Applied Mathematics & Information Sciences Letters
An International Journal

http://dx.doi.org/10.12785/amisl/010302

Bipolar fuzzy S-acts

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Received: 6 Mar. 2013, Revised: 12 Jul. 2013, Accepted: 18 Jul. 2013
Published online: 1 Sep. 2013

Abstract: In this paper, we initiate the study of bipolar fuzzy subacts of an S-act, where S is a monoid with zero and S-acts are representations of S. We introduce the notions of pure bipolar fuzzy ideal, purely maximal bipolar fuzzy ideal and purely prime bipolar fuzzy ideal of a monoid. It is shown that the set of purely prime bipolar fuzzy ideals of S admits the structure of a topological space. We also define pure bipolar fuzzy subact of an S-act and call an S-act bipolar fuzzy normal if each of its bipolar fuzzy subact is pure. Monoids, all of which S-acts are bipolar fuzzy normal are characterized. It is shown among other results that such monoids are right weakly regular.

Keywords: bipolar fuzzy subacts, pure bipolar fuzzy ideals, bipolar fuzzy normal S-act.

1 INTRODUCTION

Zadeh in his pioneering paper [1] of 1965, first introduced the notion of a fuzzy subset of a set. Since its inception, the fuzzy set theory has developed in many directions and found applications in a wide variety of fields, such as engineering, medical science, social science, physics, statistics, graph theory, artificial intelligence, signal processing, multiagent systems, pattern recognition, robotics, computer networks, expert systems, decision making, automata theory and so on. Fuzzy sets are a kind of useful mathematical structure to represent a collection of objects whose boundary is vague. There are several kinds of fuzzy set extensions in the fuzzy set theory, for example, intuitionistic fuzzy sets, interval-valued fuzzy sets, vague sets etc. Bipolar fuzzy sets are an extension of fuzzy sets whose membership degree range is enlarged from the interval $[0,1]$ to $[-1,1]$. Bipolar fuzzy sets have membership degrees that represent the degree of satisfaction to the property corresponding to a fuzzy set and its counter-property. In a bipolar fuzzy set, the membership degree 0 means that elements are irrelevant to the corresponding property, the membership degrees on $(0,1]$ indicate that elements somewhat satisfy the property, and the membership degrees on $[-1,0)$ indicate that elements somewhat satisfy the implicit counter-property (see [2,3,4,5]). Zhang [4,5] introduced the notion of bipolar fuzzy set. Jun and Kavikumar [6] applied the notion of bipolar fuzzy set to finite state machines, and they introduced the notion of bipolar fuzzy finite state machine. Jun and Park [7] introduced the notion of a bipolar fuzzy filter and a bipolar fuzzy closed quasi filter in BCH-algebras, and investigated several properties. Bipolar fuzzy sets and intuitionistic fuzzy sets look similar to each other. However, they are different from each other (see [2,3]).

In this paper, we initiate the study of bipolar fuzzy subacts of an S-acts, where S is a monoid with zero and S-acts are representations of S. One object of this study is to characterize monoids by the properties of their bipolar fuzzy subacts. In sections $2,3$, we give basic definitions and preliminary lemmas. Section 4 consists of pure bipolar fuzzy ideals, pure bipolar fuzzy subacts and we establish some of their basic properties. In section 5, we introduce purely maximal bipolar fuzzy ideals and purely prime bipolar fuzzy ideals of monoids S and show that the set of all purely prime bipolar fuzzy ideals of S admits the structure of a topological space. In section 6, we prove the main theorem of this paper, which provides characterizations of monoids in terms of their bipolar fuzzy S-acts and bipolar fuzzy subacts. It is proved that a monoid S is weakly regular if and only if each bipolar fuzzy subact of an S-act is pure.

2 PRELIMINARIES

In this paper, S will denote a monoid, that is, a semigroup with an identity element 1, which also contains a

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two-sided zero 0. A semigroup S is called regular if, for each $a \in S$, there exists $x \in S$ such that $a = axa$. S is called right (left) weakly regular if, for each $a \in S, a \in (aS)^2$ $\left(a \in (Sa)^2\right)$ (see [8,9]). Every regular semigroup is right (left) weakly regular but the converse is not true. However, if S is commutative, then converse also holds. A non-empty subset A of S is called a right (left) ideal of S if $AS \subseteq A$ $(SA \subseteq A)$. If A is both a left and a right ideal of S, then A is called an ideal of S.

By a right S-act M_S we mean a non-empty set M, a monoid S and a function $M \times S \longrightarrow M$, such that, if ms denotes the image of (m,s) for $m \in M$ and $s \in S$, then the following conditions hold:

(1) $(ms)t = m(st)$ for all $m \in M$ and $s,t \in S$.

(2) M contains a fixed element θ, called the zero of M, such that $m0 = \theta s = \theta$ for all $m \in M$ and $s \in S$.

(3) $m.1 = m$ for all $m \in M$.

One can similarly, define a left S-act ${}_SM$. An S-subact N_S of M_S is a non-empty subset of M such that $ns \in N$ for all $n \in N$ and $s \in S$.

From the above definition, it follows that every monoid S with zero is a right S-act as well as left S-act over itself, denoted by S_S and ${}_SS$, respectively.

A bipolar fuzzy set μ in S is an object having the form

$$\mu = \left\{(x,\ \mu^P(x),\ \mu^N(x)) : x \in S\right\}$$

where $\mu^P : S \longrightarrow [0,1]$ and $\mu^N : S \longrightarrow [-1,0]$ are mappings. The positive membership degree $\mu^P(x)$ denotes the satisfaction degree of an element x to the property corresponding to a bipolar fuzzy set $\mu = \left\{(x,\ \mu^P(x),\ \mu^N(x)) : x \in S\right\}$ and the negative membership degree $\mu^N(x)$ denotes the satisfaction degree of x to some implicit counter-property of $\mu = \left\{(x,\ \mu^P(x),\ \mu^N(x)) : x \in S\right\}$. If $\mu^P(x) \neq 0$ and $\mu^N(x) = 0$, it is the situation that x is regarded as having only positive satisfaction for $\mu = \left\{(x,\ \mu^P(x),\ \mu^N(x)) : x \in S\right\}$. If $\mu^P(x) = 0$ and $\mu^N(x) \neq 0$, it is the situation that x does not satisfy the property of $\mu = \left\{(x,\ \mu^P(x),\ \mu^N(x)) : x \in S\right\}$, but somewhat satisfies the counter-property of $\mu = \left\{(x,\ \mu^P(x),\ \mu^N(x)) : x \in S\right\}$. It is possible for an element x to be $\mu^P(x) \neq 0$ and $\mu^N(x) \neq 0$ when the membership function of the property overlaps that of its counter-property over some portion of the domain. For the sake of simplicity, we shall write $\mu = \left(\mu^P,\ \mu^N\right)$ for the bipolar fuzzy set $\mu = \left\{\left(x,\ \mu^P(x),\ \mu^N(x)\right) : x \in X\right\}$.

Now we define some set theoretical operations on bipolar fuzzy sets. Let $\mu = \left(\mu^P, \mu^N\right)$ and $\lambda = \left(\lambda^P, \lambda^N\right)$ be two bipolar fuzzy subsets of a non-empty set X. Then $\mu \subseteq \lambda$ means that $\mu^P(x) \leq \lambda^P(x)$ and $\mu^N(x) \geq \lambda^N(x)$ for all $x \in X$.

The intersection and union of two bipolar fuzzy sets $\mu = \left(\mu^P, \mu^N\right)$ and $\lambda = \left(\lambda^P, \lambda^N\right)$ are denoted and defined by, respectively

$$(\mu \cap \lambda)(x) = \left(\min\left(\mu^P(x), \lambda^P(x)\right),\ \max\left(\mu^N(x), \lambda^N(x)\right)\right),$$

$$(\mu \cup \lambda)(x) = \left(\max\left(\mu^P(x), \lambda^P(x)\right),\ \min\left(\mu^N(x), \lambda^N(x)\right)\right).$$

If A is a non-empty subset of S, then the bipolar fuzzy characteristic function of A denoted and defined by $\chi_A = \left(\chi_A^P, \chi_A^N\right)$, where χ_A^P and χ_A^N are defined by

$$\chi_A^P(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}$$

and

$$\chi_A^N(x) = \begin{cases} -1 & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}$$

for all $x \in S$.

2.1 Definition

Let $\mu = \left(\mu^P, \mu^N\right)$ and $\lambda = \left(\lambda^P, \lambda^N\right)$ be bipolar fuzzy sets in a semigroup S. Then the product $\mu \circ \lambda = \left(\mu^P \circ \lambda^P, \mu^N \circ \lambda^N\right)$ is a bipolar fuzzy set in S defined by

$$\left(\mu^P \circ \lambda^P\right)(x) = \begin{cases} \bigvee_{x=st} \mu^P(s) \wedge \lambda^P(t) \\ \text{if } \exists\, s,t \in S \text{ such that } x = st \\ 0 \quad \text{otherwise} \end{cases}$$

and

$$\left(\mu^N \circ \lambda^N\right)(x) = \begin{cases} \bigwedge_{x=st} \mu^N(s) \vee \lambda^N(t) \\ \text{if } \exists\, s,t \in S \text{ such that } x = st \\ 0 \quad \text{otherwise} \end{cases}$$

for all $x \in S$.

2.2 Lemma

Let A and B be non-empty subsets of a semigroup S and $\chi_A = \left(\chi_A^P, \chi_A^N\right)$ and $\chi_B = \left(\chi_B^P, \chi_B^N\right)$ are the bipolar fuzzy characteristic functions of A and B, respectively. Then the following holds:

(i) $\chi_A^P \circ \chi_B^P = \chi_{AB}^P$ and $\chi_A^N \circ \chi_B^N = \chi_{AB}^N$

(ii) $\chi_A^P \cap \chi_B^P = \chi_{A\cap B}^P$ and $\chi_A^N \cap \chi_B^N = \chi_{A\cap B}^N$

(iii) $\chi_A^P \cup \chi_B^P = \chi_{A\cup B}^P$ and $\chi_A^N \cup \chi_B^N = \chi_{A\cup B}^N$

*Proof.*Straightforward.

2.3 Lemma

Let $\mu=(\mu^P,\mu^N)$, $\lambda=(\lambda^P,\lambda^N)$, $\delta=(\delta^P,\delta^N)$ be bipolar fuzzy sets in S. Then the following holds.

(1) $\lambda^P\circ(\mu^P\cup\delta^P)=(\lambda^P\circ\mu^P)\cup(\lambda^P\circ\delta^P)$ and $\lambda^N\circ(\mu^N\cup\delta^N)=(\lambda^N\circ\mu^N)\cup(\lambda^N\circ\delta^N)$.

(2) $\lambda^P\circ(\mu^P\cap\delta^P)\subseteq(\lambda^P\circ\mu^P)\cap(\lambda^P\circ\delta^P)$ and $\lambda^N\circ(\mu^N\cap\delta^N)\supseteq(\lambda^N\circ\mu^N)\cap(\lambda^N\circ\delta^N)$.

Proof.(1) Let $x\in S$. Then,

$$\begin{aligned}(\lambda^P\circ(\mu^P\cup\delta^P))(x) &= \bigvee_{x=yz}\{\lambda^P(y)\wedge(\mu^P\cup\delta^P)(z)\}\\ &= \bigvee_{x=yz}\{\lambda^P(y)\wedge(\mu^P(z)\vee\delta^P(z))\}\\ &= \bigvee_{x=yz}\left\{\begin{array}{c}\{\lambda^P(y)\wedge\mu^P(z)\}\\ \vee\{\lambda^P(y)\wedge\delta^P(z)\}\end{array}\right\}\\ &= \left(\begin{array}{c}\left\{\bigvee_{x=yz}\{\lambda^P(y)\wedge\mu^P(z)\}\right\}\vee\\ \left\{\bigvee_{x=yz}\{\lambda^P(y)\wedge\delta^P(z)\}\right\}\end{array}\right)\\ &= ((\lambda^P\circ\mu^P)(x))\vee((\lambda^P\circ\delta^P)(x))\\ &= ((\lambda^P\circ\mu^P)\cup(\lambda^P\circ\delta^P))(x).\end{aligned}$$

Thus $\lambda^P\circ(\mu^P\cup\delta^P)=(\lambda^P\circ\mu^P)\cup(\lambda^P\circ\delta^P)$.

And

$$\begin{aligned}(\lambda^N\circ(\mu^N\cup\delta^N))(x) &= \bigwedge_{x=yz}\{\lambda^N(y)\vee(\mu^N\cup\delta^N)(z)\}\\ &= \bigwedge_{x=yz}\{\lambda^N(y)\vee(\mu^N(z)\wedge\delta^N(z))\}\\ &= \bigwedge_{x=yz}\left\{\begin{array}{c}\{\lambda^N(y)\vee\mu^N(z)\}\wedge\\ \{\lambda^N(y)\vee\delta^N(z)\}\end{array}\right\}\\ &= \left(\begin{array}{c}\left\{\bigwedge_{x=yz}\{\lambda^N(y)\vee\mu^N(z)\}\right\}\wedge\\ \left\{\bigwedge_{x=yz}\{\lambda^N(y)\vee\delta^N(z)\}\right\}\end{array}\right)\\ &= ((\lambda^N\circ\mu^N)(x))\wedge((\lambda^N\circ\delta^N)(x))\\ &= ((\lambda^N\circ\mu^N)\cup(\lambda^N\circ\delta^N))(x).\end{aligned}$$

Thus $\lambda^N\circ(\mu^N\cup\delta^N)=(\lambda^N\circ\mu^N)\cup(\lambda^N\circ\delta^N)$.

(2) Let $x\in S$. Then

$$\begin{aligned}(\lambda^P\circ(\mu^P\cap\delta^P))(x) &= \bigvee_{x=yz}\{\lambda^P(y)\wedge(\mu^P\cap\delta^P)(z)\}\\ &= \bigvee_{x=yz}\{\lambda^P(y)\wedge(\mu^P(z)\wedge\delta^P(z))\}\\ &= \bigvee_{x=yz}\left\{\begin{array}{c}\{(\lambda^P(y)\wedge\mu^P(z))\}\wedge\\ \{(\lambda^P(y)\wedge\delta^P(z))\}\end{array}\right\}\\ &\le \left(\begin{array}{c}\left\{\bigvee_{x=yz}\{\lambda^P(y)\wedge\mu^P(z)\}\right\}\wedge\\ \left\{\bigvee_{x=yz}\{\lambda^P(y)\wedge\delta^P(z)\}\right\}\end{array}\right)\\ &= ((\lambda^P\circ\mu^P)(x))\wedge((\lambda^P\circ\delta^P)(x))\\ &= ((\lambda^P\circ\mu^P)\cap(\lambda^P\circ\delta^P))(x).\end{aligned}$$

Thus $\lambda^P\circ(\mu^P\cap\delta^P)\subseteq(\lambda^P\circ\mu^P)\cap(\lambda^P\circ\delta^P)$.

And

$$\begin{aligned}(\lambda^N\circ(\mu^N\cap\delta^N))(x) &= \bigwedge_{x=yz}\{\lambda^N(y)\vee(\mu^N\cap\delta^N)(z)\}\\ &= \bigwedge_{x=yz}\{\lambda^N(y)\vee(\mu^N(z)\vee\delta^N(z))\}\\ &= \bigwedge_{x=yz}\left\{\begin{array}{c}\{(\lambda^N(y)\vee\mu^N(z))\}\vee\\ \{(\lambda^N(y)\vee\delta^N(z))\}\end{array}\right\}\\ &\ge \left(\begin{array}{c}\left\{\bigwedge_{x=yz}\{\lambda^N(y)\vee\mu^N(z)\}\right\}\vee\\ \left\{\bigwedge_{x=yz}\{\lambda^N(y)\vee\delta^N(z)\}\right\}\end{array}\right)\\ &= ((\lambda^N\circ\mu^N)(x))\vee((\lambda^N\circ\delta^N)(x))\\ &= ((\lambda^N\circ\mu^N)\cap(\lambda^N\circ\delta^N))(x).\end{aligned}$$

Thus $\lambda^N\circ(\mu^N\cap\delta^N)\supseteq(\lambda^N\circ\mu^N)\cap(\lambda^N\circ\delta^N)$.

3 BIPOLAR FUZZY SUBACTS

3.1 Definition

Let M_S be a right S-act with zero element θ. A bipolar fuzzy set $\mu=(\mu^P,\mu^N)$ of M is called a bipolar fuzzy subact of M_S if $\mu^P(ms)\ge\mu^P(m)$ and $\mu^N(ms)\le\mu^N(m)$ for all $m\in M$ and $s\in S$.

If $\mu=(\mu^P,\mu^N)$ is a bipolar fuzzy subact of M_S, then $\mu^P(\theta)\ge\mu^P(m)$ and $\mu^N(\theta)\le\mu^N(m)$ for all $m\in M$.

Analogously, one can define a bipolar fuzzy subact of a left S-act ${}_SM$. If $M_S=S_S$, then bipolar fuzzy subacts of S_S are called bipolar fuzzy right ideals of S. Bipolar fuzzy left ideals of S are defined in a similar way. A bipolar fuzzy subset of a monoid S, which is both a bipolar fuzzy right ideal and a bipolar fuzzy left ideal of S is called a bipolar fuzzy ideal of S.

3.2 Lemma

Let A be a non-empty subset of a right S-act M_S. Then A is a subact of M_S if and only if the bipolar fuzzy characteristic function $\chi_A=(\chi_A^P,\chi_A^N)$ of A is a bipolar fuzzy subact of M_S.

*Proof.*Let A be a non-empty subset of a right S-act M_S. For any $m\in M$ and $s\in S$, if $m\in A$ then $ms\in A$, so we have

$$\chi_A^P(m)=\chi_A^P(ms)=1$$

and

$$\chi_A^N(m)=\chi_A^N(ms)=-1.$$

If $m\notin A$ then we have,

$$\chi_A^P(ms)\ge 0=\chi_A^P(m)$$

and

$$\chi_A^N(ms) \leq 0 = \chi_A^N(m).$$

Thus we have, $\chi_A^P(ms) \geq \chi_A^P(m)$ and $\chi_A^N(ms) \leq \chi_A^N(m)$. This shows that $\chi_A = (\chi_A^P, \chi_A^N)$ is a bipolar fuzzy subact of M_S.

Conversely, suppose that $\chi_A = (\chi_A^P, \chi_A^N)$ is a bipolar fuzzy subact of M_S. Then for any $m \in A$ and $s \in S$, we have $\chi_A^p(ms) \geq \chi_A^P(m) = 1$ and $\chi_A^N(ms) \leq \chi_A^N(m) = -1$. Thus $\chi_A^p(ms) = 1$ and $\chi_A^N(ms) = -1$. This implies that $ms \in A$. Hence A is a subact of M_S.

3.3 Corollary

Let A be a non-empty subset of a semigroup S. Then A is a right (left) ideal of S if and only if the bipolar fuzzy characteristic function $\chi_A = (\chi_A^P, \chi_A^N)$ of A is a bipolar fuzzy right (left) ideal of S.

3.4 Definition

Let $\mu = (\mu^P, \mu^N)$ be a bipolar fuzzy set in a right S-act M_S and $\lambda = (\lambda^P, \lambda^N)$ be a bipolar fuzzy set in S. Then the product $\mu \circ \lambda = (\mu^P \circ \lambda^P, \mu^N \circ \lambda^N)$ is a bipolar fuzzy set in M_S defined by

$$(\mu^P \circ \lambda^P)(m) = \begin{cases} \bigvee_{m=ns} \{\mu^P(n) \wedge \lambda^P(s)\} \\ \text{if } \exists\, n \in M \text{ and } s \in S \text{ such that } m = ns \\ 0 \quad \text{otherwise} \end{cases}$$

and

$$(\mu^N \circ \lambda^N)(m) = \begin{cases} \bigwedge_{m=ns} \{\mu^N(n) \vee \lambda^N(s)\} \\ \text{if } \exists\, n \in M \text{ and } s \in S \text{ such that } m = ns \\ 0 \quad \text{otherwise} \end{cases}$$

for all $m \in M$.

3.5 Proposition

If $\mu = (\mu^P, \mu^N)$ is a bipolar fuzzy subact of a right S-act M_S and $\lambda = (\lambda^P, \lambda^N)$ is a bipolar fuzzy right ideal of S, then $\mu \circ \lambda = (\mu^P \circ \lambda^P, \mu^N \circ \lambda^N)$ is a bipolar fuzzy subact of M_S.

*Proof.*Let $\mu = (\mu^P, \mu^N)$ be a bipolar fuzzy subact of a right S-act M_S and $\lambda = (\lambda^P, \lambda^N)$ be a bipolar fuzzy right ideal of S. Let $m \in M$ and $s \in S$. Then

$$\begin{aligned} (\mu^P \circ \lambda^P)(m) &= \bigvee_{m=nt} \{\mu^P(n) \wedge \lambda^P(t)\} \\ &\leq \bigvee_{m=nt} \{\mu^P(n) \wedge \lambda^P(ts)\} \text{ for every } s \in S \\ &\leq \bigvee_{ms=ar} \{\mu^P(a) \wedge \lambda^P(r)\} \\ &= (\mu^P \circ \lambda^P)(ms) \end{aligned}$$

and

$$\begin{aligned} (\mu^N \circ \lambda^N)(m) &= \bigwedge_{m=nt} \{\mu^N(n) \vee \lambda^N(t)\} \\ &\geq \bigwedge_{m=nt} \{\mu^N(n) \vee \lambda^N(ts)\} \text{ for every } s \in S \\ &\geq \bigwedge_{ms=ar} \{\mu^N(a) \vee \lambda^N(r)\} \\ &= (\mu^N \circ \lambda^N)(ms). \end{aligned}$$

Thus $\mu \circ \lambda$ is a bipolar fuzzy subact of M_S.

3.6 Corollary

If $\mu = (\mu^P, \mu^N)$ and $\lambda = (\lambda^P, \lambda^N)$ are bipolar fuzzy right ideals of a semigroup S, then $\mu \circ \lambda = (\mu^P \circ \lambda^P, \mu^N \circ \lambda^N)$ is a bipolar fuzzy right ideal S.

4 PURE BIPOLAR FUZZY IDEALS AND PURE BIPOLAR FUZZY SUBACTS

4.1 Definition [10]

An ideal I of a semigroup S is called right pure if for each $x \in I$ there exists $y \in I$ such that $x = xy$.

4.2 Proposition [10]

An ideal I of a semigroup S is right pure if and only if $J \cap I = JI$ for any right ideal J of S.

4.3 Definition [9]

An S-subact N_S of a right S-act M_S is pure if $N \cap MI = NI$ for each ideal I of S, M_S is called normal if each of its subact is pure.

Extending the above notions to the case of bipolar fuzzy sets, we have the following definitions.

4.4 Definition

A bipolar fuzzy ideal $\lambda = (\lambda^P, \lambda^N)$ of a semigroup S is called a pure bipolar fuzzy ideal of S if $\mu \circ \lambda = \mu \cap \lambda$, that is $\mu^P \wedge \lambda^P = \mu^P \circ \lambda^P$ and $\mu^N \vee \lambda^N = \mu^N \circ \lambda^N$, for each bipolar fuzzy right ideal $\mu = (\mu^P, \mu^N)$ of S.

More generally, if $\lambda = (\lambda^P, \lambda^N)$ is a bipolar fuzzy subact of a right S-act M_S, then $\lambda = (\lambda^P, \lambda^N)$ is called a pure bipolar fuzzy subact of M_S, if for each bipolar fuzzy ideal $\mu = (\mu^P, \mu^N)$ of S, $\lambda \circ \mu = \lambda \cap (\chi_M \circ \mu)$, that is $\lambda^P \wedge (\chi^P_M \circ \mu^P) = \lambda^P \circ \mu^P$ and $\lambda^N \vee (\chi^N_M \circ \mu^N) = \lambda^N \circ \mu^N$.

M_S is called bipolar fuzzy normal if each bipolar fuzzy subact of M_S is a pure bipolar fuzzy subact. In particular, S is bipolar fuzzy normal if S_S is bipolar fuzzy normal.

4.5 Proposition

The following assertions are equivalent for an ideal A of a semigroup S:

(1) A is right pure in S.

(2) The bipolar fuzzy set $\chi_A = (\chi^P_A, \chi^N_A)$ is a pure bipolar fuzzy ideal of S.

Proof.(1) $\Longrightarrow$ (2) Suppose A be a right pure ideal in S and $\lambda = (\lambda^P, \lambda^N)$ be a bipolar fuzzy right ideal of S. Then by Corollary 3.3, $\chi_A = (\chi^P_A, \chi^N_A)$ is a bipolar fuzzy ideal of S. Let $x \in S$. Then

$$\begin{aligned}(\lambda^P \circ \chi^P_A)(x) &= \bigvee_{x=yz} \{\lambda^P(y) \wedge \chi^P_A(z)\} \\ &\leq \bigvee_{x=yz} \{\lambda^P(yz) \wedge \chi^P_A(yz)\} \\ &= \bigvee_{x=yz} \{\lambda^P(x) \wedge \chi^P_A(x)\} \\ &= (\lambda^P \wedge \chi^P_A)(x), \qquad (i)\end{aligned}$$

and

$$\begin{aligned}(\lambda^N \circ \chi^N_A)(x) &= \bigwedge_{x=yz} \{\lambda^N(y) \vee \chi^N_A(z)\} \\ &\geq \bigwedge_{x=yz} \{\lambda^N(yz) \vee \chi^N_A(yz)\} \\ &= \bigwedge_{x=yz} \{\lambda^N(x) \vee \chi^N_A(x)\} \\ &= (\lambda^N \vee \chi^N_A)(x). \qquad (ii)\end{aligned}$$

If $x \notin A$, then $(\lambda^P \wedge \chi^P_A)(x) = 0 \leq (\lambda^P \circ \chi^P_A)(x)$ and $(\lambda^N \vee \chi^N_A)(x) = 0 \geq (\lambda^N \circ \chi^N_A)(x)$.

If $x \in A$, then by hypothesis there exists $y \in A$ such that $x = xy$, so we have

$$\begin{aligned}(\lambda^P \circ \chi^P_A)(x) &= \bigvee_{x=ab} \{\lambda^P(a) \wedge \chi^P_A(b)\} \\ &\geq \lambda^P(x) \wedge \chi^P_A(y) \\ (\text{since } \chi^P_A(x) &= 1 = \chi^P_A(y)\ ) \\ &= \lambda^P(x) \wedge \chi^P_A(x) \\ &= (\lambda^P \wedge \chi^P_A)(x),\end{aligned}$$

and

$$\begin{aligned}(\lambda^N \circ \chi^N_A)(x) &= \bigwedge_{x=ab} \{\lambda^N(a) \vee \chi^N_A(b)\} \\ &\leq \lambda^N(x) \vee \chi^N_A(y) \\ (\text{since } \chi^N_A(x) &= -1 = \chi^N_A(y)\ ) \\ &= \lambda^N(x) \vee \chi^N_A(x) \\ &= (\lambda^N \vee \chi^N_A)(x).\end{aligned}$$

Hence in any case $(\lambda^P \circ \chi^P_A) \geq (\lambda^P \wedge \chi^P_A)$ and $(\lambda^N \circ \chi^N_A) \leq (\lambda^N \vee \chi^N_A)$. Thus $(\lambda^P \circ \chi^P_A) = (\lambda^P \wedge \chi^P_A)$ and $(\lambda^N \circ \chi^N_A) = (\lambda^N \vee \chi^N_A)$, that is $\chi_A = (\chi^P_A, \chi^N_A)$ is a pure bipolar fuzzy ideal of S.

(2) $\Longrightarrow$ (1) Suppose that $\chi_A = (\chi^P_A, \chi^N_A)$ is a pure bipolar fuzzy ideal of S. We show that A is a right pure ideal in S. Let B be a right ideal of S. Then by Corollary 3.3, $\chi_B = (\chi^P_B, \chi^N_B)$ is a bipolar fuzzy right ideal of S. Thus by hypothesis $(\chi^P_B \circ \chi^P_A) = (\chi^P_B \wedge \chi^P_A)$ and $(\chi^N_B \circ \chi^N_A) = (\chi^N_B \vee \chi^N_A)$. By Lemma 2.2, $\chi^P_{BA} = (\chi^P_B \circ \chi^P_A) = (\chi^P_B \wedge \chi^P_A) = \chi^P_{B\cap A}$ and $\chi^N_{BA} = (\chi^N_B \circ \chi^N_A) = (\chi^N_B \vee \chi^N_A) = \chi^N_{B\cap A}$, that is, $BA = B \cap A$. Hence by Proposition 4.2, A is a pure ideal in S.

4.6 Proposition

The following assertions for a monoid S with zero are true:

(1) The bipolar fuzzy sets $\zeta = (\zeta^P, \zeta^N)$ and $\phi = (\phi^P, \phi^N)$ are pure bipolar fuzzy ideals of S, where ζ^P, ζ^N, ϕ^P and ϕ^N are defined as,

$\zeta^P(x) = 1$, $\zeta^N(x) = -1$ for all $x \in S$,

$$\phi^P(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

and $\phi^N(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ -1 & \text{if } x = 0 \end{cases}$, respectively.

(2) If $\lambda = (\lambda^P, \lambda^N)$ and $\mu = (\mu^P, \mu^N)$ are pure bipolar fuzzy ideals of S, then so is $\lambda \cap \mu$.

(3) If $\{\mu_i = (\mu^P_i, \mu^N_i) \ : i \in I\}$ is a family of pure bipolar fuzzy ideals of S, then so is $\bigcup_{i\in I} \mu_i$.

Proof.(1) We show that for any bipolar fuzzy right ideal $\lambda = (\lambda^P, \lambda^N)$ of S, we have $\lambda^P \circ \zeta^P = \lambda^P \wedge \zeta^P$, $\lambda^N \circ \zeta^N = \lambda^N \wedge \zeta^N$ and $\lambda^P \circ \phi^P = \lambda^P \wedge \phi^P$, $\lambda^N \circ \phi^N = \lambda^N \wedge \phi^N$. For this, let $x \in S$. Then

$$\begin{aligned}(\lambda^P \circ \zeta^P)(x) &= \bigvee_{x=yz} \{\lambda^P(y) \wedge \zeta^P(z)\} \\ &\leq \bigvee_{x=yz} \{\lambda^P(yz) \wedge \zeta^P(yz)\} \\ &= \bigvee_{x=yz} \{\lambda^P(x) \wedge \zeta^P(x)\} \\ &= \lambda^P(x) \wedge \zeta^P(x) \\ &= (\lambda^P \wedge \zeta^P)(x) \qquad (i)\end{aligned}$$

and

$$\begin{aligned}\left(\lambda^N \circ \zeta^N\right)(x) &= \bigwedge_{x=yz} \left\{\lambda^N(y) \vee \zeta^N(z)\right\} \\ &\geq \bigwedge_{x=yz} \left\{\lambda^N(yz) \vee \zeta^N(yz)\right\} \\ &= \bigwedge_{x=yz} \left\{\lambda^N(x) \vee \zeta^N(x)\right\} \\ &= \lambda^N(x) \vee \zeta^N(x) \\ &= \left(\lambda^N \vee \zeta^N\right)(x). \qquad (ii)\end{aligned}$$

Again for $x \in S$, we have

$$\begin{aligned}\left(\lambda^P \wedge \zeta^P\right)(x) &= \lambda^P(x) \wedge \zeta^P(x) = \lambda^P(x) \wedge 1 \\ &= \lambda^P(x) \wedge \zeta^P(1) \\ &\leq \bigvee_{x=st} \left\{\lambda^P(s) \wedge \zeta^P(t)\right\} \\ &= \left(\lambda^P \circ \zeta^P\right)(x), \qquad (iii)\end{aligned}$$

and

$$\begin{aligned}\left(\lambda^N \vee \zeta^N\right)(x) &= \lambda^N(x) \vee \zeta^N(x) = \lambda^N(x) \vee -1 \\ &= \lambda^N(x) \vee \zeta^N(1) \\ &\geq \bigwedge_{x=st} \left\{\lambda^N(s) \vee \zeta^N(t)\right\} \\ &= \left(\lambda^N \circ \zeta^N\right)(x). \qquad (iv)\end{aligned}$$

From $(i), (ii), (iii)$ and (iv), we have that $\zeta = \left(\zeta^P, \zeta^N\right)$ is a pure bipolar fuzzy ideal of S.

Now we show that $\phi = \left(\phi^P, \phi^N\right)$ is a pure bipolar fuzzy ideal of S. Let $x \in S$. Then

$$\begin{aligned}\left(\lambda^P \circ \phi^P\right)(x) &= \bigvee_{x=yz} \left\{\lambda^P(y) \wedge \phi^P(z)\right\} \\ &\leq \bigvee_{x=yz} \left\{\lambda^P(yz) \wedge \phi^P(yz)\right\} \\ &= \bigvee_{x=yz} \left\{\lambda^P(x) \wedge \phi^P(x)\right\} \\ &= \lambda^P(x) \wedge \phi^P(x) \\ &= \left(\lambda^P \wedge \phi^P\right)(x) \qquad (v)\end{aligned}$$

and

$$\begin{aligned}\left(\lambda^N \circ \phi^N\right)(x) &= \bigwedge_{x=yz} \left\{\lambda^N(y) \vee \phi^N(z)\right\} \\ &\geq \bigwedge_{x=yz} \left\{\lambda^N(yz) \vee \phi^N(yz)\right\} \\ &= \bigwedge_{x=yz} \left\{\lambda^N(x) \vee \phi^N(x)\right\} \\ &= \lambda^N(x) \vee \phi^N(x) \\ &= \left(\lambda^N \vee \phi^N\right)(x). \qquad (vi)\end{aligned}$$

Again for $x \neq 0$,

$$\begin{aligned}\left(\lambda^P \wedge \phi^P\right)(x) &= \lambda^P(x) \wedge \phi^P(x) = \lambda^P(x) \wedge 0 \\ &= 0 \leq \left(\lambda^P \circ \phi^P\right)(x), \qquad (vii)\end{aligned}$$

and

$$\begin{aligned}\left(\lambda^N \vee \phi^N\right)(x) &= \lambda^N(x) \vee \phi^N(x) = \lambda^N(x) \vee 0 \\ &= 0 \geq \left(\lambda^N \circ \phi^N\right)(x). \qquad (viii)\end{aligned}$$

If $x = 0$, then

$$\begin{aligned}\left(\lambda^P \wedge \phi^P\right)(0) &= \lambda^P(0) \wedge \phi^P(0) \\ &= \bigvee_{0=yz} \left\{\lambda^P(y) \wedge \phi^P(z)\right\} \\ &= \left(\lambda^P \circ \phi^P\right)(0), \qquad (ix)\end{aligned}$$

and

$$\begin{aligned}\left(\lambda^N \vee \phi^N\right)(0) &= \lambda^N(0) \vee \phi^N(0) \\ &= \bigwedge_{0=yz} \left\{\lambda^N(y) \vee \phi^N(z)\right\} \\ &= \left(\lambda^N \circ \phi^N\right)(0). \qquad (x)\end{aligned}$$

From $(v), (vi), (vii), (viii), (ix)$ and (x), we have that $\phi = \left(\phi^P, \phi^N\right)$ is a pure bipolar fuzzy ideal of S.

(2) Let $\lambda = \left(\lambda^P, \lambda^N\right)$ and $\mu = \left(\mu^P, \mu^N\right)$ be pure bipolar fuzzy ideals of S. We show that $\gamma^P \circ \left(\mu^P \wedge \lambda^P\right) = \gamma^P \wedge \left(\mu^P \wedge \lambda^P\right)$ and $\gamma^N \circ \left(\mu^N \vee \lambda^N\right) = \gamma^N \vee \left(\mu^N \vee \lambda^N\right)$ for every bipolar fuzzy right ideal $\gamma = \left(\gamma^P, \gamma^N\right)$ of S.

Now

$$\begin{aligned}\gamma^P \wedge \left(\mu^P \wedge \lambda^P\right) &= \left(\gamma^P \wedge \mu^P\right) \wedge \lambda^P \\ &= \left(\gamma^P \circ \mu^P\right) \wedge \lambda^P\end{aligned}$$

(because μ is a pure bipolar fuzzy ideal of S), and

$$\begin{aligned}\gamma^N \vee \left(\mu^N \vee \lambda^N\right) &= \left(\gamma^N \vee \mu^N\right) \vee \lambda^N \\ &= \left(\gamma^N \circ \mu^N\right) \vee \lambda^N\end{aligned}$$

(because μ is a pure bipolar fuzzy ideal of S).

By Corollary 3.6, $\gamma \circ \mu$ is a bipolar fuzzy right ideal of S. Since λ is a pure bipolar fuzzy ideal of S, we have $\left(\gamma^P \circ \mu^P\right) \wedge \lambda^P = \left(\gamma^P \circ \mu^P\right) \circ \lambda^P$ and $\left(\gamma^N \circ \mu^N\right) \vee \lambda^N = \left(\gamma^N \circ \mu^N\right) \circ \lambda^N$. Thus we have,

$$\begin{aligned}\gamma^P \wedge \left(\mu^P \wedge \lambda^P\right) &= \left(\gamma^P \circ \mu^P\right) \circ \lambda^P \\ &= \gamma^P \circ \left(\mu^P \circ \lambda^P\right) \\ &= \gamma^P \circ \left(\mu^P \wedge \lambda^P\right) \qquad (xi)\end{aligned}$$

(because λ and μ are pure bipolar fuzzy ideals of S), and

$$\begin{aligned}\gamma^N \vee \left(\mu^N \vee \lambda^N\right) &= \left(\gamma^N \circ \mu^N\right) \circ \lambda^N \\ &= \gamma^N \circ \left(\mu^N \circ \lambda^N\right) \\ &= \gamma^N \circ \left(\mu^N \vee \lambda^N\right) \qquad (xii)\end{aligned}$$

(because λ and μ are pure bipolar fuzzy ideals of S).

Equations (xi) and (xii) imply that $\lambda \cap \mu$ is a pure bipolar fuzzy ideal of S.

(3) Let $\left\{\mu_i = \left(\mu_i^P, \mu_i^N\right) : i \in I\right\}$ be a family of pure bipolar fuzzy ideals of S and $\gamma = \left(\gamma^P, \gamma^N\right)$ be a bipolar fuzzy right ideal of S. Then we show that $\gamma^P \circ \left(\bigvee_{i \in I} \mu_i^P\right) = \gamma^P \wedge \left(\bigvee_{i \in I} \mu_i^P\right)$ and $\gamma^N \circ \left(\bigwedge_{i \in I} \mu_i^N\right) = \gamma^N \vee \left(\bigwedge_{i \in I} \mu_i^N\right)$.

Let $x \in S$. Then

$$\begin{aligned}\left(\gamma^P \circ \left(\bigvee_{i\in I}\mu_i^P\right)\right)(x) &= \bigvee_{x=yz}\left\{\gamma^P(y) \wedge \left(\bigvee_{i\in I}\mu_i^P\right)(z)\right\} \\ &\leq \bigvee_{x=yz}\left\{\gamma^P(yz) \wedge \left(\bigvee_{i\in I}\mu_i^P\right)(yz)\right\} \\ &= \bigvee_{x=yz}\left\{\gamma^P(x) \wedge \left(\bigvee_{i\in I}\mu_i^P\right)(x)\right\} \\ &= \gamma^P(x) \wedge \left(\bigvee_{i\in I}\mu_i^P\right)(x) \\ &= \left(\gamma^P \wedge \left(\bigvee_{i\in I}\mu_i^P\right)\right)(x),\end{aligned}$$

and

$$\begin{aligned}\left(\gamma^N \circ \left(\bigwedge_{i\in I}\mu_i^N\right)\right)(x) &= \bigwedge_{x=yz}\left\{\gamma^N(y) \vee \left(\bigwedge_{i\in I}\mu_i^N\right)(z)\right\} \\ &\geq \bigwedge_{x=yz}\left\{\gamma^N(yz) \vee \left(\bigwedge_{i\in I}\mu_i^N\right)(yz)\right\} \\ &= \bigwedge_{x=yz}\left\{\gamma^N(x) \vee \left(\bigwedge_{i\in I}\mu_i^N\right)(x)\right\} \\ &= \gamma^N(x) \vee \left(\bigwedge_{i\in I}\mu_i^N\right)(x) \\ &= \left(\gamma^N \vee \left(\bigwedge_{i\in I}\mu_i^N\right)\right)(x).\end{aligned}$$

Again,

$$\begin{aligned}\left(\gamma^P \wedge \left(\bigvee_{i\in I}\mu_i^P\right)\right)(x) &= \gamma^P(x) \wedge \left(\bigvee_{i\in I}\mu_i^P\right)(x) \\ &= \gamma^P(x) \wedge \left(\bigvee_{i\in I}\mu_i^P(x)\right) \\ &= \bigvee_{i\in I}\left(\gamma^P(x) \wedge \left(\mu_i^P(x)\right)\right) \\ &= \bigvee_{i\in I}\left(\left(\gamma^P \wedge \mu_i^P\right)(x)\right) \\ &= \bigvee_{i\in I}\left(\left(\gamma^P \circ \mu_i^P\right)(x)\right)\end{aligned}$$

(because each μ_i is pure bipolar fuzzy ideal of S)
Also,

$$\begin{aligned}\left(\gamma^P \circ \mu_i^P\right)(x) &= \bigvee_{x=yz}\left\{\gamma^P(y) \wedge \left(\mu_i^P\right)(z)\right\} \\ &\leq \bigvee_{x=yz}\left\{\gamma^P(y) \wedge \left(\bigvee_{i\in I}\mu_i^P\right)(z)\right\} \\ &= \left\{\gamma^P \circ \left(\bigvee_{i\in I}\mu_i^P\right)\right\}(x).\end{aligned}$$

Thus we have, $\left(\gamma^P \wedge \left(\bigvee_{i\in I}\mu_i^P\right)\right)(x) \leq \left\{\gamma^P \circ \left(\bigvee_{i\in I}\mu_i^P\right)\right\}(x)$.

And,

$$\begin{aligned}\left(\gamma^N \vee \left(\bigwedge_{i\in I}\mu_i^N\right)\right)(x) &= \gamma^N(x) \vee \left(\bigwedge_{i\in I}\mu_i^N\right)(x) \\ &= \gamma^N(x) \vee \left(\bigwedge_{i\in I}\mu_i^N(x)\right) \\ &= \bigwedge_{i\in I}\left(\gamma^N(x) \vee \left(\mu_i^N(x)\right)\right) \\ &= \bigwedge_{i\in I}\left(\left(\gamma^N \vee \mu_i^N\right)(x)\right) \\ &= \bigwedge_{i\in I}\left(\left(\gamma^N \circ \mu_i^N\right)(x)\right)\end{aligned}$$

(because each μ_i is pure bipolar fuzzy ideal of S)
Also,

$$\begin{aligned}\left(\gamma^N \circ \mu_i^N\right)(x) &= \bigwedge_{x=yz}\left\{\gamma^N(y) \vee \left(\mu_i^N\right)(z)\right\} \\ &\geq \bigwedge_{x=yz}\left\{\gamma^N(y) \vee \left(\bigwedge_{i\in I}\mu_i^N\right)(z)\right\} \\ &= \left\{\gamma^N \circ \left(\bigwedge_{i\in I}\mu_i^N\right)\right\}(x).\end{aligned}$$

Thus we have,

$$\left(\gamma^N \vee \left(\bigwedge_{i\in I}\mu_i^N\right)\right)(x) \geq \left\{\gamma^N \circ \left(\bigwedge_{i\in I}\mu_i^N\right)\right\}(x).$$

Consequently,

$$\left(\gamma^P \wedge \left(\bigvee_{i\in I}\mu_i^P\right)\right)(x) = \left\{\gamma^P \circ \left(\bigvee_{i\in I}\mu_i^P\right)\right\}(x)$$

and

$$\left(\gamma^N \vee \left(\bigwedge_{i\in I}\mu_i^N\right)\right)(x) = \left\{\gamma^N \circ \left(\bigwedge_{i\in I}\mu_i^N\right)\right\}(x).$$

Hence $\bigcup_{i\in I}\mu_i$ is a pure bipolar fuzzy ideal of S.

From the above Proposition, it follows that the set of pure bipolar fuzzy ideals of S is a distributive lattice.

5 PURE BIPOLAR FUZZY SPECTRUM OF S

We begin with the following definitions.

5.1 Definition

Let $\mu = \left(\mu^P, \mu^N\right)$ be a pure bipolar fuzzy ideal of S. Then μ is called purely maximal bipolar fuzzy ideal if μ is a maximal element in the lattice of proper pure bipolar fuzzy ideals of S.

5.2 Definition

A proper pure bipolar fuzzy ideal $\mu = (\mu^P, \mu^N)$ of S is called purely prime bipolar fuzzy ideal if for any bipolar fuzzy ideals $\lambda = (\lambda^P, \lambda^N)$ and $\gamma = (\gamma^P, \gamma^N)$ of S, $\lambda \cap \gamma \subseteq \mu$ implies $\lambda \subseteq \mu$ or $\gamma \subseteq \mu$, that is if $\lambda^P \wedge \gamma^P \leq \mu^P$ and $\lambda^N \vee \gamma^N \geq \mu^N$ then $\lambda^P \leq \mu^P$ and $\lambda^N \geq \mu^N$ or $\gamma^P \leq \mu^P$ and $\gamma^N \geq \mu^N$.

5.3 Proposition

Any purely maximal bipolar fuzzy ideal of S is purely prime bipolar fuzzy ideal.

*Proof.*Let $\mu = (\mu^P, \mu^N)$ be a purely maximal bipolar fuzzy ideal of S and $\lambda = (\lambda^P, \lambda^N)$ and $\gamma = (\gamma^P, \gamma^N)$ are pure bipolar fuzzy ideals of S such that $\lambda \cap \gamma \subseteq \mu$. Let us assume that $\lambda \nsubseteq \mu$. By Proposition 4.6, $\mu \cup \lambda$ is a pure bipolar fuzzy ideal of S. Since μ is purely maximal bipolar fuzzy ideal of S, we have $\mu^P \vee \lambda^P = \zeta^P$ and $\mu^N \wedge \lambda^N = \zeta^N$. Then

$$\begin{aligned}\gamma^P &= \gamma^P \wedge \zeta^P = \gamma^P \wedge (\mu^P \vee \lambda^P) \\ &= (\gamma^P \wedge \mu^P) \vee (\gamma^P \wedge \lambda^P) \\ &\leq \mu^P \vee \mu^P = \mu^P\end{aligned}$$

and

$$\begin{aligned}\gamma^N &= \gamma^N \vee \zeta^N = \gamma^N \vee (\mu^N \wedge \lambda^N) \\ &= (\gamma^N \vee \mu^N) \wedge (\gamma^N \vee \lambda^N) \\ &\geq \mu^N \wedge \mu^N = \mu^N.\end{aligned}$$

Hence μ is purely prime bipolar fuzzy ideal.

In the rest of this section, $PBF(S)$ will denote the set of all pure bipolar fuzzy ideals $\lambda = (\lambda^P, \lambda^N)$ of S which has the property $\lambda^P(0) = 1$ and $\lambda^N(0) = -1$, and $PPBF(S)$ the set of all purely prime bipolar fuzzy ideals contained in $PBF(S)$. As remarked earlier, $PBF(S)$ is a lattice with a least element ϕ and greatest element ζ. For any $\mu \in PBF(S)$, we define

$$\theta_\mu = \{\lambda \in PPBF(S) : \mu \nsubseteq \lambda\}.$$

Thus θ_μ is a subset of $PPBF(S)$, for each $\mu \in PBF(S)$. We will show that the set $PPBF(S)$, together with the subsets θ_μ $(\mu \in PBF(S))$, forms a topological space.

5.4 Theorem

The set $PPBF(S)$ together with the subsets θ_μ $(\mu \in PBF(S))$, forms a topological space.

*Proof.*Let $\tau = \{\theta_\mu : \mu \in PBF(S)\}$. By Proposition 4.6, ϕ is pure bipolar fuzzy ideal of S. The subset

$$\theta_\phi = \{\lambda \in PPBF(S) : \phi \nsubseteq \lambda\} = \emptyset.$$

Thus the empty subset

$$\emptyset = \theta_\phi \in \tau.$$

On the other hand, for the pure bipolar fuzzy ideal ζ of S,

$$\theta_\zeta = \{\lambda \in PPBF(S) : \zeta \nsubseteq \lambda\} = PPBF(S)$$

This is true since purely prime bipolar fuzzy ideals are proper. Hence the whole set

$$PPBF(S) = \theta_\zeta \in \tau.$$

Let θ_μ, $\theta_\gamma \in \tau$, where $\mu, \gamma \in PBF(S)$. Since

$$\begin{aligned}\theta_\mu \cap \theta_\gamma &= \{\lambda \in PPBF(S) : \mu \nsubseteq \lambda \text{ and } \gamma \nsubseteq \lambda\} \\ &= \{\lambda \in PPBF(S) : \mu \cap \gamma \nsubseteq \lambda\}\end{aligned}$$

because λ is purely prime bipolar fuzzy ideal. It follows that $\theta_\mu \cap \theta_\gamma = \theta_{\mu \cap \gamma} \in \tau$, because $\mu \cap \gamma \in PBF(S)$ by Proposition 4.6. Let us now consider a family $\{\mu_k : k \in K\}$ of pure bipolar fuzzy ideals of S. Since

$$\begin{aligned}\underset{k\in K}{\cup} \theta_{\mu_k} &= \left\{\begin{array}{c}\lambda \in PPBF(S) : \\ \text{there exists } k \in K \text{ such that } \mu_k \nsubseteq \lambda\end{array}\right\} \\ &= \left\{\lambda \in PPBF(S) : \bigcup_{k\in K} \mu_k \nsubseteq \lambda\right\} = \theta_{\underset{k\in K}{\cup}\mu_k}.\end{aligned}$$

Now since $\bigcup_{k\in K} \mu_k$ is a purely prime bipolar fuzzy ideal of S, it follows that $\underset{k\in K}{\cup} \theta_{\mu_k} \in \tau$. Thus the set $PPBF(S)$ together with the subsets θ_μ $(\mu \in PBF(S))$, forms a topological space.

6 MONOIDS ALL OF WHOSE BIPOLAR FUZZY S-ACTS ARE NORMAL

Recall that a semigroup S is called regular if for each $a \in S$, there exists $x \in S$ such that $a = axa$. S is called right weakly regular if, for each $a \in S$, $a \in (aS)^2$ $(c.f.$ [9]$)$. Thus if S is commutative then S is right weakly regular if and only S is regular.

We prove the following characterization theorem for a monoid S.

6.1 Theorem

The following assertions for a monoid S with zero are equivalent:

(1) S is right weakly regular.
(2) Each right ideal of S is idempotent.
(3) Each (two-sided) ideal of S is right pure.
(4) Each right S-act is normal.
(5) Each bipolar fuzzy right ideal of S is idempotent.
(6) Each bipolar fuzzy ideal of S is a pure bipolar fuzzy ideal.

(7) Each right S-act is bipolar fuzzy normal.

If in addition, S is commutative, then the above statements are also equivalent to:

(8) S is regular.

Proof.(1) $\iff$ (2) $\iff$ (3) $\iff$ (4) Follows from ([9], Proposition 3.4 and Theorem 4).

(1) $\implies$ (5) Suppose that S is right weakly regular and $\mu = (\mu^P, \mu^N)$ be a bipolar fuzzy right ideal of S. Let $x \in S$. Then,

$$\begin{aligned}(\mu^P \circ \mu^P)(x) &= \bigvee_{x=yz} \{\mu^P(y) \wedge \mu^P(z)\} \\ &\leq \bigvee_{x=yz} \{\mu^P(yz) \wedge \mu^P(z)\} \\ &\quad \text{(because } \mu \text{ is a bipolar fuzzy right ideal of } S\text{)} \\ &= \bigvee_{x=yz} \{\mu^P(x) \wedge \mu^P(z)\} \\ &\leq \bigvee_{x=yz} \{\mu^P(x)\} \\ &= \mu^P(x),\end{aligned}$$

and

$$\begin{aligned}(\mu^N \circ \mu^N)(x) &= \bigwedge_{x=yz} \{\mu^N(y) \vee \mu^N(z)\} \\ &\geq \bigwedge_{x=yz} \{\mu^N(yz) \vee \mu^N(z)\} \\ &\quad \text{(because } \mu \text{ is a bipolar fuzzy right ideal of } S\text{)} \\ &= \bigwedge_{x=yz} \{\mu^N(x) \vee \mu^N(z)\} \\ &\geq \bigwedge_{x=yz} \{\mu^N(x)\} \\ &= \mu^N(x).\end{aligned}$$

Thus $(\mu^P \circ \mu^P)(x) \leq \mu^P(x)$ and $(\mu^N \circ \mu^N)(x) \geq \mu^N(x)$.

Conversely, suppose $x \in S$. Since S is right weakly regular so there exist $a, b \in S$ such that $x = xaxb$. Thus we have,

$$\begin{aligned}\mu^P(x) &= \mu^P(x) \wedge \mu^P(x) \\ &\leq \mu^P(xa) \wedge \mu^P(xb) \\ &\quad \text{(because } \mu \text{ is a bipolar fuzzy right ideal of } S\text{)} \\ &\leq \bigvee_{x=yz} \{\mu^P(y) \wedge \mu^P(z)\} = (\mu^P \circ \mu^P)(x),\end{aligned}$$

and

$$\begin{aligned}\mu^N(x) &= \mu^N(x) \vee \mu^N(x) \\ &\geq \mu^N(xa) \vee \mu^N(xb) \\ &\quad \text{(because } \mu \text{ is a bipolar fuzzy right ideal of } S\text{)} \\ &\geq \bigwedge_{x=yz} \{\mu^N(y) \vee \mu^N(z)\} = (\mu^N \circ \mu^N)(x).\end{aligned}$$

Thus $\mu^P(x) \leq (\mu^P \circ \mu^P)(x)$ and $\mu^N(x) \geq (\mu^N \circ \mu^N)(x)$. Hence $\mu^P = \mu^P \circ \mu^P$ and $\mu^N = \mu^N \circ \mu^N$.

(5) $\implies$ (2) Let A be a right ideal of S. Then the bipolar fuzzy set $\chi_A = (\chi_A^P, \chi_A^N)$ is a bipolar fuzzy right ideal of S. Hence by hypothesis $\chi_A^P = \chi_A^P \circ \chi_A^P$ and $\chi_A^N = \chi_A^N \circ \chi_A^N$. Since $\chi_A^P \circ \chi_A^P = \chi_{AA}^P = \chi_{A^2}^P$ and $\chi_A^N \circ \chi_A^N = \chi_{AA}^N = \chi_{A^2}^N$, we have

$$\chi_{A^2}^P = \chi_A^P \text{ and } \chi_{A^2}^N = \chi_A^N$$
$$\Rightarrow A^2 = A.$$

Hence every right ideal of S is idempotent.

(1) $\implies$ (6) Suppose that S is right weakly regular semigroup. Let $\mu = (\mu^P, \mu^N)$ be a bipolar fuzzy ideal and $\lambda = (\lambda^P, \lambda^N)$ be a bipolar fuzzy right ideal of S. Then for $x \in S$,

$$\begin{aligned}(\lambda^P \circ \mu^P)(x) &= \bigvee_{x=yz} \{\lambda^P(y) \wedge \mu^P(z)\} \\ &\leq \bigvee_{x=yz} \{\lambda^P(yz) \wedge \mu^P(yz)\} \\ &\quad \text{(because } \mu \text{ is a bipolar fuzzy ideal and } \lambda \text{ is a bipolar fuzzy right ideal of } S\text{)} \\ &= \bigvee_{x=yz} \{\lambda^P(x) \wedge \mu^P(x)\} \\ &= \lambda^P(x) \wedge \mu^P(x) = (\lambda^P \wedge \mu^P)(x),\end{aligned}$$

and

$$\begin{aligned}(\lambda^N \circ \mu^N)(x) &= \bigwedge_{x=yz} \{\lambda^N(y) \vee \mu^N(z)\} \\ &\geq \bigwedge_{x=yz} \{\lambda^N(yz) \vee \mu^N(yz)\} \\ &\quad \text{(because } \mu \text{ is a bipolar fuzzy ideal and } \lambda \text{ is a bipolar fuzzy right ideal of } S\text{)} \\ &= \bigwedge_{x=yz} \{\lambda^N(x) \vee \mu^N(x)\} \\ &= \lambda^N(x) \vee \mu^N(x) = (\lambda^N \vee \mu^N)(x).\end{aligned}$$

Now, let $x \in S$. Since S is right weakly regular, so there exist $a, b \in S$ such that $x = xaxb$. Thus we have,

$$\begin{aligned}(\lambda^P \wedge \mu^P)(x) &= \lambda^P(x) \wedge \mu^P(x) \\ &\leq \lambda^P(xa) \wedge \mu^P(xb) \\ &\quad \text{(because } \mu \text{ and } \lambda \text{ are bipolar fuzzy right ideals of } S\text{)} \\ &\leq \bigvee_{x=yz} \{\lambda^P(y) \wedge \mu^P(z)\} \\ &= (\lambda^P \circ \mu^P)(x),\end{aligned}$$

and

$$\begin{aligned}(\lambda^N \vee \mu^N)(x) &= \lambda^N(x) \vee \mu^N(x)\\ &\geq \lambda^N(xa) \vee \mu^N(xb)\\ &\quad \text{(because } \mu \text{ and } \lambda \text{ are bipolar fuzzy right ideals of } S\text{)}\\ &\geq \bigwedge_{x=yz}\{\lambda^N(y) \vee \mu^N(z)\}\\ &= (\lambda^N \circ \mu^N)(x).\end{aligned}$$

Hence $\lambda^P \wedge \mu^P = \lambda^P \circ \mu^P$ and $\lambda^N \vee \mu^N = \lambda^N \circ \mu^N$. This implies μ is a pure bipolar fuzzy ideal of S.

(6) $\Longrightarrow$ (3) Let A be an ideal of S. Then the bipolar fuzzy set $\chi_A = (\chi_A^P, \chi_A^N)$ is a bipolar fuzzy ideal of S. Hence by hypothesis $\chi_A = (\chi_A^P, \chi_A^N)$ is a pure bipolar fuzzy ideal of S. By Proposition 4.5, A is a right pure ideal of S.

(1) $\Longrightarrow$ (7) Suppose that S is right weakly regular semigroup. Let $\mu = (\mu^P, \mu^N)$ be a bipolar fuzzy subact of a right S-act M_S and $\lambda = (\lambda^P, \lambda^N)$ be a bipolar fuzzy ideal of S. We show that, $\mu^P \wedge (\chi_M^P \circ \lambda^P) = \mu^P \circ \lambda^P$ and $\mu^N \vee (\chi_M^N \circ \lambda^N) = \mu^N \circ \lambda^N$. Let $x \in M$. Then we have,

$$\begin{aligned}(\mu^P \circ \lambda^P)(x) &= \bigvee_{x=yz}\{\mu^P(y) \wedge \lambda^P(z)\}\\ &\leq \bigvee_{x=yz}\{\mu^P(yz) \wedge \lambda^P(z)\}\\ &\quad \text{(because } \mu \text{ is abipolar fuzzy subact of the right } S\text{-act } M_S\text{)}\\ &= \bigvee_{x=yz}\{\mu^P(x) \wedge \lambda^P(z)\} \qquad (i)\end{aligned}$$

also

$$\begin{aligned}(\chi_M^P \circ \lambda^P)(x) &= \bigvee_{x=yz}\{\chi_M^P(y) \wedge \lambda^P(z)\}\\ &= \bigvee_{x=yz}\{1 \wedge \lambda^P(z)\}\\ &= \bigvee_{x=yz}\lambda^P(z). \qquad (ii)\end{aligned}$$

Thus

$$\begin{aligned}(\mu^P \circ \lambda^P)(x) &\leq \bigvee_{x=yz}\{\mu^P(x) \wedge \lambda^P(z)\}\\ &= \mu^P(x) \wedge \left\{\bigvee_{x=yz}\lambda^P(z)\right\}\\ &= \mu^P(x) \wedge \{(\chi_M^P \circ \lambda^P)(x)\} \qquad (\text{by } (ii))\\ &= (\mu^P \wedge (\chi_M^P \circ \lambda^P))(x). \qquad (iii)\end{aligned}$$

And

$$\begin{aligned}(\mu^N \circ \lambda^N)(x) &= \bigwedge_{x=yz}\{\mu^N(y) \vee \lambda^N(z)\}\\ &\geq \bigwedge_{x=yz}\{\mu^N(yz) \vee \lambda^N(z)\}\\ &\quad \text{(because } \mu \text{ is abipolar fuzzy subact of the right } S\text{-act } M_S\text{)}\\ &= \bigwedge_{x=yz}\{\mu^N(x) \vee \lambda^N(z)\} \qquad (iv)\end{aligned}$$

also

$$\begin{aligned}(\chi_M^N \circ \lambda^N)(x) &= \bigwedge_{x=yz}\{\chi_M^N(y) \vee \lambda^N(z)\}\\ &= \bigwedge_{x=yz}\{-1 \vee \lambda^N(z)\}\\ &= \bigwedge_{x=yz}\lambda^N(z). \qquad (v)\end{aligned}$$

Thus

$$\begin{aligned}(\mu^N \circ \lambda^N)(x) &\geq \bigwedge_{x=yz}\{\mu^N(x) \vee \lambda^N(z)\}\\ &= \mu^N(x) \vee \left\{\bigwedge_{x=yz}\lambda^N(z)\right\}\\ &= \mu^N(x) \vee \{(\chi_M^N \circ \lambda^N)(x)\} \qquad (\text{by } (v))\\ &= (\mu^N \vee (\chi_M^N \circ \lambda^N))(x) \qquad (vi)\end{aligned}$$

Now

$$\begin{aligned}(\mu^P \wedge (\chi_M^P \circ \lambda^P))(x) &= \mu^P(x) \wedge (\chi_M^P \circ \lambda^P)(x)\\ &= \mu^P(x) \wedge \left(\bigvee_{x=yz}\lambda^P(z)\right) \qquad (\text{by } (ii))\\ &= \bigvee_{x=yz}\{\mu^P(x) \wedge \lambda^P(z)\}\\ &= \bigvee_{x=yz}\{\mu^P(yz) \wedge \lambda^P(z)\}\\ &\leq \bigvee_{x=yz}\{\mu^P(yza) \wedge \lambda^P(zb)\}\end{aligned}$$

($z = zazb$ for some $a, b \in S$, because S is a right weakly regular)

$$\leq \bigvee_{x=y_1z_1}\{\mu^P(y_1) \wedge \lambda^P(z_1)\} = (\mu^P \circ \lambda^P)(x). \qquad (vii)$$

And

$$\begin{aligned}(\mu^N \vee (\chi_M^N \circ \lambda^N))(x) &= \mu^N(x) \vee (\chi_M^N \circ \lambda^N)(x)\\ &= \mu^N(x) \vee \left(\bigwedge_{x=yz}\lambda^N(z)\right) \qquad (\text{by } (v))\\ &= \bigwedge_{x=yz}\{\mu^N(x) \vee \lambda^N(z)\}\\ &= \bigwedge_{x=yz}\{\mu^N(yz) \vee \lambda^N(z)\}\\ &\geq \bigwedge_{x=yz}\{\mu^N(yza) \vee \lambda^N(zb)\}\end{aligned}$$

($z = zazb$ for some $a, b \in S$, because S is a right weakly regular)

$$\geq \bigwedge_{x=y_2z_2}\{\mu^N(y_2) \vee \lambda^N(z_2)\} = (\mu^N \circ \lambda^N)(x). \qquad (viii)$$

From (iii), (vi), (vii) and $(viii)$ we have the required result.

(7) $\Longrightarrow$ (6) We are given that each right S-act is bipolar fuzzy normal. Hence S, consider as an S-act, is a bipolar fuzzy normal, that is, each bipolar fuzzy subact of S_S is a pure bipolar fuzzy subact. Let $\lambda = (\lambda^P, \lambda^N)$ be a bipolar fuzzy right ideal and $\mu = (\mu^P, \mu^N)$ be a bipolar fuzzy ideal of S. By hypothesis λ is pure bipolar fuzzy subact of S, that is $\lambda^P \wedge (\chi_S^P \circ \mu^P) = \lambda^P \circ \mu^P$ and

$\lambda^N \vee (\chi_S^N \circ \mu^N) = \lambda^N \circ \mu^N$. Clearly $\chi_S^P \circ \mu^P = \mu^P$ and $\chi_S^N \circ \mu^N = \mu^N$ which means that $\lambda^P \wedge \mu^P = \lambda^P \circ \mu^P$ and $\lambda^N \vee \mu^N = \lambda^N \circ \mu^N$. Hence μ is a pure bipolar fuzzy ideal of S.

$(1) \Longleftrightarrow (8)$ It is immediate from the definition (under the assumption that S is commutative).

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