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Periodic, hyperbolic and rational function solutions of nonlinear wave equations

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Abstract: In this paper, we implemented the functional variable method for the exact solutions of the Harry-Dym, the modified Zakharov-Kuznetsov and the Kadomtsev-Petviashvili equations. By using this scheme, we found some exact solutions of the above-mentioned equations. Then, some of the solitary solutions are converted to periodic solutions or hyperbolic function solutions by a simple transformation.

Keywords: The functional variable method, Harry-Dym equation, modified Zakharov-Kuznetsov equation, Kadomtsev-Petviashvili equations

1 Introduction

Since the soliton phenomena were first observed by John Scott Russell in 1834 and the KdV equation was solved by the inverse scattering method in 1967, finding exact solutions of nonlinear wave equations has become one of the most exciting and active areas of research investigation. The investigation into the exact solutions of nonlinear evolution equations plays an important role in mathematics, physics, and other applied science areas. The aspect of integrability of nonlinear evolution equations has recently attracted a lot of attention in nonlinear science and theoretical physics. Among these methods, inverse scattering method [1,2], Hirota bilinear tranformation [3], the tanh-sech method [4,5,6,7], sine-cosine method [8,9], Exp-function method [10,11,12,13] and $\left(\frac{G'}{G}\right)$-expansion method [14,15,16,17] were used to develop nonlinear dispersive and dissipative problems.

With the development of computer science, directly searching for solutions of nonlinear wave equations has become more and more attractive. This is due to the availability of computers symbolic system like MAPLE which allows us to perform some complicated and tedious algebraic calculation on a computer and help us to find new exact solutions of nonlinear evolution equations.

The existence of soliton- type solutions for nonlinear partial differential equations (PDEs) is of particular interest because of their extensive applications in many physics areas such as nonlinear optics, plasmas, fluid mechanics, condensed matter, electro magnetics and many more. Envelope solitons are stable nonlinear wave packets that preserve their shape when propagating in a nonlinear dispersive medium. It is also of interest to note that the formation of this type of pulses is due to an exact balance between nonlinearity and dispersion effects.

The paper is arranged as follows. In section 2, we describe functional variable method for finding exact travelling wave solutions of nonlinear evolution equations. In section 3 to section 5, we illustrate this method in detail with celebrated the Harry-Dym, modified Zakharov-Kuznetsov and Kadomtsev-Petviashvili equations. Finally, some conclusions are given.

2 The functional variable method

Zerarka et.al. have summarized for using functional variable method [18,19]. For a given nonlinear partial differential equation (PDE), written in several independent variables as

$$P(u,u_t,u_x,u_y,u_z,u_{xy},u_{yz},u_{xz},...)=0. \qquad (2.1)$$

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where the subscript denotes partial derivative, P is some function, and $u\{t,x,y,z,...\}$ is called a dependent variable or unknown function to be determined.

We firstly introduce the new wave variable as $\xi = k(x+ct)$ or $\xi = x-ct$.

The nonlinear partial differential equation can be converted to an ordinary differential equation (ODE) like

$$Q(U,U_\xi,U_{\xi\xi},U_{\xi\xi\xi},U_{\xi\xi\xi\xi},...) = 0. \tag{2.2}$$

Let us make a transformation in which the unknown function U is consedered as a functional variable in the form

$$U_\xi = F(U) \tag{2.3}$$

and some successively derivatives of U are

$$\begin{aligned} U_{\xi\xi} &= \tfrac{1}{2}(F^2)', \\ U_{\xi\xi\xi} &= \tfrac{1}{2}(F^2)''\sqrt{F^2}, \\ U_{\xi\xi\xi\xi} &= \tfrac{1}{2}\left[(F^2)'''F^2+(F^2)''(F^2)'\right], \\ &\vdots \end{aligned} \tag{2.4}$$

where "′" stands for $\frac{d}{dU}$. The ODE (2.2) can be reduced in terms of U,F and its derivatives upon using the expressions of (2.4) into (2.2) gives

$$R(U,F,F',F'',F''',F^{(4)},...) = 0 \tag{2.5}$$

The key idea of this particular form (2.5) is of special interest because it admits analytical solutions for a large class of nonlinear wave type equations. After integration, the Eq.(2.5) provides the expression of F, and this in turn together with (2.3) give the appropriate solutions to the original problem.

In order to illustrate how the method works we examine some examples treated by other approaches. This matter is exposed in the following section.

3 The Harry-Dym equation

Consider the Harry-Dym equation [20]

$$u_t = u^3 u_{xxx}. \tag{3.1}$$

The Harry–Dym (HD) equation that has nonlinearity and dispersion coupled together was discovered by H. Dym in 1973–1974 while its first appearance in the literature occurred in a 1975 paper of Kruskal, [21] where it was named after its discoverer. It arises, e.g., in the analysis of the Saffman–Taylor problem with surface tension [22].

Using the wave variable $\xi = x-ct$, the Eq. (3.1) is carried to an ODE

$$cU_\xi + U^3 U_{\xi\xi\xi} = 0. \tag{3.2}$$

Integrating (3.2) with respect to ξ and considering the zero constants for intergation we obtain

$$U_{\xi\xi} - \frac{c}{2U^2} = 0, \tag{3.3}$$

then we use the transformation

$$U_\xi = F(U), \tag{3.4}$$

that will convert Eq. (3.3) to

$$\frac{(F^2(U))'}{2} - \frac{c}{2U^2} = 0. \tag{3.5}$$

where the prime denotes differentiation with respect to ξ. According to Eq. (2.4), we get from (3.5) the expression of the function $F(U)$ reads

$$F(U) = \sqrt{\tfrac{-c}{U}}. \tag{3.6}$$

If we get

$$U_\xi = F(U) \tag{3.7}$$

then we setting the constants of integration to zero and we can obtain the below result

$$U(\xi) = \left(\frac{3}{2}\sqrt{-c}\,\xi\right)^{2/3}. \tag{3.8}$$

We can easily obtain following exact solutions

$$u(x,t) = \left[\frac{3}{2}\sqrt{-c}(x-ct)\right]^{2/3}, \quad c<0, \tag{3.9}$$

Comparing our results and Mokhtari's results [20] with Hereman's results [23] then it can be seen that the results are same.

4 The modified Zakharov-Kuznetsov equations

Let us consider the (2+1)-dimensional modified Zakharov-Kuznetsov equations [24]:

$$u_t + u^2 u_x + u_{xxx} + u_{xyy} = 0. \tag{4.1}$$

The solitary wave solutions of the mZK equation had been constructed via a direct method in [25]. The Lie group analysis is used to carry out the integration of this equation in [26]. The first integral method was used to construct travelling wave solutions of this equation in [27].

Using the wave variable $\xi = x+y-ct$ and proceeding as before we find

$$-cU_\xi + U^2 U_\xi + 2U_{\xi\xi\xi} = 0, \tag{4.2}$$

Integrating the equation (4.2) and neglecting constants of integration, we find

$$-cU + \frac{U^3}{3} + 2U_{\xi\xi} = 0. \tag{4.3}$$

Following the Eq. (2.4), it is easy to deduce from (4.3) the expression of the function $F(U)$ reads

$$F(U)=\sqrt{\frac{c}{2}}U\sqrt{1-\frac{1}{6c}U^2}, \tag{4.4}$$

or

$$F(U)=\sqrt{\frac{c}{2}}U\sqrt{1-c_1U^2}, \tag{4.5}$$

where $c_1=\frac{1}{6c}$.

The solution of the Eq.(4.3) is obtained as

$$U(\xi)=\frac{1}{\sqrt{-c_1}}csch(\sqrt{\frac{c}{2}}\xi). \tag{4.6}$$

We can easily obtain following hyperbolic solutions

$$u_1(x,y,t)=\sqrt{-6c}csch[\frac{\sqrt{2c}}{2}(x+y-ct)], \tag{4.7}$$

$$u_2(x,y,t)=-\sqrt{6c}sech[\frac{\sqrt{2c}}{2}(x+y-ct)]. \tag{4.8}$$

For $c<0$, it is easy to see that solutions (4.7) and (4.8) can reduce to complex solutions as follows:

$$u_3(x,y,t)=-\sqrt{6c}\csc[\frac{\sqrt{-2c}}{2}(x+y-ct)], \quad c<0, \tag{4.9}$$

$$u_4(x,y,t)=-\sqrt{6c}\sec[\frac{\sqrt{-2c}}{2}(x+y-ct)] \quad c<0. \tag{4.10}$$

Comparing our results and Wazwaz's results [25], Adem's results [26] with Tascan's results [27] then it can be seen that the results are same.

5 Kadomtsev-Petviashvili equation

We next consider Kadomtsev-Petviashvili equations [28]

$$\begin{aligned} &u_t-u_{xxx}-6buu_x-3v_y=0,\\ &u_y=v_x. \end{aligned} \tag{5.1}$$

Using the wave variable $\xi=x+y-ct$, the system (5.1) is carried to a system of ODEs

$$\begin{aligned} &-cU_\xi-U_{\xi\xi\xi}-6bUU_\xi-3V_\xi=0,\\ &V_\xi=U_\xi. \end{aligned} \tag{5.2}$$

Integrating the second equation in the system and neglecting the constant of integration we find

$$U=V. \tag{5.3}$$

Substituting (5.3) into the first equation of the system and integrating we find

$$-cU-U_{\xi\xi}-3bU^2-3U=0. \tag{5.4}$$

Following the Eq. (2.4), it is easy to deduce from (5.4) the expression of the function $F(U)$ reads

$$F(U)=\sqrt{U^2(-2bU-3-c)}. \tag{5.5}$$

If we get

$$U_\xi=F(U) \tag{5.6}$$

then we setting the constants of integration to zero and we can obtain the below result

$$V(\xi)=U(\xi)=-\frac{c+3}{2b}\sec^2(\frac{\sqrt{c+3}}{2}\xi), \tag{5.7}$$

We can easily obtain following periodic solutions

$$v_1(x,y,t)=u_1(x,y,t)=-\tfrac{c+3}{2b}\sec^2[\tfrac{\sqrt{c+3}}{2}(x+y-ct)], \quad c+3>0 \tag{5.8}$$

$$v_2(x,y,t)=u_2(x,y,t)=-\tfrac{c+3}{2b}\csc^2[\tfrac{\sqrt{c+3}}{2}(x+y-ct)], \quad c+3>0. \tag{5.9}$$

For $c+3<0$, it is easy to see that solutions (5.8)-(5.9) can reduce to hyperbolic solutions as follows:

$$v_3(x,y,t)=u_3(x,y,t)=-\frac{c+3}{2b}sech^2[\frac{\sqrt{-c-3}}{2}(x+y-ct)], \tag{5.10}$$

$$v_4(x,y,t)=u_4(x,y,t)=\frac{c+3}{2b}csch^2[\frac{\sqrt{-c-3}}{2}(x+y-ct)]. \tag{5.11}$$

Comparing our results and Wazwaz's results [29], Bekir's results [16] with Wang's results [30] then it can be seen that the results are same. All the solutions reported in this paper have been verified with Maple by putting them back into the original Eq. (5.1).

6 Conclusion

As a result, many exact solutions are obtained with the help of symbolic system Maple including soliton solutions presented by hyperbolic functions sech and cosech, periodic solutions presented by sec and cosec and rational solutions. The functional variable method was succesfully used to establish exact solutions. This method handle nonlinear wave equation effectively. This method has many advantages: it is direct and concise. It is shown that the algorithm can be also applied to other NLPDEs in mathematical physics. On the other hand, we will extend this method to seek soliton-like solutions for some PDEs in the forthcoming works.

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