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Flux Observability for Hyperbolic Systems

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Abstract: The aim of this paper is to introduce the concept of boundary gradient observability for distributed hyperbolic systems evolving in spatial domain Ω, and which the gradient state is to be observed on a boundary subregion Γ of $\partial\Omega$. We give definitions and characterizations and some properties of this kind of regional boundary observability. To explore this notion we describe an new approach to solve this problem, which is performed through numerical examples and simulations.

Keywords: Distributed systems, hyperbolic systems, regional gradient observability, boundary observability, regional reconstruction, strategic sensor

1 Introduction

Many real problems in the control and observation of distributed systems can be reformulated as a problem of analysis for infinite dimensional systems. New and interesting notions has been introduced and developed. Among the most important is observability which has been widely developed by Curtain and Zwart [1] and the references therein. But in practical situations one is interested in the reconstruction of the systems state in a restricted given subregion, from the knowledge of the system dynamics and the output function [13]. This concept was introduced by El Jai and Zerrik [5,15] and extended to the case where the subregion is a part of the boundary of the system evolution domain [8,10,15].

In many real world problem, one is interested by the knowledge directly of the state gradient without the knowledge of its state. This concept was introduced and developed by Zerrik et al.[14,**?**] and Boutoulout et al.[8, 9,10,11] and interesting results were obtained only for parabolic systems but little has been done for hyperbolic ones. It is also plausible in real problems that the subregion of interest may be a portion $\Gamma \subset \partial\Omega$, rather than an actual subregion ω. Our interest is to extend the notion of regional gradient observability for hyperbolic systems developed initially in internal domain by [12] to the case where the subregion of interest is a part of the boundary of system evolution domain. Technically, the distinguishing difficulty is that the relevant restriction map is now a trace map and can not be expected to be continuous. More precisely we have to develop tools to reconstruct directly the initial state and speed gradient on the boundary of the system evolution domain. This leads to the so-called regional boundary gradient observability.

The paper is organized as follows : First we give some fundamental results related to regional boundary gradient observability for hyperbolic systems. In the next section we give characterization of boundary gradient strategic sensors. The established results are also applied to a two-dimensional diffusion system. Section 4 we give a reconstruction method for the gradient on a subregion $\Gamma \subset \partial\Omega$. At last the simulations show that there exists a relation between the choice of sensors location and the boundary subregion target.

2 Gradient observability

2.1 Problem statement

Let Ω be an open and bounded domain of $\mathbb{R}^n (n = 1,2,3)$ with regular boundary $\partial\Omega$. For $T > 0$, we denote $Q = \Omega\times]0,T[$, $\Sigma = \partial\Omega\times]0,T[$ and we consider the following hyperbolic system defined by

$$\begin{cases} \dfrac{\partial^2 y}{\partial t^2}(x,t) = \mathscr{A} y(x,t) & \text{in } Q \\ y(x,0) = y^0(x), \quad \dfrac{\partial y}{\partial t}(x,0) = y^1(x) & \text{in } \Omega \\ y(\xi,t) = 0 & \text{on } \Sigma \end{cases} \qquad (1)$$

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with y^0 and y^1 are the initial conditions, the measurement given by the output function

$$z(t) = Cy(t), \quad t \in]0,T[\tag{2}$$

where $\mathscr{A} = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(a_{ij}\frac{\partial}{\partial x_j})$ with $a_{ij} \in D(Q)$.
Suppose that $\mathscr{A}$ is elliptic and a second order differential operator, i.e., there exists $\alpha > 0$ such that

$$\sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j, \geq \alpha \sum_{i=1}^{n} |\xi_i|^2 \qquad \forall \xi = (\xi_1, ..., \xi_n) \in \mathbb{R}^n$$

and $C : H^2(\Omega) \cap H_0^1(\Omega) \longrightarrow \mathbb{R}^q$ is a linear operator where q depends on the number of the considered sensors. We are interested in the reconstruction of the gradient on a portion Γ of the boundary domain $\partial\Omega$ where a distributed parameter system governed by hyperbolic equations evolving in a spatial domain Ω.
Let's consider the observation space $\mathscr{O} = L^2(0,T;\mathbb{R}^q)$ and assume that

$$(y^0, y^1) \in X = (H^2(\Omega) \cap H_0^1(\Omega)) \times (H^2(\Omega) \cap H_0^1(\Omega)).$$

Without loss of generality we denote $y(t) := y(x,t)$.
Let's consider

$$\bar{y}(t) = \begin{bmatrix} y(t) \\ \frac{\partial y(t)}{\partial t} \end{bmatrix}, \qquad \bar{y}^0 = \begin{bmatrix} y^0 \\ y^1 \end{bmatrix}, \qquad \bar{\mathscr{A}} = \begin{bmatrix} 0 & I \\ \mathscr{A} & 0 \end{bmatrix}$$

For $(y_1, y_2) \in \mathscr{F} = L^2(\Omega) \times L^2(\Omega)$, the system (1) is equivalent to

$$\begin{cases} \frac{\partial \bar{y}}{\partial t}(t) = \bar{\mathscr{A}}\bar{y}(t) & 0 < t < T \\ \bar{y}(0) = \bar{y}^0 \end{cases} \tag{3}$$

with measurements given by the output function

$$\bar{z}(t) = \bar{C}\bar{y}(t) \tag{4}$$

with $\bar{C} = (C, 0)$, the system (3) has a unique solution that can be expressed as $\bar{y}(t) = \bar{S}(t)\bar{y}^0$, where $(\bar{S}(t))_{t\geq 0}$ is a strongly continuous semigroup generated by the operator $\bar{\mathscr{A}}$.
The initial conditions (y^0, y^1) and their gradients $(\nabla y^0, \nabla y^1)$ are assumed to be unknown.
For $\omega \subset \Omega$, open, regular and of positive Lebesgue measure, the problem of regional gradient observability is the possibility to reconstruct directly the initial gradient $(\nabla y^0, \nabla y^1)$ on ω, without the knowledge of the state (y^0, y^1).
For the case where C is unbounded, some precautions must be taken as $D(C) \subset (H^2(\Omega) \cap H_0^1(\Omega))$ and $S(t)(D(C)) \subset D(C)\ \forall t \geq 0$ [4].
Let's denote Φ_{m_j} a basis of eigenfunctions of the operator $\mathscr{A}$, with Dirichlet conditions and associated to the eigenvalues λ_m with the multiplicity r_m, then for any $(y_1, y_2) \in \mathscr{F}$

$$\bar{S}(t)\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \sum_{m=1}^{\infty}\sum_{j=1}^{r_m}[\langle y_1, \Phi_{m_j}\rangle \cos(-\lambda_m)^{\frac{1}{2}} t \\ +(-\lambda_m)^{-\frac{1}{2}}\langle y_2, \Phi_{m_j}\rangle \sin(-\lambda_m)^{\frac{1}{2}} t]\Phi_{m_j} \\ \\ \sum_{m=1}^{\infty}\sum_{j=1}^{r_m}[-(-\lambda_m)^{\frac{1}{2}}\langle y_1, \Phi_{m_j}\rangle \sin(-\lambda_m)^{\frac{1}{2}} t \\ +\langle y_2, \Phi_{m_j}\rangle \cos(-\lambda_m)^{\frac{1}{2}} t]\Phi_{m_j} \end{pmatrix}$$

then the output function can be expressed as follows

$$\bar{z}(t) = \bar{C}\bar{S}(t)\bar{y}^0 = \bar{K}(t)\bar{y}^0, \ t \in]0,T[$$

Where $\bar{K}$ is the observation operator defined by

$$\begin{aligned} \bar{K} : X &\longrightarrow \mathscr{O} \\ \bar{z} &\longmapsto \bar{C}\bar{S}(.)\bar{z} \end{aligned}$$

which is linear bounded, with the adjoint $\bar{K}^*$ given by

$$\begin{aligned} \bar{K}^* : \mathscr{O} &\longrightarrow X \\ \bar{z}^* &\longmapsto \int_0^T \bar{S}^*(t)\bar{C}^*\bar{z}^*(t)dt \end{aligned}$$

Let's consider the operator $\bar{\nabla}$ given as follows

$$\begin{aligned} \bar{\nabla} : (H^2(\Omega) \cap H_0^1(\Omega)) \times (H^2(\Omega) \cap H_0^1(\Omega)) &\longrightarrow (L^2(\Omega))^n \times (L^2(\Omega))^n \\ (y_1, y_2) &\longmapsto \bar{\nabla}(y_1, y_2) = (\nabla y_1, \nabla y_2) \end{aligned}$$

where

$$\begin{aligned} \nabla : H^2(\Omega) \cap H_0^1(\Omega) &\longrightarrow (L^2(\Omega))^n \\ y &\longmapsto \nabla y = (\frac{\partial y}{\partial x_1}, \ldots, \frac{\partial y}{\partial x_n}) \end{aligned}$$

We denote by $\bar{\nabla}^*$ (resp. ∇^*) the adjoint of $\bar{\nabla}$ (resp. ∇).
Let $\bar{\nabla}\bar{y}_1^0$ be the restriction of the trace of $\bar{\nabla}\bar{y}^0$ to Γ with $\bar{\nabla}\bar{y}^0 = (\nabla y^0, \nabla y^1)$ and $\bar{\nabla}\bar{y}_1^0 = (\nabla y_1^0, \nabla y_1^1)$.
Let's consider the trace operator

$$\begin{aligned} \bar{\gamma} : (H^1(\Omega))^n \times (H^1(\Omega))^n &\longrightarrow (H^{\frac{1}{2}}(\partial\Omega))^n \times (H^{\frac{1}{2}}(\partial\Omega))^n \\ (y^1, y^2) &\longmapsto \bar{\gamma}(y^1, y^2) = (\gamma y^1, \gamma y^2) \end{aligned}$$

with

$$\begin{aligned} \gamma : (H^1(\Omega))^n &\longrightarrow (H^{\frac{1}{2}}(\partial\Omega))^n \\ z &\longmapsto \gamma z = (\gamma_0 z_1, \ldots, \gamma_0 z_n) \end{aligned}$$

and $\gamma_0 : H^1(\Omega) \longrightarrow H^{\frac{1}{2}}(\partial\Omega)$ is the trace operator of order zero which is linear, continuous, and surjective.
γ^* (resp. γ_0^*) denote the adjoint of operator γ (resp. γ_0)
For $\Gamma \subset \partial\Omega$, consider

$$\begin{aligned} \bar{\chi}_\Gamma : (H^{\frac{1}{2}}(\partial\Omega))^n \times (H^{\frac{1}{2}}(\partial\Omega))^n &\to (H^{\frac{1}{2}}(\Gamma))^n \times (H^{\frac{1}{2}}(\Gamma))^n \\ (\xi, \xi') &\mapsto \bar{\chi}_\Gamma(\xi, \xi') = (\xi, \xi')|_\Gamma \end{aligned}$$

$$\begin{aligned} \chi_\Gamma : (H^{\frac{1}{2}}(\partial\Omega))^n &\to (H^{\frac{1}{2}}(\Gamma))^n \\ \xi &\mapsto \chi_\Gamma \xi = \xi|_\Gamma \end{aligned} \quad \text{and} \quad \begin{aligned} \tilde{\chi}_\Gamma : H^{\frac{1}{2}}(\partial\Omega) &\to H^{\frac{1}{2}}(\Gamma) \\ \xi &\mapsto \tilde{\chi}_\Gamma \xi = \xi|_\Gamma \end{aligned}$$

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with $\overline{\chi}_{\Gamma}^{*}$ (resp. χ_{Γ}^{*} and $\tilde{\chi}_{\Gamma}^{*}$) is the adjoint of $\overline{\chi}_{\Gamma}$ (resp. χ_{Γ} and $\tilde{\chi}_{\Gamma}$) which is the restriction operator, consider also the operator :

$$\overline{\chi}_{\omega} : (H^1(\Omega))^n \times (H^1(\Omega))^n \longrightarrow (H^1(\omega))^n \times (H^1(\omega))^n$$
$$(y_1, y_2) \longmapsto \overline{\chi}_{\omega}(y_1, y_2) = (\chi_{\omega} y_1, \chi_{\omega} y_2)$$

where

$$\chi_{\omega} : (H^1(\Omega))^n \to (H^1(\omega))^n \quad \text{and} \quad \tilde{\chi}_{\omega} : H^1(\Omega) \to H^1(\omega)$$
$$\xi \mapsto \chi_{\omega}\xi = \xi|_{\omega} \qquad \xi \mapsto \tilde{\chi}_{\omega}\xi = \xi|_{\omega}$$

and $\overline{\chi}_{\omega}^{*}$ denotes its adjoint operator, finally we reconstruct the operator as follows

$$\overline{H} = \overline{\chi}_{\Gamma}\overline{\gamma}\overline{\nabla}\overline{K}^{*} \quad \text{from} \quad \mathscr{O} \quad \text{into} \quad (H^{\frac{1}{2}}(\Gamma))^n \times (H^{\frac{1}{2}}(\Gamma))^n$$

Let's recall some definitions and results related to the internal regional gradient observability. Then the following will apply.

– The system (1) together with the output (2) is said to be exactly gradient observable in ω if $Im(\overline{\chi}_{\omega}\overline{\nabla}\overline{K}^{*}) = (H^1(\omega))^n \times (H^1(\omega))^n$.
– The system (1) together with the output (2), is said to be weakly gradient observable in ω if $\overline{Im(\overline{\chi}_{\omega}\overline{\nabla}\overline{K}^{*})} = (H^1(\omega))^n \times (H^1(\omega))^n$.
– A sequence of sensors is said to be gradient strategic in ω (or G-strategic in ω) if the observed system is weakly gradient observable on ω. For more details, see [12] and [16].

The regional boundary gradient observability explores the reconstruction of the gradient in the particular case where the subregion ω is a subset of the boundary. More precisely we have to rebuild the $\overline{\nabla}\overline{y}_1^0$ component of the initial gradient of an unknown portion of the boundary.

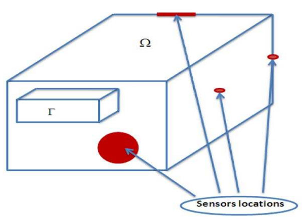

Fig. 1: The domain Ω, the subregion Γ, and the sensors locations.

Definition 2.1. The system (1) together with the output (2) is said to be exactly (resp. weakly) boundary gradient observable on Γ or exactly (resp. weakly) G-observable on Γ if $Im\overline{H} = (H^{\frac{1}{2}}(\Gamma))^n \times (H^{\frac{1}{2}}(\Gamma))^n$ (resp. $\overline{Im\overline{H}} = (H^{\frac{1}{2}}(\Gamma))^n \times (H^{\frac{1}{2}}(\Gamma))^n$.)

Definition 2.2.

1)-A sensor (D, f) is said to be gradient strategic on Γ or G-strategic on Γ if the observed system is weakly G-observable on Γ,

2)-A sequence of sensors $(D_i, f_i)_{1\le i\le q}$ is said to be gradient strategic on Γ or G-strategic on Γ if there is at least one sensor (D_{i_0}, f_{i_0}) which is G-strategic on Γ.

Remark.

1)-If the system (1) together with the output (2) is exactly G-observable on Γ then it is weakly G-observable on Γ.
2)-For $\Gamma_2 \subset \Gamma_1 \subset \partial\Omega$ the system (1) together with the output (2) is exactly (resp. weakly) G-observable on Γ_1 then it is exactly (resp. weakly) G-observable on Γ_2.
3)-One can find states that are G-observable on Γ but not G-observable in the whole domain Ω. This is illustrated through the following example.

2.2 Example

Let's consider the two-dimensional system described in $\Omega =]0,1[\times]0,1[$ by the following equation

$$\begin{cases} \dfrac{\partial^2 y}{\partial t^2}(x_1,x_2,t) = \dfrac{\partial^2 y}{\partial x_1^2}(x_1,x_2,t) + \dfrac{\partial^2 y}{\partial x_2^2}(x_1,x_2,t) & \text{in} \quad Q, \\ y(x_1,x_2,0) = y^0(x_1,x_2), \dfrac{\partial y}{\partial t}(x_1,x_2,0) = y^1(x_1,x_2) & \text{in} \quad \Omega, \\ y(\zeta,\eta,t) = 0 & \text{on} \quad \Sigma, \end{cases} \tag{5}$$

The measurements are given by the output function

$$z(t) = \int_D y(x_1,x_2,t) f(x_1,x_2) dx_1 dx_2, \tag{6}$$

Where $D =]0,1[\times\{1/2\}$ is the sensor support and

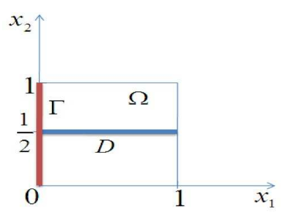

Fig. 2: The domain Ω, the subregion Γ, and sensors support D.

$f(x_1,x_2) = \sin(2\pi x_1)$ is the function repartition of the measures.
Let's consider $\Gamma = \{0\} \times [0,1]$ and

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$\overline{g}(x_1,x_2) = \begin{bmatrix} g^0(x_1,x_2) \\ g^1(x_1,x_2) \end{bmatrix}$ the gradient to be observed, with

$g^0(x_1,x_2) = (\pi\cos(\pi x_1)\sin(\pi x_2), 2\pi\sin(\pi x_1)\cos(\pi x_2))$
$g^1(x_1,x_2) = (-\pi\sin(\pi x_1)\cos(\pi x_2), -2\pi\cos(\pi x_1)\sin(\pi x_2))$
the gradient to be observed on Γ, then we have the following result:

Proposition 2.1. The gradient $\overline{g}$ is not weakly G-observable in the whole domain Ω, but it is weakly G-observable on Γ.

Proof. We have

$$\overline{K}\overline{\nabla}^*(\overline{g}) = \overline{CS}(t)\overline{\nabla}^*\overline{g}$$
$$= \sum_{m,j=1}^{\infty}\Big[\langle\nabla^* g^0, \Phi_{m_j}\rangle_{(H^2(\Omega)\cap H_0^1(\Omega))}\cos(-\lambda_{m_j})^{\frac{1}{2}}t$$
$$+(-\lambda_{m_j})^{-\frac{1}{2}}\langle\nabla^* g^1, \Phi_{m_j}\rangle_{(H^2(\Omega)\cap H_0^1(\Omega))}\sin(-\lambda_{m_j})^{\frac{1}{2}}t\Big]\langle\Phi_{m_j}, f\rangle$$
$$= \sum_{m,j=1}^{\infty}\frac{-1}{\lambda_{m_j}}[\frac{1}{2}m+j]\pi^2\cos(-\lambda_{m_j})^{\frac{1}{2}}t\sin(\frac{j\pi}{2})\delta_{m1}\delta_{m2}\delta_{j1}$$
$$+\sum_{m,j\in 2\mathbb{N}}\frac{1}{\lambda_{m_j}(-\lambda_{m_j})^{\frac{1}{2}}}[\frac{8m^2j+16j^2m}{(m^2-1)(j^2-1)}]\sin(-\lambda_{m_j})^{\frac{1}{2}}t\sin(\frac{j\pi}{2})\delta_{m2}$$

for $m=2$ and $j\in 2\mathbb{N}$, we have $\overline{K}\overline{\nabla}^*(\overline{g}) = 0$, Then the system (5) together with (6) is not weakly G-observable in Ω.
We show that the restriction of $\overline{g}$ to the subregion Γ is G-observable on Γ.
We have

$$\overline{K}\overline{\nabla}^*\overline{\gamma}^*\overline{\chi}_\Gamma^*\overline{\chi}_\Gamma\overline{\gamma}(\overline{g}) = \sum_{m,j=1}^{\infty}[\langle\chi_\Gamma\gamma g^0, \chi_\Gamma\gamma\nabla\Phi_{m_j}\rangle_{(H^{\frac{1}{2}}(\Gamma))^n}\cos(\sqrt{-\lambda_{m_j}}t)$$
$$+\sqrt{-\lambda_{m_j}}\langle\chi_\Gamma\gamma g^1, \chi_\Gamma\gamma\nabla\Phi_{m_j}\rangle_{(H^{\frac{1}{2}}(\Gamma))^n}\sin(\sqrt{-\lambda_{m_j}}t)]\langle\Phi_{m_j}, f\rangle_{L^2(D)}$$
$$= \sum_{m,j=1}^{\infty}\Big[\Big(\langle\tilde{\chi}_\Gamma\gamma_0 g_1^0, \tilde{\chi}_\Gamma\gamma_0\frac{\partial\Phi_{m_j}}{\partial x_1}\rangle_{H^{\frac{1}{2}}(\Gamma)}$$
$$+\langle\tilde{\chi}_\Gamma\gamma_0 g_2^0, \tilde{\chi}_\Gamma\gamma_0\frac{\partial\Phi_{m_j}}{\partial x_2}\rangle_{H^{\frac{1}{2}}(\Gamma)}\Big)\cos(\sqrt{-\lambda_{m_j}}t)$$
$$+\sqrt{-\lambda_{m_j}}\Big(\langle\tilde{\chi}_\Gamma\gamma_0 g_1^1, \tilde{\chi}_\Gamma\gamma_0\frac{\partial\Phi_{m_j}}{\partial x_1}\rangle_{H^{\frac{1}{2}}(\Gamma)}$$
$$+\langle\tilde{\chi}_\Gamma\gamma_0 g_2^1, \tilde{\chi}_\Gamma\gamma_0\frac{\partial\Phi_{m_j}}{\partial x_2}\rangle_{H^{\frac{1}{2}}(\Gamma)}\Big)\sin(\sqrt{-\lambda_{m_j}}t)\Big]\langle\Phi_{m_j}, f\rangle_{L^2(D)}$$
$$= \sum_{m,j=1}^{\infty}2m\pi^2\cos(\sqrt{-\lambda_{m_j}}t)\sin(\frac{j\pi}{2})\delta_{m2}\delta_{j1}$$
$$= 4\pi^2\cos(\sqrt{-\lambda_{21}}t) \neq 0$$

Then $\overline{g}$ is G-observable on Γ. ■

2.3 Characterizations

The gradient observability on Γ is characterized by the following results. **Proposition 2.2.**

1)-The system (1) together with the output (2) is exactly G-observable on Γ if and only if,
$\exists\alpha > 0, \|\overline{z}^*\|_{(H^{\frac{1}{2}}(\Gamma))^n\times(H^{\frac{1}{2}}(\Gamma))^n} \leq \alpha\|\overline{K}\overline{\nabla}^*\overline{\alpha}^*\overline{\chi}_\Gamma^*\overline{z}^*\|_{\mathscr{O}}$,
$\forall\overline{z}^* \in (H^{\frac{1}{2}}(\Gamma))^n\times(H^{\frac{1}{2}}(\Gamma))^n$.

2)-The system (1) together with the output (2) is weakly G-observable on Γ if and only if the operator $\overline{H}\overline{H}^*$ is positive definite.

Proof.

1)-Let's consider $h = Id_{(H^{\frac{1}{2}}(\Gamma))^n\times(H^{\frac{1}{2}}(\Gamma))^n}$ and $g = \overline{\chi}_\Gamma\overline{\alpha}\overline{\nabla}\overline{K}^*$, since the system (1) together with the output (2) is exactly G-observable on Γ, we have $Imh \subset Img$, which is equivalent to the fact that there exists $\alpha > 0$, such that
$\|\overline{h}^*\overline{z}^*\|_{(H^{\frac{1}{2}}(\Gamma))^n\times(H^{\frac{1}{2}}(\Gamma))^n} \leq \alpha\|\overline{g}^*\overline{z}^*\|_{\mathscr{O}\times\mathscr{O}}$,
$\forall\overline{z}^* \in (H^{\frac{1}{2}}(\Gamma))^n\times(H^{\frac{1}{2}}(\Gamma))^n$.

2)-Let's consider $\overline{z}^* \in (H^{\frac{1}{2}}(\Gamma))^n\times(H^{\frac{1}{2}}(\Gamma))^n$ such that $\langle\overline{H}\overline{H}^*\overline{z}^*, \overline{z}^*\rangle = 0$
then $\|\overline{H}^*\overline{z}^*\| = 0$ and therefore $\overline{z}^* \in Ker\overline{H}^*$, consequently, $\overline{z}^* = 0$, and then $\overline{H}\overline{H}^*$ is positive definite.
Conversely, let's consider $\overline{z}^* \in (H^{\frac{1}{2}}(\Gamma))^n\times(H^{\frac{1}{2}}(\Gamma))^n$ such that $\|\overline{H}^*\overline{z}^*\| = 0$, then $\langle\overline{H}\overline{H}^*\overline{z}^*, \overline{z}^*\rangle = 0$ and since $\overline{H}\overline{H}^*$ is positive definite, then $\overline{z}^* = 0$.
So, the system (1) together with the output (2) is weakly G-observable on Γ.

Here we show that it is possible to link the internal gradient observability and the boundary one. The boundary regional G-observability can be reduced as internal regional G-observability, we have the following result.

Proposition 2.3. Assume that $\omega \subset \Omega$ such that $\Gamma \subset \partial\omega\cap\partial\Omega$, then if the system (1) together with the output (2) is exactly (resp.weakly) G-observable in ω, then it is exactly (resp.weakly) G-observable on Γ.

Proof. Let's consider
$\overline{y} = (\overline{y}_1^1,\ldots,\overline{y}_n^1,\overline{y}_n^2,\ldots,\overline{y}_n^2) \in (H^{\frac{1}{2}}(\Gamma))^n\times(H^{\frac{1}{2}}(\Gamma))^n$,
$\tilde{y} = (\tilde{y}_1^1,\ldots,\tilde{y}_n^1,\tilde{y}_n^2,\ldots,\tilde{y}_n^2)$ is a continuous extension of $\overline{y}$ to $\partial\Omega$ such that $\tilde{y} \in (H^{\frac{1}{2}}(\partial\Omega))^n\times(H^{\frac{1}{2}}(\partial\Omega))^n$ with the following transformation

$$\overline{\mathscr{R}} : (H^{\frac{1}{2}}(\partial\Omega))^n\times(H^{\frac{1}{2}}(\partial\Omega))^n \longrightarrow (H^1(\Omega))^n\times(H^1(\Omega))^n$$
$$(z_1^1,\ldots,z_n^1,z_1^2,\ldots,z_n^2) \longrightarrow (\tilde{\mathscr{R}}z_1^1,\ldots,\tilde{\mathscr{R}}z_n^1,\tilde{\mathscr{R}}z_1^2,\ldots,\tilde{\mathscr{R}}z_n^2)$$

where $\tilde{\mathscr{R}} : H^{\frac{1}{2}}(\partial\Omega) \longrightarrow H^1(\Omega)$, such that $\gamma_0\tilde{\mathscr{R}}g = g$ $\forall g \in H^{\frac{1}{2}}(\partial\Omega)$, then $\overline{\mathscr{R}}$ verify $\overline{\gamma}\overline{\mathscr{R}}\tilde{y} = \tilde{y}$ $\forall\tilde{y} \in (H^{\frac{1}{2}}(\partial\Omega))^n$(see[2])
then using the trace theorem (see[1]), there exists
$\overline{\mathscr{R}}\tilde{y} = (\tilde{\mathscr{R}}\tilde{y}_1^1,\ldots,\tilde{\mathscr{R}}\tilde{y}_n^1,\tilde{\mathscr{R}}\tilde{y}_1^2,\ldots,\tilde{\mathscr{R}}\tilde{y}_n^2) \in (H^1(\Omega))^n\times(H^1(\Omega))^n$
with a bounded support such that $\overline{\gamma}\overline{\mathscr{R}}\tilde{y} = \tilde{y}$.
Let $\overline{\overline{\chi}}_\Gamma$ be the map restriction from
$(H^{\frac{1}{2}}(\partial\omega))^n\times(H^{\frac{1}{2}}(\partial\omega))^n \longrightarrow (H^{\frac{1}{2}}(\Gamma))^n\times(H^{\frac{1}{2}}(\Gamma))^n$,
and $\overline{\overline{\gamma}}$ be the trace mapping from
$(H^1(\omega))^n\times(H^1(\omega))^n \longrightarrow (H^{\frac{1}{2}}(\partial\omega))^n\times(H^{\frac{1}{2}}(\partial\omega))^n$

- –Since the system (1) together with the output (2) is exactly G-observable in ω (see[12]), there exists $\overline{z} \in \mathscr{O}$ such as $\overline{\chi}_\omega\overline{\mathscr{R}}\tilde{y} = \overline{\chi}_\omega\overline{\nabla}\overline{K}^*\overline{z}$, or $\Gamma \subset \partial\omega\cap\partial\Omega$ then $\overline{y} = \overline{\overline{\chi}}_\Gamma\overline{\overline{\gamma}}\overline{\chi}_\omega\overline{\nabla}\overline{K}^*\overline{z}$ and the system (1) together with the output (2) is exactly G-observable in Γ.
- –If the system (1) together with the output (2) is weakly G-observable in ω, then $\forall\varepsilon > 0$, there exist $\overline{z} \in \mathscr{O}$ such as $\|\overline{\chi}_\omega\overline{\mathscr{R}}\tilde{y} - \overline{\chi}_\omega\overline{\nabla}\overline{K}^*\overline{z}\|_{(H^1(\omega))^n\times(H^1(\omega))^n} \leq \varepsilon$
or $\overline{\overline{\gamma}}$ is continuous, then
$\|\overline{\overline{\gamma}}\overline{\chi}_\omega\overline{\mathscr{R}}\tilde{y} - \overline{\overline{\gamma}}\overline{\chi}_\omega\overline{\nabla}\overline{K}^*\overline{z}\|_{(H^{\frac{1}{2}}(\partial\omega))^n\times(H^{\frac{1}{2}}(\partial\omega))^n} \leq \varepsilon$

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finaly $\|\overline{y}-\overline{\overline{\chi}}_{\Gamma}\overline{\overline{\gamma}\chi_{\omega}}\overline{\nabla}\overline{K}^*\overline{z}\|_{(H^{\frac{1}{2}}(\Gamma))^n\times(H^1(\Gamma))^n}\leq\varepsilon$

Therefore the system (1) together with the output (2) is weakly G-observable on Γ.

In conclusion we have shown that the regional boundary gradient observability can be reduced as internal regional gradient observability.

3 Γ-strategic sensors

We propose to give a characterization of sensors making a Γ-weakly observability. Let's consider the system (1) and assume that the measurements are given by way of q sensors $(D_i,f_i)_{1\leq i\leq q}$. The output equation is then given by:

$$z(t)=Cy(t)=\big(z_1(t),z_2(t),\dots,z_q(t)\big)$$

with $D_i=\{b_i\}$ and $f=\delta(.-b_i)$ in the case of pointwise sensor.
and $D_i\subset\overline{\Omega}$ with $f\in L^2(D_i)$ for the zonal sensor.
We assume that $(\tilde{\chi}_{\Gamma}\gamma_0\Phi_{m_j})_{1\leq j\leq r_m,\,m\geq 1}$ form a complete set in $H^{\frac{1}{2}}(\Gamma)$.
More we assume that $r=\sup\limits_{m} r_m<\infty$.
We have the following proposition

Proposition 3.1. The sequence of sensors $(D_i,f_i)_{1\leq i\leq q}$ is G-strategic on Γ if

- $q\geq r$
- $rankG_m=r_m,\ \forall m\geq 1$

where

$$(G_m)_{ij}=\begin{cases}\displaystyle\sum_{k=1}^{n}\langle\frac{\partial\Phi_{m_j}}{\partial x_k},f_i\rangle & \text{in zone case}\\ \displaystyle\sum_{k=1}^{n}\frac{\partial\Phi_{m_j}}{\partial x_k}(b_i) & \text{in pointwise case}\end{cases}$$

with $1\leq i\leq q$ and $1\leq j\leq r_m$.

Proof. We show that if rank $G_m=r_m$, $\forall m\geq 1$, then the system (1) together with the output (2) is weakly G-observable on Γ.
We suppose that $\quad Ker\overline{K}\overline{\nabla}^*\overline{\gamma}^*\overline{\chi}_{\Gamma}^*\neq\{0\}$
i.e. there exists $z^*=(z^{1*},z^{2*})\in(H^{\frac{1}{2}}(\Gamma))^n\times(H^{\frac{1}{2}}(\Gamma))^n$ as $(z^{1*},z^{2*})\neq 0$ and $\overline{K}\overline{\nabla}^*\overline{\gamma}^*\overline{\chi}_{\Gamma}^*z^*=0.$
with

$$\begin{aligned}\overline{\gamma}^*\overline{\chi}_{\Gamma}^*z^*&=(\gamma^*\chi_{\Gamma}^*z^{1*},\gamma^*\chi_{\Gamma}^*z^{2*})\\&=(\gamma_0^*\tilde{\chi}_{\Gamma}^*z_1^{1*},\dots,\gamma_0^*\tilde{\chi}_{\Gamma}^*z_n^{1*},\gamma_0^*\tilde{\chi}_{\Gamma}^*z_1^{2*},\dots,\gamma_0^*\tilde{\chi}_{\Gamma}^*z_n^{2*})\end{aligned}$$

then

$$\begin{aligned}&\overline{K}\overline{\nabla}^*\overline{\gamma}^*\overline{\chi}_{\Gamma}^*(z^{1*},z^{2*})\\&=\overline{K}\overline{\nabla}^*(\gamma_0^*\tilde{\chi}_{\Gamma}^*z_1^{1*},\dots,\gamma_0^*\tilde{\chi}_{\Gamma}^*z_n^{1*},\gamma_0^*\tilde{\chi}_{\Gamma}^*z_1^{2*},\dots,\gamma_0^*\tilde{\chi}_{\Gamma}^*z_n^{2*})\\&=\sum_{m\geq 1}\sum_{j=1}^{r_m}\sum_{k=1}^{n}\Big[\langle\gamma_0^*\tilde{\chi}_{\Gamma}^*z_k^{1*},\Phi_{m_j}\rangle\cos(-\lambda_m)^{\frac{1}{2}}t\\&+(-\lambda_m)^{-\frac{1}{2}}\langle\gamma_0^*\tilde{\chi}_{\Gamma}^*z_k^{2*},\Phi_{m_j}\rangle\sin(-\lambda_m)^{\frac{1}{2}}t\Big]\langle\frac{\partial\Phi_{m_j}}{\partial x_k},f_i\rangle\\&=0\quad\forall i=1\dots q\end{aligned}$$

for T large enough the functions $\{\sin(-\lambda_n)^{\frac{1}{2}}(.),\cos(-\lambda_n)^{\frac{1}{2}}(.)\}_{n\geq 1}$ constitute a complete orthonormal set in $L^2(0,T)$, then

$$\begin{cases}\displaystyle\sum_{j=1}^{r_m}\langle\gamma_0^*\tilde{\chi}_{\Gamma}^*z_k^{1*},\Phi_{m_j}\rangle\sum_{k=1}^{n}\langle\frac{\partial\Phi_{m_j}}{\partial x_k},f_i\rangle_{L^2(D_i)}=0,\ \forall m\geq 1,\forall i=1,\dots,q\\ \displaystyle\sum_{j=1}^{r_m}\langle\gamma_0^*\tilde{\chi}_{\Gamma}^*z_k^{2*},\Phi_{m_j}\rangle\sum_{k=1}^{n}\langle\frac{\partial\Phi_{m_j}}{\partial x_k},f_i\rangle_{L^2(D_i)}=0,\ \forall m\geq 1,\forall i=1,\dots,q\end{cases}$$

and then

$$\begin{cases}\displaystyle\sum_{j=1}^{r_m}\langle z_k^{1*},\tilde{\chi}_{\Gamma}\gamma_0\Phi_{m_j}\rangle\sum_{k=1}^{n}\langle\frac{\partial\Phi_{m_j}}{\partial x_k},f_i\rangle_{L^2(D_i)}=0,\ \forall m\geq 1,\forall i=1,\dots,q\\ \displaystyle\sum_{j=1}^{r_m}\langle z_k^{2*},\tilde{\chi}_{\Gamma}\gamma_0\Phi_{m_j}\rangle\sum_{k=1}^{n}\langle\frac{\partial\Phi_{m_j}}{\partial x_k},f_i\rangle_{L^2(D_i)}=0,\ \forall m\geq 1,\forall i=1,\dots,q\end{cases}$$

but $z^{1*}\in(H^{\frac{1}{2}}(\Gamma))^n$ if $z^{1*}\neq 0$ and $(\gamma_0^*\tilde{\chi}_{\Gamma}{}^*\Phi_{m_j})_{1\leq j\leq r_m\ m\geq 1}$ form a complete set in $H^{\frac{1}{2}}(\Gamma)$, then

$$z_k^{1*}=\sum_{m\geq 1}\sum_{j=1}^{r_m}\langle z_k^{1*},\tilde{\chi}_{\Gamma}\gamma_0\Phi_{m_j}\rangle_{H^{\frac{1}{2}}(\Gamma)}\tilde{\chi}_{\Gamma}\gamma_0\Phi_{m_j}\quad\forall k=1,\dots,n$$

if $z^{1*}\neq 0$ then there exists $1\leq k_0\leq n, m_1\geq 1$ and $1\leq j\leq r_{m_1}$ with $\langle z_k^{1*},\tilde{\chi}_{\Gamma}\gamma_0\Phi_{m_j}\rangle_{H^{\frac{1}{2}}(\Gamma)}\neq 0$
Let's consider, then

$$z_{m_1}^1=\begin{bmatrix}\langle z_1^{1*},\tilde{\chi}_{\Gamma}\gamma_0\Phi_{m_1 1}\rangle_{H^{\frac{1}{2}}(\Gamma)} & \dots\langle z_{k_0}^{1*},\tilde{\chi}_{\Gamma}\gamma_0\Phi_{m_1 1}\rangle_{H^{\frac{1}{2}}(\Gamma)} & \dots\langle z_n^{1*},\tilde{\chi}_{\Gamma}\gamma_0\Phi_{m_1 1}\rangle_{H^{\frac{1}{2}}(\Gamma)}\\ & \vdots & \\ \langle z_1^{1*},\tilde{\chi}_{\Gamma}\gamma_0\Phi_{m_1 r_{m_1}}\rangle_{H^{\frac{1}{2}}(\Gamma)} & \dots\langle z_{k_0}^{1*},\tilde{\chi}_{\Gamma}\gamma_0\Phi_{m_1 r_{m_1}}\rangle_{H^{\frac{1}{2}}(\Gamma)} & \dots\langle z_n^{1*},\tilde{\chi}_{\Gamma}\gamma_0\Phi_{m_1 r_{m_1}}\rangle_{H^{\frac{1}{2}}(\Gamma)}\end{bmatrix}$$

then we obtain $G_{m_1}z_{m_1}^1=0$, but $z_{m_1}^1\neq 0$, then $rankG_{m_1}\neq r_{m_1}$ this is contradiction with $rankG_m=r_m$, $\forall m\geq 1$,
the same think for $z^{2*}\in(H^{\frac{1}{2}}(\Gamma))^n$ and $z^{2*}\neq 0$.
Finally $Ker\overline{K}\overline{\nabla}^*\overline{\gamma}^*\overline{\chi}_{\Gamma}^*=\{0\}$, then the system (1) together with the output (2) is weakly G-observable on Γ.

4 Regional boundary gradient reconstruction

We take $A=\Delta$, the system (1) is then written as

$$\begin{cases}\dfrac{\partial^2 y}{\partial t^2}(x,t)=\Delta y(x,t) & \text{in } Q\\ y(x,0)=y^0(x),\ \dfrac{\partial y}{\partial t}(x,0)=y^1(x) & \text{in } \Omega\\ y(\xi,t)=0 & \text{on } \Sigma\end{cases}\tag{7}$$

Let's consider the set

$$\begin{aligned}\mathscr{G}&=\{(h^1,h^2)\in(L^2(\Omega))^n\times(L^2(\Omega))^n|\,h^1=h^2=0\,\mathrm{sur}\,\Omega\setminus\omega\}\\&\cap\{\overline{\nabla}(f^1,f^2)=(\nabla f^1,\nabla f^2)|\,(f^1,f^2)\in\big(H^2(\Omega)\cap H_0^1(\Omega)\big)^2\}\end{aligned}$$

where ω be a subset of Ω such that $\Gamma \subset \partial\omega \cap \partial\Omega$.
It is known that if the system (1) together with (2) is weakly G-observable in ω, then it is weakly G-observable on Γ (see [**??**]). This result links the internal regional gradient observability in ω with the boundary case.
We decompose the initial gradient $\overline{\nabla} \bar{y}^0$ in the form

$$\overline{\nabla}\bar{y}^0 = \begin{cases} \overline{\nabla}\bar{y}^0_1 \text{ in } & \omega \\ \overline{\nabla}\bar{y}^0_2 \text{ in } & \Omega\backslash\omega \end{cases}$$

We present an approach which allows the reconstruction of the initial gradient $\tilde{\overline{\nabla}}\bar{y}^0_1$ on Γ based on the internal regional gradient observability techniques (see[17]) and Hilbert Uniqueness Method (see[3]).
In the following, we proceed to reconstruct the initial gradient $\overline{\nabla}\bar{y}^0_1 = (\nabla y^0_1, \nabla y^1_1)$ in the subregion ω and then we deduce its trace $\tilde{\overline{\nabla}}\bar{y}^0_1$ on $\Gamma \subset \partial\Omega \cap \partial\omega$.
For $(\varphi^0,\varphi^1) \in (H^2(\Omega)\cap H^1_0(\Omega)) \times (H^2(\Omega)\cap H^1_0(\Omega))$, we consider the following system

$$\begin{cases} \dfrac{\partial^2\varphi}{\partial t^2}(x,t) = \Delta\varphi(x,t) & \text{in } Q \\ \varphi(x,0) = \varphi^0(x), \dfrac{\partial\varphi}{\partial t}(x,0) = \varphi^1(x) & \text{in } \Omega \\ \varphi(\xi,t) = 0 & \text{on } \Sigma \end{cases} \tag{8}$$

which admits a unique solution

$\varphi \in C(0,T;H^2(\Omega)) \cap C^1(0,T;H^1_0(\Omega)) \cap C^2(0,T;L^2(\Omega))$(see [2])

We develop our reconstruction approach in the case where the system (1) is observed by means of pointwise sensor.
In the following, we shall consider two kind of measurements.

4.1 State measurement case

Here we consider the system (1) with the output function

$$z(t) = y(b,t), \quad b \in \Omega, \quad t \in]0,T[\tag{9}$$

For $(\tilde{\varphi}^0,\tilde{\varphi}^1) \in \tilde{\mathscr{G}}$, there exists a unique $(\varphi^0,\varphi^1) \in (H^2(\Omega)\cap H^1_0(\Omega))^2$ such that $\overline{\nabla}(\varphi^0,\varphi^1) = (\nabla\varphi^0,\nabla\varphi^1) = (\tilde{\varphi}^0,\tilde{\varphi}^1)$.
Then we consider the semi-norm on $\tilde{\mathscr{G}}$ defined by

$$(\tilde{\varphi}^0,\tilde{\varphi}^1) \longmapsto \|(\tilde{\varphi}^0,\tilde{\varphi}^1)\|_{\tilde{\mathscr{G}}} = \left[\int_0^T \left(\sum_{k=1}^n \frac{\partial\varphi}{\partial x_k}(b,t)\right)^2 dt\right]^{\frac{1}{2}}, \tag{10}$$

where φ is the solution of (8).
We introduce the auxiliary system

$$\begin{cases} \dfrac{\partial^2\tilde{\psi}}{\partial t^2}(x,t) = \Delta\tilde{\psi}(x,t) + \displaystyle\sum_{k=1}^n \frac{\partial\varphi}{\partial x_k}(b,t)\delta(x-b) & \text{in } Q \\ \tilde{\psi}(x,T) = 0, \dfrac{\partial\tilde{\psi}}{\partial t}(x,T) = 0 & \text{in } \Omega \\ \dfrac{\partial\tilde{\psi}}{\partial\nu}(\xi,t) = 0 & \text{on } \Sigma \end{cases} \tag{11}$$

The solution $\tilde{\psi}$ of (11) is in $C(0,T;H^1_0(\Omega)) \cap C^1(0,T;L^2(\Omega))$ (see [2]), where $\dfrac{\partial\tilde{\psi}}{\partial\nu}$ denotes the conormal with respect to Δ.
When the semi norm is a norm (see [4]), we also denote by $\tilde{\mathscr{G}}$ the completion of $\tilde{\mathscr{G}}$ and consider the operator

$$\begin{aligned} \Lambda : \tilde{\mathscr{G}} &\longrightarrow \tilde{\mathscr{G}}^* \\ (\tilde{\varphi}^0,\tilde{\varphi}^1) &\longmapsto \mathscr{P}(-\tilde{\Psi}^1,\tilde{\Psi}^0) \end{aligned}$$

where $\mathscr{P} = \overline{\chi}^*_\omega\overline{\chi}_\omega$ and $\begin{cases} \tilde{\Psi}^1 = (\tilde{\psi}^1,\ldots,\tilde{\psi}^1) \\ \tilde{\Psi}^0 = (\tilde{\psi}^0,\ldots,\tilde{\psi}^0) \end{cases}$
with $\tilde{\Psi}(x,0) = \tilde{\Psi}^0(x)$ and $\dfrac{\partial\tilde{\Psi}}{\partial t}(x,0) = \tilde{\Psi}^1(x)$
We introduce the system

$$\begin{cases} \dfrac{\partial^2\bar{\psi}}{\partial t^2}(x,t) = \Delta\bar{\psi}(x,t) + \displaystyle\sum_{k=1}^n \frac{\partial y}{\partial x_k}(b,t)\delta(x-b) & \text{in } Q \\ \bar{\psi}(x,T) = 0, \dfrac{\partial\bar{\psi}}{\partial t}(x,T) = 0 & \text{in } \Omega \\ \dfrac{\partial\bar{\psi}}{\partial\nu}(\xi,t) = 0 & \text{on } \Sigma \end{cases} \tag{12}$$

If $(\tilde{\varphi}_0,\tilde{\varphi}_1)$ is chosen such that $\tilde{\psi}^1 = \bar{\psi}^1$ and $\tilde{\psi}^0 = \bar{\psi}^0$ in ω, then the system (12) looks like the adjoint of the system (1), and the regional gradient observability in ω amounts to the conditions for solving the equation

$$\Lambda(\tilde{\varphi}_0,\tilde{\varphi}_1) = \mathscr{P}(-\bar{\Psi}^1,\bar{\Psi}^0) \tag{13}$$

where $\begin{cases} \bar{\Psi}^1 = (\dfrac{\partial\bar{\psi}}{\partial t}(0),\ldots,\dfrac{\partial\bar{\psi}}{\partial t}(0)) \\ \bar{\Psi}^0 = (\bar{\psi}(0),\ldots,\bar{\psi}(0)) \end{cases}$

with $\bar{\psi}$ being the solution of (12) **Remark.** Among choice of $\tilde{\varphi}^0$ and $\tilde{\varphi}^1$ who realizes $\tilde{\psi}^0 = \bar{\psi}^0$ and $\tilde{\psi}^1 = \bar{\psi}^1$ in ω, where $\tilde{\psi}^0 = \nabla y^0_1$ and $\tilde{\psi}^1 = \nabla y^1_1$, this choice is not unique but if we show that the operator Λ is an isomorphism then (13) admit a unique solution $(\tilde{\varphi}^0,\tilde{\varphi}^1)$ which will coincide with $(\nabla y^0_1, \nabla y^1_1)$ in ω.

Proposition 4.1. If the sensor (b,δ_b) is G-strategic in ω, then the semi norm (10) becomes a norm and the equation (13) has a unique solution $(\tilde{\varphi}^0,\tilde{\varphi}^1)$ which corresponds to

$(\nabla y_1^0, \nabla y_1^1)$ and then $\bar{\chi}_\Gamma \bar{\gamma}(\tilde{\varphi}^0, \tilde{\varphi}^1)$ is the initial gradient to be observed on Γ.

Proof.
If the system (1) together with the output (2) is weakly G-observable in ω, then (10) defines a norm in $\tilde{\mathscr{G}}$.
Let's consider (Φ_m) the eigenfunctions of the operator Δ, without loss of generality, we assume that the eigenvalues λ_m are simple.
Let's consider $(\tilde{\varphi}^0, \tilde{\varphi}^1) \in \tilde{\mathscr{G}}$ such as $\|(\tilde{\varphi}^0, \tilde{\varphi}^1)\|_{\mathscr{G}} = 0$, we show that $(\tilde{\varphi}^0, \tilde{\varphi}^1) = (0,0)$ which gives

$$\sum_{i\geq 1}\left[\begin{array}{l}\langle \varphi^0, \Phi_i\rangle \cos(-\lambda_i)^{\frac{1}{2}} t \\ +(-\lambda_i)^{-\frac{1}{2}}\langle \varphi^1, \Phi_i\rangle \sin(-\lambda_i)^{\frac{1}{2}} t\end{array}\right] \sum_{k=1}^{n} \frac{\partial \Phi_i}{\partial x_k}(b) = 0$$

for T large enough the functions $\{(\sin(-\lambda_i)^{\frac{1}{2}} t)_{i\geq 1}; (\cos(-\lambda_i)^{\frac{1}{2}} t)_{i\geq 1}\}$ form a complete orthonormal set in $L^2(0,T)$. we obtain :

$$\langle \varphi^0, \Phi_i\rangle_{L^2(\Omega)} \sum_{k=1}^{n} \frac{\partial \Phi_i}{\partial x_k}(b) = 0 \quad \forall i \geq 1$$

and

$$\langle \varphi^1, \Phi_i\rangle_{L^2(\Omega)} \sum_{k=1}^{n} \frac{\partial \Phi_i}{\partial x_k}(b) = 0 \quad \forall i \geq 1$$

But the sensor (b, δ_b) is G-strategic, then $\sum_{k=1}^{n} \frac{\partial \Phi_i}{\partial x_k}(b) \neq 0$, $\forall i \geq 1$, then $\langle \varphi^0, \Phi_i\rangle = \langle \varphi^1, \Phi_i\rangle = 0$, $\forall i \geq 1$ which implies $(\varphi^0, \varphi^1) = (0,0)$, then $(\tilde{\varphi}^0, \tilde{\varphi}^1) = (0,0)$.
We show that Λ is an isomorphism.
Multiplying (11) by $\frac{\partial \varphi}{\partial x_k}$ and integrating over Q, we obtain

$$\begin{aligned}
&\int_0^T \langle \frac{\partial \varphi}{\partial x_k}(x,t), \frac{\partial^2 \tilde{\psi}}{\partial t^2}(x,t)\rangle_{L^2(\Omega)} dt \\
&= \int_0^T \langle \frac{\partial \varphi}{\partial x_k}(x,t), \Delta \tilde{\psi}(x,t)\rangle_{L^2(\Omega)} dt \\
&+ \int_0^T \langle \frac{\partial \varphi}{\partial x_k}(x,t), \sum_{l=1}^{n} \frac{\partial \varphi}{\partial x_l}(b,t)\delta(x-b)\rangle_{L^2(\Omega)} dt
\end{aligned}$$

which gives

$$\begin{aligned}
&\left[\langle \frac{\partial \varphi}{\partial x_k}(x,t), \frac{\partial \tilde{\psi}}{\partial t}(x,t)\rangle_{L^2(\Omega)}\right]_0^T - \left[\langle \frac{\partial}{x_k}(\frac{\partial \varphi}{\partial t}(x,t)), \tilde{\psi}(x,t)\rangle_{L^2(\Omega)}\right]_0^T \\
&\quad + \int_0^T \langle \frac{\partial}{\partial x_k} \frac{\partial^2 \varphi}{\partial t^2}(x,t), \tilde{\psi}(x,t)\rangle_{L^2(\Omega)} dt \\
&\quad = \int_0^T \langle \frac{\partial \varphi}{\partial x_k}(x,t), \Delta \tilde{\psi}(x,t)\rangle_{L^2(\Omega)} dt \\
&\quad + \int_0^T \frac{\partial \varphi}{\partial x_k}(b,t) \sum_{l=1}^{n} \frac{\partial \varphi}{\partial x_l}(b,t) dt
\end{aligned}$$

with the final condition, we obtain

$$\begin{aligned}
-\langle \frac{\partial \varphi}{\partial x_k}(x,0), \frac{\partial \tilde{\psi}}{\partial t}(x,0)\rangle_{L^2(\Omega)} &+ \langle \frac{\partial}{\partial x_k} \frac{\partial \varphi}{\partial t}(x,0), \tilde{\psi}(x,0)\rangle_{L^2(\Omega)} \\
&+ \langle \Delta \frac{\partial \varphi}{\partial x_k}(x,t), \tilde{\psi}(x,t)\rangle_{L^2(Q)} \\
&= \langle \frac{\partial \varphi}{\partial x_k}(x,t), \Delta \tilde{\psi}(x,t)\rangle_{L^2(Q)} \\
&+ \int_0^T \frac{\partial \varphi}{\partial x_k}(b,t) \sum_{l=1}^{n} \frac{\partial \varphi}{\partial x_l}(b,t) dt
\end{aligned}$$

Using Green formula, we obtain

$$\begin{aligned}
-\langle \frac{\partial \varphi}{\partial x_k}(x,0), \frac{\partial \tilde{\psi}}{\partial t}(x,0)\rangle_{L^2(\Omega)} &+ \langle \frac{\partial}{\partial x_k} \frac{\partial \varphi}{\partial t}(x,0), \tilde{\psi}(x,0)\rangle_{L^2(\Omega)} \\
&= \int_0^T \frac{\partial \varphi}{\partial x_k}(b,t) \sum_{l=1}^{n} \frac{\partial \varphi}{\partial x_l}(b,t) dt
\end{aligned}$$

and then

$$\begin{aligned}
&\langle (-\frac{\partial \tilde{\psi}}{\partial x_k}(x,0), \tilde{\psi}(x,0)), (\frac{\partial \varphi}{\partial x_k}(x,0), \frac{\partial}{\partial t} \frac{\partial \varphi}{\partial x_k}(x,0))\rangle_{L^2(\Omega)} \\
&\quad = \int_0^T \frac{\partial \varphi}{\partial x_k}(b,t) \sum_{l=1}^{n} \frac{\partial \varphi}{\partial x_l}(b,t) dt
\end{aligned}$$

Thus

$$\begin{aligned}
&\sum_{k=1}^{n} \langle (-\frac{\partial \tilde{\psi}}{\partial x_k}(x,0), \tilde{\psi}(x,0)), (\frac{\partial \varphi}{\partial x_k}(x,0), \frac{\partial}{\partial t} \frac{\partial \varphi}{\partial x_k}(x,0))\rangle_{L^2(\Omega)} \\
&\quad = \sum_{k=1}^{n} \int_0^T \frac{\partial \varphi}{\partial x_k}(b,t) \sum_{l=1}^{n} \frac{\partial \varphi}{\partial x_l}(b,t) dt
\end{aligned}$$

Finally

$$\begin{aligned}
\langle \Lambda(\tilde{\varphi}^0, \tilde{\varphi}^1), (\tilde{\varphi}^0, \tilde{\varphi}^1)\rangle &= \int_0^T \left(\sum_{l=1}^{n} \frac{\partial \varphi}{\partial x_l}(b,t)\right)^2 dt \\
&= \|(\tilde{\varphi}^0, \tilde{\varphi}^1)\|^2_{\tilde{\mathscr{G}}}, \quad \forall (\tilde{\varphi}^0, \tilde{\varphi}^1) \in \mathscr{G}
\end{aligned}$$

which proves that Λ is an isomorphism and (10) has a unique solution which corresponds to the gradient of the initial state to be estimated in the subregion ω.

4.2 Speed measurement case

Here we consider the system (1) augmented with the output function

$$z(t) = \frac{\partial y}{\partial t}(b,t), \quad b \in \Omega, \quad t \in]0,T[\qquad (14)$$

For$(\tilde{\varphi}^0, \tilde{\varphi}^1) \in \tilde{\mathscr{G}}$, the system (8) produces the solution φ.
We consider the semi-norm on $\tilde{\mathscr{G}}$ defined by

$$(\tilde{\varphi}^0, \tilde{\varphi}^1) \longmapsto \|(\tilde{\varphi}^0, \tilde{\varphi}^1)\|_{\tilde{\mathscr{G}}} = \left[\int_0^T \left(\sum_{k=1}^{n} \frac{\partial^2 \varphi}{\partial x_k \partial t}(b,t)\right)^2 dt\right]^{\frac{1}{2}} \qquad (15)$$

We introduce the auxiliary system

$$\begin{cases}
\frac{\partial^2 \tilde{\psi}}{\partial t^2}(x,t) = \Delta \tilde{\psi}(x,t) + \sum_{k=1}^{n} \frac{\partial^2 \varphi}{\partial x_k \partial t}(b,t)\delta(x-b) & \text{in } Q \\
\tilde{\psi}(x,T) = 0, \frac{\partial \tilde{\psi}}{\partial t}(x,T) = 0 & \text{in } \Omega \\
\frac{\partial \tilde{\psi}}{\partial \nu}(\xi,t) = 0 & \text{on } \Sigma
\end{cases} \qquad (16)$$

The solution $\tilde{\psi}$ of (16) is in $C(0,T;H_0^1(\Omega)) \cap C^1(0,T;L^2(\Omega))$(see [2]).
The resolution of the system (16) provides $\tilde{\psi}(x,0) = \tilde{\psi}^0(x)$

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and $\dfrac{\partial \tilde{\psi}}{\partial t}(x,0) = \tilde{\psi}^1(x)$.
When the semi norm (15) is a norm, we also denote by $\mathscr{G}$ the completion of $\mathscr{G}$ and we consider the operator

$$\begin{aligned} \Lambda_1 : \mathscr{G} &\longrightarrow \mathscr{G}^* \\ (\tilde{\varphi}^0, \tilde{\varphi}^1) &\longmapsto \mathscr{P}(\Delta\tilde{\Psi}^0, -\tilde{\Psi}^1) \end{aligned}$$

where $\mathscr{P} = \overline{\chi}^*_\omega \overline{\chi}_\omega$ and $\begin{cases} \tilde{\Psi}^1 = (\tilde{\psi}^1(0), \dots, \tilde{\psi}^1(0)) \\ \tilde{\Psi}^0 = (\tilde{\psi}^0(0), \dots, \tilde{\psi}^0(0)) \end{cases}$

With $\tilde{\Psi}(x,0) = \tilde{\Psi}^0(x)$ and $\dfrac{\partial \tilde{\Psi}}{\partial t}(x,0) = \tilde{\Psi}^1(x)$
We introduce the system

$$\begin{cases} \dfrac{\partial^2 \overline{\psi}}{\partial t^2}(x,t) = \Delta \overline{\psi}(x,t) + \displaystyle\sum_{k=1}^{n} \dfrac{\partial^2 y}{\partial x_k \partial t}(b,t)\delta(x-b) & \text{in } Q \\ \overline{\psi}(x,T) = 0, \dfrac{\partial \overline{\psi}}{\partial t}(x,T) = 0 & \text{in } \Omega \\ \dfrac{\partial \overline{\psi}}{\partial \nu}(\xi,t) = 0 & \text{on } \Sigma \end{cases} \tag{17}$$

If $(\tilde{\varphi}^0, \tilde{\varphi}^1)$ is chosen such that $\tilde{\psi}^1 = \overline{\psi}^1$ and $\tilde{\psi}^0 = \overline{\psi}^0$ in ω, then the system (17) looks like the adjoint of the system (1), and the regional gradient observability amounts to the conditions for solving the equation

$$\Lambda_1(\tilde{\varphi}^0, \tilde{\varphi}^1) = \mathscr{P}(\Delta\overline{\Psi}^0, -\overline{\Psi}^1) \tag{18}$$

where

$$\begin{cases} \overline{\Psi}^1 = (\dfrac{\partial \overline{\psi}}{\partial t}(0), \dots, \dfrac{\partial \overline{\psi}}{\partial t}(0)) \\ \\ \overline{\Psi}^0 = (\overline{\psi}(0), \dots, \overline{\psi}(0)) \end{cases}$$

with $\overline{\psi}$ being the solution of (17)

Remark. Among choice of $\tilde{\varphi}^0$ and $\tilde{\varphi}^1$ who realizes $\tilde{\psi}^0 = \overline{\psi}^0$ and $\tilde{\psi}^1 = \overline{\psi}^1$ in ω.
For $\tilde{\psi}^0 = \nabla y_1^0$ and $\tilde{\psi}^1 = \nabla y_1^1$, this choice is not unique but if we show that the operator Λ is an isomorphism then (18) admit a unique solution $(\tilde{\varphi}^0, \tilde{\varphi}^1)$ which will coincide with $(\nabla y_1^0, \nabla y_1^1)$ in ω,

Proposition 4.2. If the sensor (b, δ_b) is G-strategic in ω, then the semi norm (15) becomes a norm and the equation (18) has a unique solution $(\tilde{\varphi}^0, \tilde{\varphi}^1)$ and then $\chi_\Gamma \gamma(\tilde{\varphi}^0, \tilde{\varphi}^1)$ correspondent to $(\nabla y_1^0, \nabla y_1^1)$ is the initial gradient to be observed on Γ.

Proof.
With minor technical modifications, the proof is simular to the state measurement one.

5 Numerical approach

In this section, we shall give a numerical approach, which will reconstruct the initial gradient ∇y_1^0 and ∇y_1^1 in ω.
We consider the system (1) observed by a pointwise sensor located at $b \in \Omega$.

5.1 State measurement case

We consider the system (1) toghether with the output function (9).
The resolution of the equation (18) is equivalent to the minimization of the functional

$$\begin{aligned} \mathscr{R}(\tilde{\varphi}^0, \tilde{\varphi}^1) &= \frac{1}{2}\langle \Lambda(\tilde{\varphi}^0, \tilde{\varphi}^1), (\tilde{\varphi}^0, \tilde{\varphi}^1)\rangle \\ &\quad - \langle \mathscr{P}(-\overline{\Psi}'(0), \overline{\Psi}(0)), (\tilde{\varphi}^0, \tilde{\varphi}^1)\rangle \\ &= \frac{1}{2}\int_0^T (\sum_{l=1}^{n} \frac{\partial \varphi}{\partial x_l}(b,t))^2 dt + \langle \overline{\Psi}^1, \tilde{\varphi}^0\rangle - \langle \overline{\Psi}^0, \tilde{\varphi}^1\rangle \end{aligned} \tag{19}$$

the minimization of (26) is equivalent to finding

$$\inf_{(\varphi^0,\varphi^1)} \left(\begin{aligned} &\frac{T}{4}\sum_{m=1}^{\infty}\left[\langle \varphi^0, \Phi_m\rangle^2 - \frac{1}{\lambda_m}\langle \varphi^1, \Phi_m\rangle^2\right]\Big(\sum_{l=1}^{n}\frac{\partial \Phi_m}{\partial x_l}(b)\Big)^2 \\ &+ \sum_{m=1}^{\infty}\Big(\langle \varphi^0, \Phi_m\rangle \sum_{l=1}^{n}\langle \overline{\psi}^1, \frac{\partial \Phi_m}{\partial x_l}\rangle - \langle \varphi^1, \Phi_m\rangle \sum_{l=1}^{n}\langle \overline{\psi}^0, \frac{\partial \Phi_m}{\partial x_l}\rangle\Big) \end{aligned} \right)$$

with separation of the variables we obtain

$$\begin{cases} \inf_{\varphi^0} \displaystyle\sum_{m=1}^{\infty} \left[\begin{aligned} &\frac{T}{4}\langle \varphi^0, \Phi_m\rangle^2 \Big(\sum_{l=1}^{n}\frac{\partial \Phi_m}{\partial x_l}(b)\Big)^2 \\ &+\langle \varphi^0, \Phi_m\rangle \sum_{l=1}^{n}\langle \overline{\psi}^1, \frac{\partial \Phi_m}{\partial x_l}\rangle \end{aligned} \right] \\ \\ \inf_{\varphi^1} \displaystyle\sum_{m=1}^{\infty} \left[\begin{aligned} &\frac{-T}{4}\frac{1}{\lambda_m}\langle \varphi^1, \Phi_m\rangle^2 \Big(\sum_{l=1}^{n}\frac{\partial \Phi_m}{\partial x_l}(b)\Big)^2 \\ &-\langle \varphi^1, \Phi_m\rangle \sum_{l=1}^{n}\langle \overline{\psi}^0, \frac{\partial \Phi_m}{\partial x_l}\rangle \end{aligned} \right] \end{cases}$$

which is equivalent to

$$\langle \varphi^0, \Phi_m\rangle = -\frac{2}{T}\frac{\langle \overline{\Psi}^1, \nabla\Phi_m\rangle}{\Big(\displaystyle\sum_{l=1}^{n}\frac{\partial \Phi_m}{\partial x_l}(b)\Big)^2} \quad \forall m \geq 1$$

and

$$\langle \varphi^1, \Phi_m\rangle = -\frac{2\lambda_m}{T}\frac{\langle \overline{\Psi}^0, \nabla\Phi_m\rangle}{\Big(\displaystyle\sum_{l=1}^{n}\frac{\partial \Phi_m}{\partial x_l}(b)\Big)^2} \quad \forall m \geq 1$$

Then the initial gradient ∇y_1^0 and ∇y_1^1 can be approximated by the following formula :

$$\tilde{\nabla} y_1^0(x) = \begin{cases} -\dfrac{2}{T}\displaystyle\sum_{m=1}^{\infty}\left[\frac{\langle \overline{\Psi}^1, \nabla\Phi_m\rangle_{(L^2(\omega))^n}}{(\displaystyle\sum_{l=1}^{n}\frac{\partial \Phi_m}{\partial x_l}(b))^2}\right]\nabla\Phi_m(x) & x \in \omega \\ 0 & x \in \Omega\backslash\omega \end{cases} \tag{20}$$

$$\tilde{\nabla} y_1^1(x) = \begin{cases} -\dfrac{2}{T}\displaystyle\sum_{m=1}^{\infty}\left[\lambda_m\frac{\langle \overline{\Psi}^0, \nabla\Phi_m\rangle_{(L^2(\omega))^n}}{(\displaystyle\sum_{l=1}^{n}\frac{\partial \Phi_m}{\partial x_l}(b))^2}\right]\nabla\Phi_m(x) & x \in \omega \\ 0 & x \in \Omega\backslash\omega \end{cases} \tag{21}$$

Remark.
1)- In the case of a zonal sensor (D, f), with similar developments as in the case of pointwise sensor, we obtain

$$\tilde{\nabla} y_1^0(x) = \begin{cases} -\dfrac{2}{T}\displaystyle\sum_{m=1}^{\infty}\left[\frac{\langle \overline{\Psi}^1, \nabla\Phi_m\rangle_{(L^2(\omega))^n}}{\Big(\displaystyle\sum_{k=1}^{n}\langle\frac{\partial \Phi_m}{\partial x_k}, f\rangle_{L^2(D)}\Big)^2}\right]\nabla\Phi_m(x) & x \in \omega \\ 0 & x \in \Omega\backslash\omega \end{cases} \tag{22}$$

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and

$$\tilde{\nabla} y_1^1(x) = \begin{cases} -\frac{2}{T}\sum_{m=1}^{\infty}\left[\lambda_m \frac{\langle \bar{\Psi}^0, \nabla\Phi_m\rangle_{(L^2(\omega))^n}}{\left(\sum_{k=1}^{n}\langle \frac{\partial \Phi_m}{\partial x_k}, f\rangle_{L^2(D)}\right)^2}\right]\nabla\Phi_m(x) & x\in\omega \\ 0 & x\in\Omega\backslash\omega \end{cases} \quad (23)$$

2)- In the case where many pointwise sensors $(b_j, \delta(.-b_j))_{j=1,...,q}$ are G-strategic in ω, then for T large enough, the initial gradient ∇y_1^0 and ∇y_1^1 can be approximated by:

$$\tilde{\nabla} y_1^0(x) = \begin{cases} -\frac{2}{T}\sum_{m=1}^{\infty}\left[\frac{\langle \bar{\Psi}^1, \nabla\Phi_m\rangle_{(L^2(\omega))^n}}{\sum_{j=1}^{q}(\sum_{l=1}^{n}\frac{\partial \Phi_m}{\partial x_l}(b_j))^2}\right]\nabla\Phi_m(x) & x\in\omega \\ 0 & x\in\Omega\backslash\omega \end{cases} \quad (24)$$

and

$$\tilde{\nabla} y_1^1(x) = \begin{cases} -\frac{2}{T}\sum_{m=1}^{\infty}\left[\lambda_m\frac{\langle \bar{\Psi}^0, \nabla\Phi_m\rangle_{(L^2(\omega))^n}}{\sum_{j=1}^{q}(\sum_{l=1}^{n}\frac{\partial \Phi_m}{\partial x_l}(b_j))^2}\right]\nabla\Phi_m(x) & x\in\omega \\ 0 & x\in\Omega\backslash\omega \end{cases} \quad (25)$$

5.2 Speed measurement case

We consider the system (1) together with the output function (14). The resolution of equation (18) is equivalent to the minimization of the functional

$$\begin{aligned} \mathscr{R}(\tilde{\varphi}^0,\tilde{\varphi}^1) &= \frac{1}{2}\langle \Lambda_1(\tilde{\varphi}^0,\tilde{\varphi}^1),(\tilde{\varphi}^0,\tilde{\varphi}^1)\rangle \\ &-\langle \mathscr{P}(\Delta\bar{\Psi}(0),-\bar{\Psi}'(0)),(\tilde{\varphi}^0,\tilde{\varphi}^1)\rangle \\ &= \frac{1}{2}\int_0^T\left(\sum_{l=1}^{n}\frac{\partial\varphi}{\partial x_l}(b,t)\right)^2 dt - \langle \Delta\bar{\Psi}^0,\tilde{\varphi}^0\rangle \\ &+\langle \bar{\Psi}^1,\tilde{\varphi}^1\rangle \end{aligned} \quad (26)$$

with $\mathscr{P}(\Delta\bar{\Psi}(0),\bar{\Psi}^1(0)) = \langle \Delta\bar{\Psi}^0,\bar{\Psi}^1\rangle$, then we give the following proposition

Proposition 5.1. If the pointwise sensors (b,δ_b) is G-strategic, then for T large enough, the initial gradient ∇y_1^0 and ∇y_1^1 can be approximated by:

$$\tilde{\nabla} y_1^0(x) = \begin{cases} -\frac{2}{T}\sum_{m=1}^{\infty}\left[\frac{\langle \Delta\bar{\Psi}^0, \nabla\Phi_m\rangle_{(L^2(\omega))^n}}{\lambda_m(\sum_{l=1}^{n}\frac{\partial \Phi_m}{\partial x_l}(b))^2}\right]\nabla\Phi_m(x) & x\in\omega \\ 0 & x\in\Omega\backslash\omega \end{cases} \quad (27)$$

and

$$\tilde{\nabla} y_1^1(x) = \begin{cases} -\frac{2}{T}\sum_{m=1}^{\infty}\left[\frac{\langle \bar{\Psi}^1, \nabla\Phi_m\rangle_{(L^2(\omega))^n}}{(\sum_{l=1}^{n}\frac{\partial \Phi_m}{\partial x_l}(b))^2}\right]\nabla\Phi_m(x) & x\in\omega \\ 0 & x\in\Omega\backslash\omega \end{cases} \quad (28)$$

Proof. With minor technical modifications, the proof is similar to the state measurement one.
We define the final error

$$\mathscr{E}^2 = \| \nabla y_1^0 - \tilde{\nabla} y_1^0 \|^2_{L^2(\omega)} + \| \nabla y_1^1 - \tilde{\nabla} y_1^1 \|^2_{L^2(\omega)}$$

The good choice of the truncation M will be such that $\mathscr{E} \le \varepsilon(\varepsilon > 0)$, and we have the following algorithm.

Algorithm:

Step 1 : Choice of the sensor location b and ε the test error, the truncation M.
Step 2 : Repeat
- ⊖ Computation of $\tilde{\nabla} y_1^0$ and $\tilde{\nabla} y_1^1$ by the formulae ((20) and (21)) or ((27) and (28)).
- ⊖ $M \leftarrow M+1$.

Until $\mathscr{E} \le \varepsilon$.
Step 3 : The estimated initial gradient conditions $\tilde{\nabla} y_1^0$ and $\tilde{\nabla} y_1^1$ corresponds to the initial gradient conditions to be observed in the subregion ω.
Step 4 : The restriction of $\tilde{\nabla} y_1^0$ and $\tilde{\nabla} y_1^1$ to Γ corresponds to ∇y_1^0 and ∇y_1^1 to be reconstructed on Γ.

6 Simulation results

In this section we develop numerical example which illustrate the efficiency of the previous approach. The results are related to the choice of the subregion and the gradient to be observed. Consider the two-dimensional diffusion process described in $\Omega =]0,1[\times]0,1[$ by

$$\begin{cases} \frac{\partial^2 y}{\partial t^2}(x,t) = \left[\frac{\partial^2 y}{\partial x_1^2}(x_1,x_2,t) + \frac{\partial^2 y}{\partial x_2^2}(x_1,x_2,t)\right] & \text{in }]0,1[\times]0,T[\\ y(x_1,x_2,0) = y^0(x_1,x_2), \frac{\partial y}{\partial t}(x_1,x_2,0) = y^1(x_1,x_2) & \text{in }]0,1[\\ y(\xi,\eta,t) = 0 & \text{on }]0,T[\end{cases} \quad (29)$$

The system (29) is augmented with the output function described by a pointwise sensor located in (b_1,b_2) where $b_1 = 0.21$, $b_2 = 0.78$ and $T = 3$

$$z(t) = y(b_1,b_2,t) \text{ with } t\in[0,T] \quad (30)$$

Let's consider $\Gamma = \{0\}\times[0,1]$ and $\omega =]0,0.3[\times]0,1[$ the subregion target and

$$\begin{cases} \nabla y^0(x_1,x_2) = & A\Big((2x_1-1)x_2(x_2-1);(2x_2-1)x_1(x_1-1)\Big) \\ \nabla y^1(x_1,x_2) = & B\Big((2x_1-1)\sin(\frac{5\pi x_2}{2});x_1(x_1-1)\frac{5\pi}{2}\cos(\frac{5\pi x_2}{2})\Big) \end{cases}$$

being the gradient of the initial state to be observed on Γ with A and B are selected for numerical considerations. Using the previous algorithm, we obtain the following results:
with $A = 0.055$ and $B = 0.05$
The reconstruction is observed with error equals to: 7.016×10^{-7} for ∇y_1^0 and 3.12×10^{-4} for ∇y_1^1.

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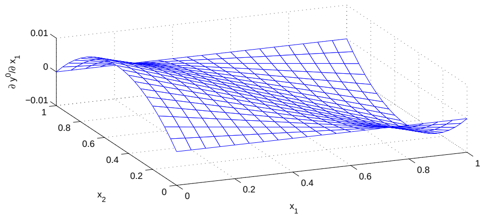

Fig. 3: The exact state gradient $\frac{\partial y^0}{\partial x_1}$ in ω.

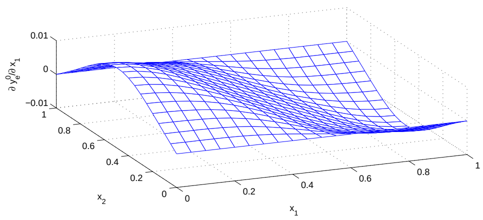

Fig. 4: The estimated state gradient $\frac{\partial y_e^0}{\partial x_1}$ in ω.

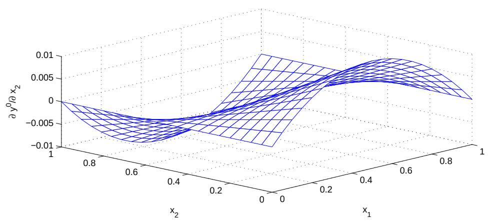

Fig. 5: The exact state gradient $\frac{\partial y^0}{\partial x_2}$ in ω.

We note that from figure 5 and figure 6 (resp. figure 9 and figure 10) the trace of the initial state gradient $\frac{\partial y^0}{\partial x_2}$ and $\frac{\partial y_e^0}{\partial x_2}$ (resp. initial speed gradient $\frac{\partial y^1}{\partial x_2}$ and $\frac{\partial y_e^1}{\partial x_2}$) vanish on Γ.

6.1 Reconstruction error- subregion area

Here we study numerically the dependence of the gradient reconstruction error with respect to the subregion area of ω, we have the following table.

From Table1, we note that the reconstruction error and the subregion area increase or decrease. This means that the larger the subregion error is the greater the error is. The weakly G-observability is realized by means of one pointwise sensor located at $b = (0.21, 0.78)$. The results are similar for other types of sensors.

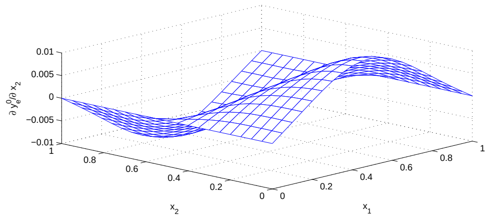

Fig. 6: The estimated state gradient $\frac{\partial y_e^0}{\partial x_2}$ in ω.

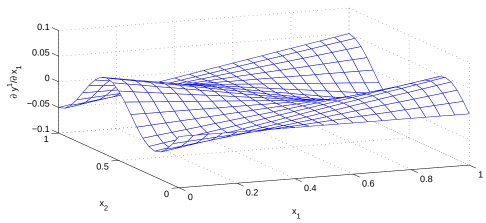

Fig. 7: The exact speed gradient $\frac{\partial y^1}{\partial x_1}$ in ω.

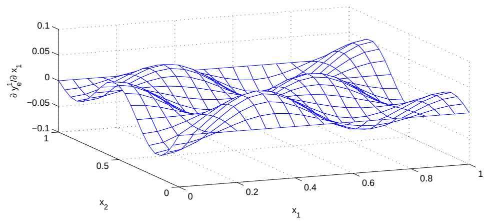

Fig. 8: The estimated speed gradient $\frac{\partial y_e^1}{\partial x_1}$ in ω.

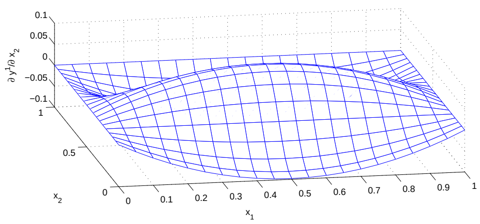

Fig. 9: The exact speed gradient $\frac{\partial y^1}{\partial x_2}$ in ω.

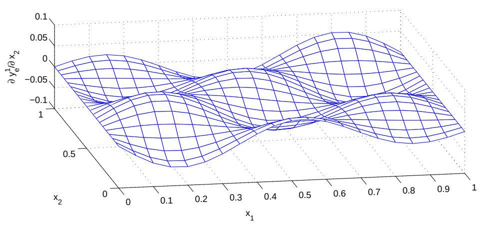

Fig. 10: The estimated speed gradient $\frac{\partial y_e^1}{\partial x_2}$ in ω.

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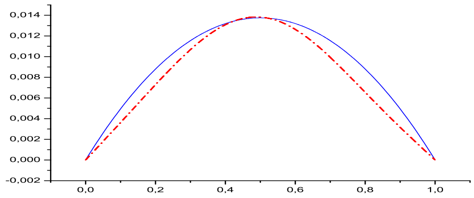

Fig. 11: The exact state gradient $\frac{\partial y^0}{\partial x_1}$ (continuous line) and estimated state gradient $\frac{\partial y_e^0}{\partial x_1}$ (dashed line) on Γ.

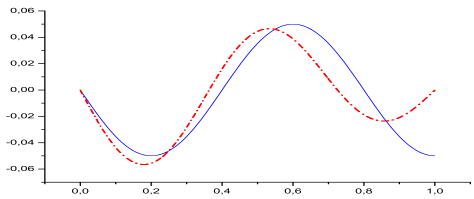

Fig. 12: The exact speed gradient $\frac{\partial y^1}{\partial x_1}$ (continuous line) and estimated speed gradient $\frac{\partial y_e^1}{\partial x_1}$ (dashed line) on Γ.

Table 1: The reconstruction error with respect to the subregion area.

The subregion	Reconstruction error
$]0,0.1[\times]0,1[$	6.5679×10^{-5}
$]0,0.2[\times]0,1[$	1.2725×10^{-4}
$]0,0.3[\times]0,1[$	3.1270×10^{-4}
$]0,0.4[\times]0,1[$	7.5275×10^{-4}
$]0,0.5[\times]0,1[$	1.4252×10^{-3}
$]0,0.6[\times]0,1[$	2.4406×10^{-3}
$]0,0.7[\times]0,1[$	3.4433×10^{-3}
$]0,0.8[\times]0,1[$	3.6327×10^{-3}

7 Conclusion

This work has extended the usual results on regional gradient observability for hyperbolic systems to the case where the gradient is to be observed on a part of the boundary of the geometrical domain where the system is defined. We developed a technical approach that leads to a implementable gradient reconstruction algorithm and the obtained result are successfully tested through numerical example and simulations. The problem of characterization of the sensors that make the system boundary observable is of great interest and the work is under consideration and will be the subject of the feature paper.

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