Mathematical Sciences Letters
An International Journal

http://dx.doi.org/10.12785/msl/020306

On the Number of 1-factors of Bipartite Graphs

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Received: 28 Feb. 2013, Revised: 2 May. 2013, Accepted: 5 May. 2013
Published online: 1 Sep. 2013

Abstract: In this paper, we investigated relationships between the Fibonacci, Lucas, Padovan numbers and 1-factors of some bipartite graphs with upper Hessenberg adjacency matrix. We calculated permanent of these upper Hessenberg matrices by contraction method and show that their permanents are equal to elements of the Fibonacci, Lucas and Padovan numbers. At the end of the paper, we give some Maple 13 procedure in order to calculate numbers of 1-factors of above-mentioned bipartite graphs.

Keywords: Fibonacci Sequence, Lucas Sequence, Padovan Sequence, 1-Factor, Permanent.

1 Introduction

The well-known *Fibonacci sequence* $\{F_n\}$ is defined by the recurrence relation, for $n > 2$

$$F_n = F_{n-1} + F_{n-2},$$

where $F_1 = F_2 = 1$.

The well-known *Lucas sequence* $\{L_n\}$ is defined by the recurrence relation, for $n > 2$

$$L_n = L_{n-1} + L_{n-2},$$

where $L_1 = 1, L_2 = 3$.

The *Padovan sequence* $\{P_n\}$ is defined by the recurrence relation, for $n > 2$

$$P_n = P_{n-2} + P_{n-3},$$

where $P_0 = P_1 = P_2 = 1$ [1].

The first few values of these sequences are given below:

n	0	1	2	3	4	5	6	7	8	9	10	$\cdots$
F_n	0	1	1	2	3	5	8	13	21	34	55	$\cdots$
L_n	2	1	3	4	7	11	18	29	47	76	123	$\cdots$
P_n	1	1	1	2	2	3	4	5	7	9	12	$\cdots$.

The *permanent* of an n-square matrix $A = [a_{ij}]$ is defined by

$$perA = \sum_{\sigma \varepsilon S_n} \prod_{i=1}^{n} a_{i\sigma(i)}$$

where the summation extends over all permutations σ of the symmetric group S_n.

Let $A = [a_{ij}]$ be an $m \times n$ real matrix with row vectors $r_1, r_2, \ldots, r_m$. We say A is *contractible* on column (resp. row) k if column (resp. row) k contains exactly two nonzero entries. Suppose A is contractible on column k with $a_{ik} \neq 0 \neq a_{jk}$ and $i \neq j$. Then the $(m-1) \times (n-1)$ matrix $A_{ij:k}$ obtained from A by replacing row i with $a_{jk}r_i + a_{ik}r_j$ and deleting row j and column k is called the contraction of A on column k relative to rows i and j. If A is contractible on row k with $a_{ki} \neq 0 \neq a_{kj}$ and $i \neq j$, then the matrix $A_{k:ij} = \left[A^T_{ij:k}\right]^T$ is called the *contraction* of A on row k relative to columns i and j. We say that A can be contracted to a matrix B if either $B = A$ or there exist matrices $A_0, A_1, \ldots, A_t$ $(t \geq 1)$ such that $A_0 = A$, $A_t = B$, and A_r is a contraction of A_{r-1} for $r = 1, \ldots, t$. One can find the following fact in [2]: Let A be a nonnegative integral matrix of order n for $n > 1$ and let B be a contraction of A. Then

$$perA = perB. \quad (1)$$

It is known that there are a lot of relations between permanents of matrices and well-known number sequences. For example, in [3], Minc defines generalized

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Fibonacci numbers of order r as

$$f(n,r)=\begin{cases} 0 & \text{if } n<0,\\ 1 & \text{if } n=0,\\ \sum_{k=1}^{r} f(n-k,r) & \text{if } n>0. \end{cases}$$

Minc also defines the $n\times n$ super-diagonal $(0,1)-$matrix $F(n,r)=[f_{ij}]$, where

$$f_{ij}=\begin{cases} 1 \text{ if } -1\le j-i\le r-2,\\ 0 \text{ otherwise,}\end{cases}$$

and proves that $per(F(n,r))=f(n,r-1)$.

In [4], the authors denote the matrix $F(n,r)$ in [3] as $\mathscr{F}^{(n,k)}$ and obtain permanent of this matrix, the same result in [3, Theorem 2], by applying contraction to the matrix $\mathscr{F}^{(n,k)}$.

In [5], Kilic defines the $n\times n$ super-diagonal $(0,1,2)$-matrix $S(k,n)$ as:

$$S(k,n)=\begin{bmatrix} 2 & 1 & \cdots & 1 & 0 & \cdots & 0\\ 1 & 2 & 1 & \ldots & 1 & \ddots & \vdots\\ 0 & 1 & 2 & 1 & \ldots & \ddots & 0\\ \vdots & \ddots & \ddots & \ddots & \ddots & & 1\\ \vdots & & \ddots & \ddots & \ddots & \ddots & \vdots\\ \vdots & & & \ddots & \ddots & \ddots & 1\\ 0 & \cdots & \cdots & \cdots & 0 & 1 & 2 \end{bmatrix}$$

and proves that the permanent of $S(k,n)$ equals to the $(n+1)th$ generalized k-Pell number.

In [6], authors define the $n-$square $(0,1)-$matrix as

$$H(n)=\begin{bmatrix} 1 & 1 & 1 & 0 & \cdots & \cdots & 0\\ 1 & 0 & 1 & 1 & & & \\ & 1 & 0 & 1 & 1 & & \\ & & \ddots & \ddots & \ddots & \ddots & 0\\ & & & 1 & 0 & 1 & 1\\ & 0 & & & 1 & 0 & 1\\ & & & & & 1 & 0 \end{bmatrix},$$

and by using contraction method, they obtain $perH(n)=P_{n-2}$, where P_n is the nth Padovan number.

A *bipartite graph* G is a graph whose vertex set V can be partitioned into two two subsets V_1 and V_2 such that every edge of G joins a vertex in V_1 and a vertex in V_2. A $1-$*factor* (or *perfect matching*) of a graph with $2n$ vertices is a spanning subgraph of G in which every vertex has degree 1. The enumeration or actual constuction of $1-$factors of a bipartite graph has many applications, for example, in maximal flow problems and in assignment and scheduling problems. Let $A(G)$ be adjacency matrix of the bipartite graph G, and let $\mu(G)$ denote the number of $1-$factors of G. Then, one can find the following fact in [7]: $\mu(G)=\sqrt{perA(G)}$. Also, one can find more applications of permanents in [7].

Let G be a bipartite graph whose vertex set V is partitioned into two subsets V_1 and V_2 such that $|V_1|=|V_2|=n$. We construct the *bipartite adjacent matrix* $B(G)=[b_{ij}]$ of G as following: $b_{ij}=1$ if and only if G contains an edge from $v_i\in V_1$ to $v_j\in V_2$, and otherwise. Then, in [7], the number of $1-$factors of bipartite graph G equals the permanent of its bipartite adjaceny matrix.

In [8], Lee defines bipartite adjacency matrix of bipartite graph $G(\mathscr{L}^{(n,k)})$ the following way: Let $S_n^{(k)}=[s_{ij}]$ be the $n\times n$ $(0,1)-$matrix defined by $s_{ij}=1$ if and only if $-1\le j-i\le k-1$. For $k<n$, $\mathscr{L}^{(n,k)}=S_n^{(k)}-\sum_{j=2}^{k}E_{1j}+E_{1k+1}$, where E_{ij} denotes the $n\times n$ matrix with 1 in the (i,j) position and zeros elsewhere. Clearly,

$$\mathscr{L}^{(n,k)}=\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & 0 & \cdots & 0\\ 1 & 1 & 1 & \cdots & 1 & 1 & 0 & 0 & 0 & \cdots & 0\\ 0 & 1 & 1 & \cdots & 1 & 1 & 1 & 0 & 0 & \cdots & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & \cdots & 0 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 & 0\\ 0 & \cdots & \cdots & 0 & 1 & 1 & 1 & \cdots & 1 & 1 & 1\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 & 1 & 1\\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 & 1 \end{bmatrix},$$

and he also shows that the numbers of $1-$factors of $G(\mathscr{L}^{(n,2)})$ and $G(\mathscr{L}^{(n,k)})$ is L_{n-1} and $l_{n-1}^{(k)}$, where L_n and $l_n^{(k)}$ are nth Lucas and $k-$Lucas numbers, respectively.

In [9], the authors define bipartite adjacency matrices of $G(V_n)$ and $G(W_n)$ bipartite graphs as follows:

$$V_n=\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & \cdots & 1\\ 1 & 1 & 1 & 0 & 0 & \cdots & 0\\ 0 & 1 & 1 & 1 & 0 & \cdots & 0\\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots\\ \vdots & & \ddots & \ddots & \ddots & \ddots & 0\\ \vdots & & & \ddots & \ddots & \ddots & 1\\ 0 & \cdots & \cdots & \cdots & 0 & 1 & 1 \end{bmatrix},\ W_n=\begin{bmatrix} 1 & 0 & 1 & 1 & 1 & \cdots & 1\\ 1 & 1 & 1 & 1 & 0 & \cdots & 0\\ 0 & 1 & 1 & 1 & 0 & \cdots & 0\\ 0 & 0 & 1 & 1 & 1 & \ddots & \vdots\\ \vdots & & \ddots & \ddots & \ddots & \ddots & 0\\ \vdots & & & \ddots & \ddots & \ddots & 1\\ 0 & \cdots & \cdots & \cdots & 0 & 1 & 1 \end{bmatrix}$$

and they show that the numbers of $1-$factors of $G(V_n)$ and $G(W_n)$ are $\sum_{i=0}^{n}F_i$ and $\sum_{i=0}^{n-2}L_i$, respectively. In other words, $perV_n=\sum_{i=0}^{n}F_i$, $perW_n=\sum_{i=0}^{n-2}L_i$.

In [10], the authors define families of square matrices such that (i) each matrix is the adjacency matrix of a bipartite graph; and (ii) the permanent of the matrices are the generalized order$-k$ Lucas numbers and a sum of consecutive generalized order$-k$ Fibonacci or Lucas numbers.

In this paper, we give families of (0,1) upper Hessenberg matrices such that each of these matrices is the bipartite adjacent matrix corresponding to a bipartite graph and then we show that the permanents of these matrices are equal to the well-known Fibonacci, Lucas and Padovan sequences. At the end of this paper, we give some Maple 13 procedures in order to calculate the numbers of 1-factors of bipartite graphs mentioned above.

2 Main Results

In this section, we consider a class of bipartite graphs. Then we show that the numbers of 1-factors of the graphs equal to Fibonacci, Lucas and Padovan numbers.

Let $U_n = [u_{ij}]$ be the $n-$square $(0,1)$ upper Hessenberg matrix defined by

$$U_n = \begin{bmatrix} 1 & 0 & 1 & 0 & \cdots & 1 & 0 & 1 & \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & 0 & 1 & 0 & \cdots & 1 & 0 & 1 & \cdots & \cdots & \cdots \\ 0 & 1 & 1 & 0 & 1 & 0 & \cdots & 1 & 0 & 1 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 1 & 1 & 0 & 1 & 0 & \cdots & 1 & 0 & 1 \\ 0 & \cdots & \cdots & 0 & 1 & 1 & 0 & 1 & 0 & \cdots & 1 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 & 1 \end{bmatrix} \quad (2)$$

where

$$u_{ij} = \begin{cases} 1 \text{ if } j-i=-1, \\ 1 \text{ if } i \leq j \text{ and } j-i \equiv 0 \ (mod 2), \\ 0 \text{ otherwise.} \end{cases} .$$

Theorem 2.1. Let $G(U_n)$ be the bipartite graph with bipartite adjacency matrix U_n as in (2) for $n \geq 3$. Then the number of $1-$factors of $G(U_n)$ is F_n.

Proof. If $n=3$, then we get

$$perU_3 = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = 2 = F_3. \quad (3)$$

Let U_n^p be the pth contraction of U_n, $1 \leq p \leq n-2$. Since the matrix U_n can be contracted on column 1, we get

$$U_n^1 = \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & 0 & 1 & 0 & \cdots & 1 & 0 & 1 & \cdots & \cdots & \cdots \\ 0 & 1 & 1 & 0 & 1 & 0 & \cdots & 1 & 0 & 1 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 1 & 1 & 0 & 1 & 0 & \cdots & 1 & 0 & 1 \\ 0 & \cdots & \cdots & 0 & 1 & 1 & 0 & 1 & 0 & \cdots & 1 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 & 1 \end{bmatrix}.$$

Since the matrix U_n^1 can be contracted on column 1, $F_2 = 1$ and $F_3 = 2$, we write

$$U_n^2 = \begin{bmatrix} 2 & 1 & 2 & 1 & \cdots & 2 & 1 & 2 & \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & 0 & 1 & 0 & \cdots & 1 & 0 & 1 & \cdots & \cdots & \cdots \\ 0 & 1 & 1 & 0 & 1 & 0 & \cdots & 1 & 0 & 1 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 1 & 1 & 0 & 1 & 0 & \cdots & 1 & 0 & 1 \\ 0 & \cdots & \cdots & 0 & 1 & 1 & 0 & 1 & 0 & \cdots & 1 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} F_3 & F_2 & F_3 & F_2 & \cdots & F_3 & F_2 & F_3 & \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & 0 & 1 & 0 & \cdots & 1 & 0 & 1 & \cdots & \cdots & \cdots \\ 0 & 1 & 1 & 0 & 1 & 0 & \cdots & 1 & 0 & 1 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 1 & 1 & 0 & 1 & 0 & \cdots & 1 & 0 & 1 \\ 0 & \cdots & \cdots & 0 & 1 & 1 & 0 & 1 & 0 & \cdots & 1 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 & 1 \end{bmatrix}.$$

Furthermore, if the matrix U_n^2 can be contracted on column 1, we write

$$U_n^3 = \begin{bmatrix} F_4 & F_3 & F_4 & F_3 & \cdots & F_4 & F_3 & F_4 & \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & 0 & 1 & 0 & \cdots & 1 & 0 & 1 & \cdots & \cdots & \cdots \\ 0 & 1 & 1 & 0 & 1 & 0 & \cdots & 1 & 0 & 1 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 1 & 1 & 0 & 1 & 0 & \cdots & 1 & 0 & 1 \\ 0 & \cdots & \cdots & 0 & 1 & 1 & 0 & 1 & 0 & \cdots & 1 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 & 1 \end{bmatrix}.$$

Continuing this process, we get

$$U_n^p = \begin{bmatrix} F_{p+1} & F_p & F_{p+1} & F_p & \cdots & F_{p+1} & F_p & F_{p+1} & \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & 0 & 1 & 0 & \cdots & 1 & 0 & 1 & \cdots & \cdots & \cdots \\ 0 & 1 & 1 & 0 & 1 & 0 & \cdots & 1 & 0 & 1 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 1 & 1 & 0 & 1 & 0 & \cdots & 1 & 0 & 1 \\ 0 & \cdots & \cdots & 0 & 1 & 1 & 0 & 1 & 0 & \cdots & 1 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 & 1 \end{bmatrix},$$

for $3 \leq p \leq n-4$. Hence,

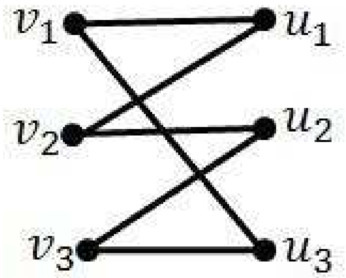
$$U_n^{n-3} = \begin{bmatrix} F_{n-2} & F_{n-3} & F_{n-2} \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix},$$

which, by contraction of U_n^{n-3} on column 1, gives

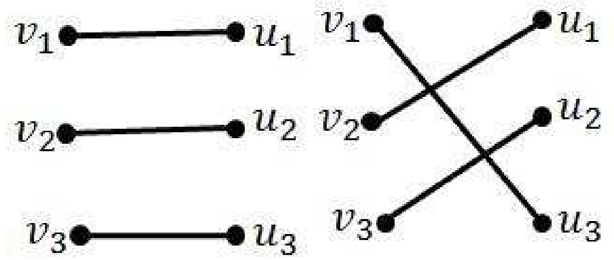
$$U_n^{n-2} = \begin{bmatrix} F_{n-2}+F_{n-3} & F_{n-2} \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} F_{n-1} & F_{n-2} \\ 1 & 1 \end{bmatrix}.$$

By applying (1), we get $perU_n = perU_n^{(n-2)} = F_n$ and the proof is completed.

Example 2.1. Let the matrix U_3, as in (3), be bipartite adjacency matrix of the graph $G(U_3)$. Then, bipartite graph $G(U_3)$ can be seen as:



and its 1-factors can be seen as:



Let $V_n = [v_{ij}]$ be the $n-$square $(0,1)$ upper Hessenberg matrix defined as the following form:

$$V_n = \begin{bmatrix} 1\,1\,1 & \cdots & \cdots\cdots\cdots & 1 \\ 1\,0\,1 & \cdots & 0 \quad 1 & \cdots\cdots \\ 0\,1 & & & \\ 0\,0 & U_{(n-2)} & & \\ \vdots\;\vdots & & & \\ 0\,0 & & & \end{bmatrix}. \qquad (4)$$

Theorem 2.2. Let $G(V_n)$ be the bipartite graph with bipartite adjacency matrix V_n as in (4) for $n \geq 3$. Then the number of $1-$factors of $G(V_n)$ is L_{n-1}.

Proof. If $n = 3$, then we get

$$perV_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} = 3 = L_2. \qquad (5)$$

Let V_n^r be the rth contraction of V_n for $1 \leq r \leq n-2$. Since the matrix V_n can be contracted on column 1, we get

$$V_n^1 = \begin{bmatrix} 1 & 2 & 1 & 2 & \cdots & 1 & 2 & 1 & \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & 0 & 1 & 0 & \cdots & 1 & 0 & 1 & \cdots & \cdots & \cdots \\ 0 & 1 & 1 & 0 & 1 & 0 & \cdots & 1 & 0 & 1 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \vdots & \cdots & 0 & 1 & 1 & 0 & 1 & 0 & \cdots & 1 & 0 & 1 \\ \vdots & \cdots & \cdots & 0 & 1 & 1 & 0 & 1 & 0 & \cdots & 1 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 & 1 \end{bmatrix}.$$

Since the matrix V_n^1 can be contracted on column 1, $L_1 = 1$ and $L_2 = 3,$ we write

$$V_n^2 = \begin{bmatrix} 3 & 1 & 3 & 1 & \cdots & 3 & 1 & 3 & \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & 0 & 1 & 0 & \cdots & 1 & 0 & 1 & \cdots & \cdots & \cdots \\ 0 & 1 & 1 & 0 & 1 & 0 & \cdots & 1 & 0 & 1 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 1 & 1 & 0 & 1 & 0 & \cdots & 1 & 0 & 1 \\ 0 & \cdots & \cdots & 0 & 1 & 1 & 0 & 1 & 0 & \cdots & 1 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} L_2 & L_1 & L_2 & L_1 & \cdots & L_2 & L_1 & L_2 & \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & 0 & 1 & 0 & \cdots & 1 & 0 & 1 & \cdots & \cdots & \cdots \\ 0 & 1 & 1 & 0 & 1 & 0 & \cdots & 1 & 0 & 1 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \vdots & \cdots & 0 & 1 & 1 & 0 & 1 & 0 & \cdots & 1 & 0 & 1 \\ \vdots & \cdots & \cdots & 0 & 1 & 1 & 0 & 1 & 0 & \cdots & 1 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 & 1 \end{bmatrix}.$$

Furthermore, if the matrix V_n^2 can be contracted on column 1, we write

$$V_n^3 = \begin{bmatrix} L_3 & L_2 & L_3 & L_2 & \cdots & L_3 & L_2 & L_3 & \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & 0 & 1 & 0 & \cdots & 1 & 0 & 1 & \cdots & \cdots & \cdots \\ 0 & 1 & 1 & 0 & 1 & 0 & \cdots & 1 & 0 & 1 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 1 & 1 & 0 & 1 & 0 & \cdots & 1 & 0 & 1 \\ 0 & \cdots & \cdots & 0 & 1 & 1 & 0 & 1 & 0 & \cdots & 1 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 & 1 \end{bmatrix}.$$

Continuing this process, we get

$$V_n^r = \begin{bmatrix} L_r & L_{r-1} & L_r & L_{r-1} & \cdots & L_r & L_{r-1} & L_r & \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & 0 & 1 & 0 & \cdots & 1 & 0 & 1 & \cdots & \cdots & \cdots \\ 0 & 1 & 1 & 0 & 1 & 0 & \cdots & 1 & 0 & 1 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 1 & 1 & 0 & 1 & 0 & \cdots & 1 & 0 & 1 \\ 0 & \cdots & \cdots & 0 & 1 & 1 & 0 & 1 & 0 & \cdots & 1 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 & 1 \end{bmatrix},$$

for $3 \leq r \leq n-4$. Hence,

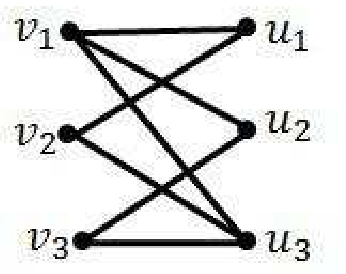
$$V_n^{n-3} = \begin{bmatrix} L_{n-3} & L_{n-4} & L_{n-3} \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix},$$

which, by contraction of V_n^{n-3} on column 1, gives

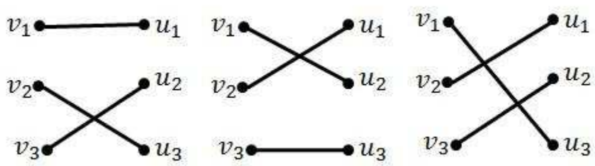
$$V_n^{n-2} = \begin{bmatrix} L_{n-3}+L_{n-4} & L_{n-3} \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} L_{n-2} & L_{n-3} \\ 1 & 1 \end{bmatrix}.$$

By applying (1), we get $perV_n = perV_n^{(n-2)} = L_{n-1}$ and the proof is completed.

Example 2.2. Let the matrix V_3, as in (5), be bipartite adjacency matrix of the graph $G(V_3)$. Then, bipartite graph $G(V_3)$ can be seen as:



and its 1-factors can be given as:



Let $W_n = [w_{ij}]$ be the $n-$square $(0,1)$ upper Hessenberg matrix defined by

$$W_n = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & \cdots & \cdots & \cdots \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & \cdots & 0 & 1 & 0 & \cdots & \cdots & \cdots \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & \cdots & 0 & 1 & 0 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & \cdots & 0 & 1 & 0 \\ 0 & \cdots & \cdots & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & \cdots & 0 & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 & 0 & 1 \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 & 0 \end{bmatrix}, \quad (6)$$

where

$$w_{ij} = \begin{cases} 1 \text{ if } i = 1, \\ 1 \text{ if } j - i = -1, \\ 1 \text{ if } j \geq i \text{ and } j - i \equiv 1 \ (mod\ 3), \\ 0 \text{ otherwise.} \end{cases}$$

Theorem 2.3. Let $G(W_n)$ be the bipartite graph with bipartite adjacency matrix W_n as in (6) for $n \geq 3$. Then the number of $1-$factors of $G(W_n)$ is P_n.

Proof. If $n = 5$, then we get

$$perW_5 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} = 3 = P_5. \quad (7)$$

Let W_n^t be the tth contraction of W_n, $1 \leq t \leq n-2$. Since the matrix W_n can be contracted on column 1, we get

$$W_n^1 = \begin{bmatrix} 1 & 2 & 1 & 1 & 2 & 1 & \cdots & 1 & 2 & 1 & \cdots & \cdots & \cdots & \cdots \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & \cdots & 0 & 1 & 0 & \cdots & \cdots & \cdots \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & \cdots & 0 & 1 & 0 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & \cdots & 0 & 1 & 0 \\ 0 & \cdots & \cdots & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & \cdots & 0 & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 & 0 & 1 \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 & 0 \end{bmatrix}.$$

Since the matrix W_n^1 can be contracted on column 1, we write

$$W_n^2 = \begin{bmatrix} 2 & 2 & 1 & 2 & 2 & 1 & \cdots & 2 & 2 & 1 & \cdots & \cdots & \cdots & \cdots \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & \cdots & 0 & 1 & 0 & \cdots & \cdots & \cdots \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & \cdots & 0 & 1 & 0 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & \cdots & 0 & 1 & 0 \\ 0 & \cdots & \cdots & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & \cdots & 0 & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 & 0 & 1 \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 & 0 \end{bmatrix}.$$

Furthermore, since the matrix W_n^2 can be contracted on column 1, $P_3 = P_4 = 2$ and $P_5 = 3$, we can write

$$W_n^3 = \begin{bmatrix} 2 & 3 & 2 & 2 & 3 & 2 & \cdots & 2 & 3 & 2 & \cdots & \cdots & \cdots & \cdots \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & \cdots & 0 & 1 & 0 & \cdots & \cdots & \cdots \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & \cdots & 0 & 1 & 0 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & \cdots & 0 & 1 & 0 \\ 0 & \cdots & \cdots & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & \cdots & 0 & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 & 0 & 1 \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} P_4 & P_5 & P_3 & P_4 & P_5 & P_3 & \cdots & P_4 & P_5 & P_3 & \cdots & \cdots & \cdots & \cdots \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & \cdots & 0 & 1 & 0 & \cdots & \cdots & \cdots \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & \cdots & 0 & 1 & 0 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & \cdots & 0 & 1 & 0 \\ 0 & \cdots & \cdots & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & \cdots & 0 & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 & 0 & 1 \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 & 0 \end{bmatrix}.$$

Continuing this process, we get

$$W_n^t = \begin{bmatrix} P_{r+1} & P_{r+2} & P_r & P_{r+1} & P_{r+2} & P_r & \cdots & P_{r+1} & P_{r+2} & P_r & \cdots & \cdots & \cdots & \cdots \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & \cdots & 0 & 1 & 0 & \cdots & \cdots & \cdots \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & \cdots & 0 & 1 & 0 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & \cdots & 0 & 1 & 0 \\ 0 & \cdots & \cdots & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & \cdots & 0 & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 & 0 & 1 \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 & 0 \end{bmatrix},$$

for $3 \leq r \leq n-4$. Hence,

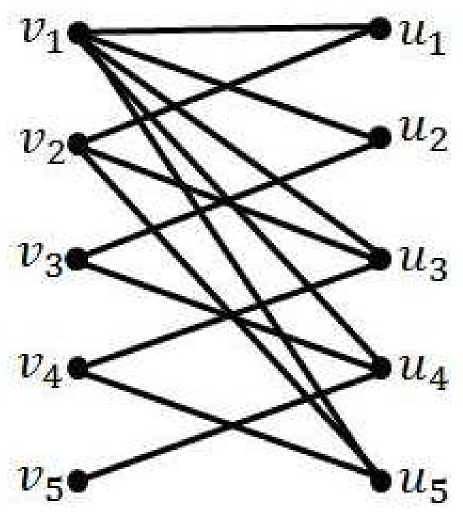
$$W_n^{n-3} = \begin{bmatrix} P_{n-2} & P_{n-1} & P_{n-3} \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

which, by contraction of W_n^{n-3} on column 1, gives

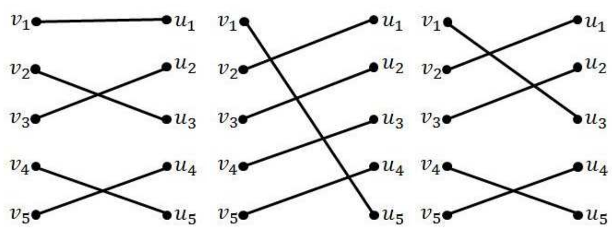
$$W_n^{n-2} = \begin{bmatrix} P_{n-1} & P_{n-2}+P_{n-3} \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} P_{n-1} & P_n \\ 1 & 0 \end{bmatrix}.$$

By applying (1), we get $perW_n = perW_n^{(n-2)} = P_n$ and the proof is completed.

Example 2.3. Let the matrix W_3, as in (7), be bipartite adjacency matrix of the graph $G(W_3)$. Then, bipartite graph $G(W_3)$ can be seen as:



and its 1-factors can be given as:



3 Conclusion

We showed that numbers of 1-factors of some bipartite graphs are equal to the well-known Fibonacci, Lucas and Padovan numbers.

Appendix A. The following procedure calculates the number of 1−factors of bipartite graph $G(U_n)$ given in Theorem 1.

```
restart:
with(LinearAlgebra):
permanent:=proc(n)
local i,j,p,u,U;
u:=(i,j)->piecewise(j>=i and j-i mod 2 =0,1,i-j=1,1,0);
U:=Matrix(n,n,u):
for p from 0 to n-2 do
print(p,U):
for j from 2 to n-p do
U[1,j]:=U[2,1]*U[1,j]+U[1,1]*U[2,j]:
od:
U:=DeleteRow(DeleteColumn(Matrix(n-p,n-p,U),1),2):
od:
print(p,eval(U)):
end proc:with(LinearAlgebra):
permanent( );
```

Appendix B. The following procedure calculates the number of 1−factors of bipartite graph $G(V_n)$ given in Theorem 2.

```
restart:
with(LinearAlgebra):
permanent:=proc(n)
local i,j,r,v,V;
v:=(i,j)->piecewise(i=1,1,i=2 and j mod 2 =1,1,i>2 and j>=i and j-i mod 2 =0,1,i-j=1,1,0);
V:=Matrix(n,n,v):
for r from 0 to n-2 do
print(r,V):
for j from 2 to n-r do
V[1,j]:=V[2,1]*V[1,j]+V[1,1]*V[2,j]:
od:
V:=DeleteRow(DeleteColumn(Matrix(n-r,n-r,V),1),2):
od:
print(r,eval(V)):
end proc:with(LinearAlgebra):
permanent( );
```

Appendix C. The following procedure calculates the number of 1−factors of bipartite graph $G(W_n)$ given in Theorem 3.

```
restart:
with(LinearAlgebra):
permanent:=proc(n)
local i,j,t,w,W;
w:=(i,j)- >piecewise(i=1,1,j>=i and j-i mod 3 =1,1,i-j=1,1,0);
W:=Matrix(n,n,w):
for t from 0 to n-2 do
print(t,W):
for j from 2 to n-t do
W[1,j]:=W[2,1]*W[1,j]+W[1,1]*W[2,j]:
od:
W:=DeleteRow(DeleteColumn(Matrix(n-t,n-t,W),1),2):
od:
print(t,eval(W)):
end proc:with(LinearAlgebra):
permanent( );
```

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