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A Result Concerning the Ratio of Consecutive Prime Numbers

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Abstract: Let p_n be the nth prime number and $\pi(n)$ be the number of primes less than or equal to n. In this note, we show that the limit of $(p_{n+1}/p_n)^{\pi(n)}$ does not exist.

Keywords: Prime number, arithmetic function, asymptotics.

1 Introduction and main results

Denote by p_n the nth prime number. The prime number theorem (see e.g. [1,2]) implies that

$$p_n \sim n \ln n, \tag{1}$$

as $n \to \infty$; i.e. $\lim_{n\to\infty} p_n/(n\ln n) = 1$. Let $\pi(n)$ be the number of primes less than or equal to n. It is easy to see that $\pi(n) \sim n/\ln n$ as $n \to \infty$. Thus, it follows from (1) that

$$\left(\frac{p_{n+1}}{p_n}\right)^{\pi(n)/n} \to 1, \tag{2}$$

as $n \to \infty$. A recent result [3] on the ratio of consecutive prime numbers shows that the limit of $(p_{n+1}/p_n)^n$ does not exist. More interesting results regarding consecutive prime numbers can be found in e.g. [4,5].

In this note, we will prove the following result:

Proposition 1.1

$$\limsup_{n\to\infty} \left(\frac{p_{n+1}}{p_n}\right)^{\pi(n)} = e, \tag{3}$$

and

$$\liminf_{n\to\infty} \left(\frac{p_{n+1}}{p_n}\right)^{\pi(n)} = 0. \tag{4}$$

For $n \geq 1$, let $a_n = (p_{n+1} - p_n)/\ln p_n$. It is well known that the limit of a_n does not exist. In particular, it is shown that [6]

$$\limsup_{n\to\infty} a_n = +\infty. \tag{5}$$

and [7]

$$\liminf_{n\to\infty} a_n = 0. \tag{6}$$

More properties of the sequence $\{a_n\}$ can be found in the monograph [8].

The following weaker corollary can be obtained from Proposition 1.1.

Corollary 1.1

$$\liminf_{n\to\infty} \frac{a_n}{\ln n} = 0. \tag{7}$$

A sequence $\{b_n\}$ is said to be an Erdős-Turán type sequence [9] if $\Delta_k = b_{k+1} - b_k$ changes its sign infinitely many times. In view of Proposition 1.1, a natural question would be to ask if $(p_{n+1}/p_n)^{\pi(n)}$ is an Erdős-Turán type sequence. For more general background, we refer the interested reader to [10] and references therein.

2 Proofs

In this section, we present the proofs of our main results.

Proof of Proposition 1.1. By the prime number theorem (1), we obtain

$$\frac{\ln p_n}{\ln n} \to 1,$$

as $n \to \infty$. Therefore,

$$\frac{\pi(n)\ln p_n}{n} \sim \frac{\pi(n)\ln n}{n} \sim 1.$$

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A more careful examination shows that

$$\frac{\pi(n)\ln p_n}{n} \sim 1+\Theta\left(\frac{\ln\ln n}{\ln n}\right),$$

and hence

$$p_n^{\pi(n)} \sim e^n \cdot e^{\Theta\left(\frac{n\ln\ln n}{\ln n}\right)},$$

as $n \to \infty$.

By some basic manipulations, we obtain

$$\begin{aligned}\left(\frac{p_{n+1}}{p_n}\right)^{\pi(n)} &= \frac{p_{n+1}^{\pi(n+1)}}{p_n^{\pi(n)}} \cdot p_{n+1}^{\pi(n)-\pi(n+1)} \\ &\sim e \cdot p_{n+1}^{\pi(n)-\pi(n+1)} \\ &\sim e(n\ln n)^{\pi(n)-\pi(n+1)}.\end{aligned}$$

Since $\pi(n)-\pi(n+1)$ equals to either -1 or 0 depending on $n+1$ is a prime number or not, we derive (3) and (4) as desired. □

Proof of Corollary 1.1. We have

$$\begin{aligned}\left(\frac{p_{n+1}}{p_n}\right)^{\pi(n)} &\sim \left(\frac{p_{n+1}}{p_n}\right)^{\frac{n}{\ln n}} \\ &= \left(\left(1+\frac{p_{n+1}-p_n}{p_n}\right)^{\frac{p_n}{(p_{n+1}-p_n)\ln n}}\right)^{\frac{(p_{n+1}-p_n)n}{p_n}} \quad (8)\end{aligned}$$

as $n \to \infty$.

From (1), it yields $(p_{n+1}-p_n)/p_n \to 0$ and

$$\frac{(p_{n+1}-p_n)n}{p_n} \sim \frac{p_{n+1}-p_n}{\ln n} \sim \frac{p_{n+1}-p_n}{\ln p_n} = a_n,$$

as $n \to \infty$.

Inserting these estimates into (8), we obtain

$$\left(\frac{p_{n+1}}{p_n}\right)^{\pi(n)} \ll e^{\frac{a_n}{\ln n}},$$

as $n \to \infty$. The result (7) then follows from (3). □

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