $p_v = v^{-1} - v$ and $p_v = 2(e - v)$, respectively, and he obtained the standard primal-dual algorithm (Darvay, 2002). Peng *et al.* 2002 observed that a new search direction can be obtained by taking $p_v = v^{-3} - v$. The same authors analysed in 2001 the case $p_v = v^{-q} - v$, where $q > 1$. The authors have also introduced a class of search directions based on self-regular proximities (Peng *et al.* 2000). For $\varphi(x) = x^2$ Darvay got $p_v = (v^{-3} - v)/2$, and for $\varphi(x) = x^{(q+1)/2}$, where $q > 1$ he obtained $p_v = 2(v^{-q} - v)/(q + 1)$. Our general approach can be particularized in such a way as to obtain the directions defined in above papers. In the following section we use a different function to develop a new primal-dual algorithm.

5 A New Primal-Dual Algorithm

In this section we take $\varphi(x) = x^{(q+1)/q},\ q \geq 1$ and we develop a new primal-dual weighted-path-following algorithm based on the appropriate search directions. Thus, making the substitution $\varphi(x) = x^{(q+1)/q}$ in (4.11) we get

$$p_v = \frac{q}{q+1}\left(v^{(-2+q)/q} - v\right) = \frac{q}{q+1}\left(v^{-(2+q)/q} - v\right). \tag{5.1}$$

And we can define a proximity measure to the central path by

$$\sigma\left(x, \frac{1}{t}\right) = \frac{\|p_v\|}{2},$$

where $\|.\|$ denotes the Euclidean norm (l_2 norm). Furthermore, let us introduce the notation

$$q_v = d_x - d_s.$$

From (4.8), (4.9) and (4.10) we get $d_x^T \cdot d_s = 0$. Hence the vectors d_x and d_s are orthogonal, and thus we obtain

$$\|p_v\| = \|q_v\|.$$

Therefore the proximity measure can be written in the form

$$\sigma\left(x, \frac{1}{t}\right) = \frac{\|q_v\|}{2}.$$

Moreover, we have

$$d_x = \frac{p_v + q_v}{2}, \qquad d_s = \frac{p_v - q_v}{2}, \qquad d_x d_s = \frac{p_v^2 - q_v^2}{4}. \tag{5.2}$$

The algorithm can be defined as follows:

Algorithm 5.1. *Let $\epsilon \geq 0$ be the accuracy parameter and $0 < \theta < 1$ the update parameter (default $\theta = 1/(2\sqrt{n})$).*

begin
    $x := e;\ t := 1;$
    while $n/t \geq \epsilon$ **do begin**
        $t := \frac{1}{1-\theta} t;$
        Compute Δx using (4.5) and $\varphi(x) = x^{(q+1)/q}$;
        $x := x + \Delta x;$
    end
end.

In the next section we shall prove that this algorithm is well defined for the default value of θ, and we will also give an upper bound for the number of iterations performed by the algorithm.

6 Complexity Analysis

From the self-dual property of the problem (SP) it follows that the duality gap is

$$2(q^T x) = 2(x^T s),$$

where x is a feasible solution of (SP), and $s = s(x)$ is the appropriate slack vector. For simplicity we also refer to $x^T s$ as the duality gap.

Remark 6.1. Let $\sigma = \sigma(x, 1/t),\ \|v\| < 1$, and introduce the vectors x_+ and s_+ such that $x_+ = x + \Delta x$ and $s_+ = s + \Delta s$. Then we have

$$(x_+)^T s_+ \leq \frac{1}{t}\left(\frac{4(q+1)}{q}\sigma^2\right).$$

In the following lemma we discuss the question of the bound on the number of iterations.

Lemma 6.1. *Let x^k be the k-th iteration of algorithm (2.1), and let $s^k = s(x^k)$ be the appropriate slack vector. Then $(x^k)s^k < \epsilon$ for*

$$k \geq \left(\frac{1}{1-\theta}\log\frac{n}{\epsilon}\right).$$

Proof. Using lemma (2) we find that

$$(x^k)^T s^k \leq \left(\frac{1}{t}\right)^k \left(\frac{4(q+1)}{q}\sigma^2\right),$$

and with suitable selection of σ, we obtain

$$(x^k)^T s^k \leq \frac{n}{(1-\theta)^k}.$$

Thus the inequality $(x^k)^T s^k \leq \epsilon$ is satisfied if

$$\frac{n}{(1-\theta)^k} < \epsilon.$$

Now taking logarithms, we may write

$$\log(n) - k\ \log(1-\theta) \leq \log\ \epsilon,$$

and using the equation $\log(1-\theta) \leq (1-\theta)$ we observe that the above inequality holds if

$$\log(n) - \log\ \epsilon \leq k\ \log(1-\theta) \leq k(1-\theta),$$

or

$$k \geq \frac{1}{1-\theta}\log\frac{n}{\epsilon}.$$

Thus, the proof is complete $\square$

For $\theta = 1/(2\sqrt{n})$ we obtain the following Theorem.

Theorem 6.1. *Let $\theta = 1/(2\sqrt{n})$. Then algorithm (2.1) requires at most*

$$o\left(\log \frac{n}{\epsilon}\right)$$

iterations

7 Conclusion

In this paper we have developed a new class of search directions for the linear optimization problem. For this purpose we have introduced a function φ, and we have used Newton's method to define new search directions. For $\varphi(x) = x^{(q+1)/q}$ these results can be used to introduce a new primal-dual polynomial algorithm for solving (SP). We have proved that the complexity of this algorithm is $o(\log(n/\epsilon))$. It is clear that the complexity of this algorithm is less than $o(\sqrt{n}\ \log(n/\epsilon))$.

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References

[1] N. Andrei, *Mathematical Programming: Interior Point Methods*, Editura Tehnica, Bucuresti, 1999. (In Romanian).

[2] Zs. Darvay, *Interior Point Methods in Linear Programming*, ELTE, Budapest, 1997. (In Hungarian).

[3] Zs. Darvay, A new algorithm for solving self-dual linear optimization problems, *Studia Universitatis Babes-Bolyai, Series Informatica* **47** (2002), 15–26.

[4] Zs. Darvay, A weighted-path-following method for linear optimization, *Studia Universitatis Babes-Bolyai, Series Informatica* **47** (2002), 3–12.

[5] C. C. Gonzaga, *An algorithm for solving linear programming problems in $O(n^3\ L)$ operations*, in: N. Megiddo, editor, Progress in Mathematical Programming: Interior Point and Related Methods, pages 1-28. Springer Verlag, New York, 1989.

[6] C. C. Gonzaga, Search directions for interior linear programming, *Algorithmica* **6** (1991), 153–181.

[7] N. K. Karmakar, A new polynomial-time algorithm for linear programming, *Combinatorica* **4** (1984), 373–395.

[8] L. G. Khachiyan, A polynomial algorithm for linear programming, *Soviet Math. Dokl.* **20** (1979), 191–194.

[9] M. Kojima, N. Megiddo, T. Noma, and A. Yoshise, *A unified approach to interior point algorithms for linear programming,* in: N. Megiddo, editor, Progress in Mathematical Programming: Interior Point and Related Methods, pages 29-47. Springer Verlag, New York, 1989.

[10] J. Peng, C. Roos, and T. Terlaky, *Self-Regular Proximities and New Search Directions for Linear and Semidefinite Optimization,* Technical report, Department of Computing and Software, McMaster University, Hamilton, Ontario, Canada, 2000.

[11] J. Peng, C. Roos, and T. Terlaky, A new and efficient large-update interior-point method for linear optimization, *Journal of Computational Technologies* **6** (2001), 61–80.

[12] J. Peng, C. Roos, and T. Terlaky, A new class of polynomial primal-dual methods for linear and semidefinite optimization, *European Journal of Operational Research* **143** (2002), 234–256.

[13] C. Roos and J.-PH. Vial, A polynomial method of approximate centers for linear programming, *Mathematical Programming* **54** (1992), 295–305.

[14] M. J. Todd and B. P. Burrell, An extension of Karmakar's algorithm for linear programming using dual variables, *Algorithmica* **1**(1986), 409–424.

[15] S. J. Wright and D. Ralph, A superlinear infeasible-interior-point algorithm for monotone complementarity problems, *Mathematics of operations Research* **21** (1996), 815–838.

[16] Y. Ye, Karmakar's algorithm and the ellipsoid method, *Operations Research Letters* **6** (1987), 177–182.

[17] Y. Ye, *Interior Point Algorithms, Theory and Analysis*, John Wiley & Sons, Chichester, UK, 1997.

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