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Regional Boundary Observability of Semilinear Hyperbolic Systems : HUM Approach

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Abstract: This paper aims to study the notion of regional observability of a distributed parameter system governed by semilinear hyperbolic equations. This original concept is interested in the reconstruction of the the state only on a subregion Γ of the boundary of the system evolution domain $\partial\Omega$. We give denition and some properties of this notion and we show that under some hypothesis,the regional boundary observability is guaranted.We show by means of Hilbert Uniqueness Method (HUM) combined with fixed point techniques that it is possible to reconstruct such a state on a desired subregion Γ .This approach leads to interesting results which are performed through numerical example and simulations.

Keywords: Distributed systems, semilinear hyperbolic systems, boundary reconstruction, regional boundary observability, fixed point, Hilbert Uniqueness Method (HUM).

1 Introduction

The control theory is highly interdisciplinary, it is a part of applied mathematics serving the most important link between mathematics and technology : complex systems in physics and mechanical engineering should be analyzed to achieve designated mission or operational requirements. Most of these devises are inherently nonlinear, indeed modern scientific inquiry and the demands of advancing technology are driving theoretical and experimental research towards control of nonlinear systems. Compelling applications have been noted and have motivated seminal studies in such wide-ranging elds as chemistry, meteorology, optical networking and computer sciences. Experience has so far shown that nonlinear systems dynamics can be incorporated within the framework of estimation and control theory but give rise to unusual models that have not yet been studied in depth. Most of theses problems request a regional study, this concept of regional analysis was introduced by (Zerrik, [8] and El Jai et al., [3]), which offers important tools for solving many real problems, particularly the concept of regional observability, which refers to problems in which the observed state of interest is not fully specified as a state, but concerns only a region ω, a portion of the spatial domain on which the system is considered. It was extended by Zerrik et al. [9] to the case where the subregion ω is a part of the boundary $\partial\Omega$ of Ω. Roughly speaking, the regional observability problem may be formulated as follows : considering an evolution system, we have to reconstruct the initial state in a given subregion ω (resp. Γ) in the whole domain Ω (resp. the boundary of the whole domain $\partial\Omega$). This became a classical problem in systems theory and there is a large literature on the topic, research in this area has been very intensive in the last two decades.

In this paper we introduce a new concept which is the regional boundary observability for hyperbolic semilinear systems, this important class of systems is an intermediate between the linear systems which are widely studied and nonlinear ones which are very close to the nature, we are interested in the knowledge of the state only in a critical subregion of the boundary of the system domain. The introduction of this concept is motivated by many real situations. The paper is organized as follows : In Section 2 we give some recalls about definitions and properties in linear case, Section 3 is devoted to the presentation of the considered system, as well as to definitions and characterizations of this new concept. Section 4 is focused on the regional reconstruction of the initial state in a portion of the boundary of the evolution domain using HUM approach. In the last section we

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develop a numerical approach, which is illustrated by simulations that lead to some conjectures.

2 The considered system

Let's consider Ω an open bounded set of $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, and $Q=\Omega\times]0,T[$, $\Sigma=\partial\Omega\times]0,T[$ and the following semilinear hyperbolic system :

$$\begin{cases} \dfrac{\partial^2 y(x,t)}{\partial t^2} = Ay(x,t)+\mathcal{N}y(x,t) & \text{in } Q \\ y(x,0)=y^0(x),\ \dfrac{\partial y(x,0)}{\partial t}=y^1(x) & \text{in } \Omega \\ \dfrac{\partial y(\xi,t)}{\partial \nu_A}=0 & \text{on } \Sigma \end{cases} \tag{1}$$

with A is a second order differential operator, which is linear and symmetric, $\dfrac{\partial y(\xi,t)}{\partial \nu_A}$ is the conormal and $\mathcal{N}$ is a nonlinear operator defined from $L^2(\Omega)$ to $L^2(\Omega)$ in order to ensure the existence and the uniqueness of the solution of (1) which is augmented with the following output function

$$z(t)=Cy(.,t) \tag{2}$$

where $C: L^2(\Omega)\to\mathbb{R}^q$ (q is the number of sensors), is the observation operator which depends on the number of the sensors.
Without loss of generality we denote by : $y(t):=y(x,t)$. Let's consider

$$\bar{y}(t)=\begin{pmatrix} y(t) \\ \dfrac{\partial y(t)}{\partial t}\end{pmatrix},\ \bar{y}^0(t)=\begin{pmatrix} y^0 \\ y^1\end{pmatrix}$$

and

$$\bar{A}=\begin{pmatrix} 0 & I \\ A & 0\end{pmatrix},\ \bar{N}\begin{pmatrix} y_1 \\ y_2\end{pmatrix}=\begin{pmatrix} 0 \\ \mathcal{N}y_1\end{pmatrix}$$

For $(y_1,y_2)\in\mathcal{F}$, with $\mathcal{F}=L^2(\Omega)\times L^2(\Omega)$, the system (1) is equivalent to the following one

$$\begin{cases} \bar{y}(t) &= \bar{A}\bar{y}(t)+\bar{N}\bar{y}(t);\ 0<t<T \\ \bar{y}(0) &= \bar{y}^0 \end{cases} \tag{3}$$

Augmented with the following output equation

$$\bar{z}(t)=\bar{C}\bar{y}(t) \tag{4}$$

with $\bar{C}=(C,0)$. The system (3) admits a unique solution which is expressed as follows : (see [7])

$$\bar{y}(t)=\bar{S}(t)\bar{y}^0+\int_0^t \bar{S}(t-s)\bar{N}\bar{y}(s)ds \tag{5}$$

$(\bar{S}(t))_{t\geq 0}$ is the semigroup endowed by the operator $\bar{A}$, which is defined as follows :

$$\bar{S}\begin{pmatrix} y_1 \\ y_2\end{pmatrix}=\begin{pmatrix} \displaystyle\sum_m\sum_{i=1}^{r_m}\left[\langle y_1,w_{mj}\rangle cos\sqrt{-\lambda_m t}+\frac{1}{\sqrt{-\lambda_m}}\langle y_2,w_{mj}\rangle sin\sqrt{-\lambda_m t}\right]w_{mj} \\ -\sqrt{-\lambda_m}\displaystyle\sum_m\sum_{i=1}^{r_m}\left[\langle y_1,w_{mj}\rangle sin\sqrt{-\lambda_m t}+\langle y_2,w_{mj}\rangle cos\sqrt{-\lambda_m t}\right]w_{mj}\end{pmatrix} \tag{6}$$

Considering the following operators :

–The observability operator :

$$\begin{aligned} \bar{K}: H^2(\Omega)\times H^1(\Omega) &\longrightarrow L^2(0,T;\mathbb{R}^q) \\ (y_1,y_2) &\longmapsto \bar{C}\bar{S}(.)(y_1,y_2)\end{aligned}$$

–The restriction operator in ω :

$$\begin{aligned} \bar{\chi}_\omega: H^2(\Omega)\times H^1(\Omega) &\longrightarrow H^2(\omega)\times H^1(\omega) \\ (y_1,y_2) &\longmapsto (y_1,y_2)_{|\omega}\end{aligned}$$

–The restriction operator on Γ :

$$\begin{aligned} \bar{\chi}_\Gamma: H^{\frac{3}{2}}(\partial\Omega)\times H^{\frac{1}{2}}(\partial\Omega) &\longrightarrow H^{\frac{3}{2}}(\Gamma)\times H^{\frac{1}{2}}(\Gamma) \\ (y_1,y_2) &\longmapsto (y_1,y_2)_{|\Gamma}\end{aligned}$$

–The trace operator γ_0 from $H^2(\Omega)\times H^1(\Omega)$ to $H^{\frac{3}{2}}(\partial\Omega)\times H^{\frac{1}{2}}(\partial\Omega)$ which is linear, continue and surjective over considered spaces.

We give the following definitions :

Definition 1(see [9]) The system (3)-(4) is said to be exactly (resp. weakly) ω-observable if

$$Im(\bar{\chi}_\omega\bar{K}^*)=H^2(\omega)\times H^1(\omega)$$

$$(\text{resp. } \overline{Im(\bar{\chi}_\omega\bar{K}^*)}=H^2(\omega)\times H^1(\omega)).$$

Definition 2(see [9]). The system (3)-(4) is said to be exactly (resp. weakly) Γ-observable if

$$Im(\bar{\chi}_\Gamma\bar{\gamma}_0\bar{K}^*)=H^{\frac{3}{2}}(\Gamma))\times H^{\frac{1}{2}}(\Gamma))$$

$$(\text{resp. } \overline{Im(\bar{\chi}_\Gamma\bar{\gamma}_0\bar{K}^*)}=H^{\frac{3}{2}}(\Gamma))\times H^{\frac{1}{2}}(\Gamma))).$$

Then we deduce the following proposition :

Proposition 3 If the system (3) augmented with the output equation (4) is exactly (resp. weakly) ω-observable then it is exactly (resp. weakly) Γ-observable. (see [9]). To the system (1) we associate the linear one defined by :

$$\begin{cases} \dfrac{\partial^2 y(x,t)}{\partial t^2} = Ay(x,t) & \text{in } Q \\ y(x,0)=y^0(x),\ \dfrac{\partial y(x,0)}{\partial t}=y^1(x) & \text{in } \Omega \\ \dfrac{\partial y(\xi,t)}{\partial \nu_A}=0 & \text{on } \Sigma \end{cases} \tag{7}$$

which admits a solution:

$$y\in C(0,T;H_0^1(\Omega))\cap C^1(0,T;L^2(\Omega)).$$

3 HUM Approach

The objective of this section is to give an extension of the Hilbert Uniqueness Method introduced in the linear case by Lions (see [4]) which allows the determination of the regional boundary initial conditions on Γ, and leads to an

algorithm which is tested through a numerical example. We consider the system (1) augmented with the output function (2), and the set G as follows:
$G = \{(\varphi_0,\varphi_1) \in D(A)\times H^1(\Omega)$ such that $\varphi_0 = \varphi_1 = 0$ on $\Omega\setminus\omega_r\}$
with $r>0$ sufficiently small and the ball $B(z,r)$, such that $F_r = \bigcup_{z\in\Gamma} B(z,r)$, and $\omega_r = F_r\cap\Omega$. We decompose the initial state and speed as following

$$y_0 = \begin{cases} y_0^1 \text{ in } \omega_r \\ y_0^2 \text{ in } \Omega\setminus\omega_r \end{cases},\ y_1 = \begin{cases} y_1^1 \text{ in } \omega_r \\ y_1^2 \text{ in } \Omega\setminus\omega_r \end{cases}$$

The aim is to reconstruct the component y_0^1 and y_1^1. We consider the system (1) supposed observed by an internal zone sensor (D,f) with $D\subset\Omega$ and $f\in L^2(D)$.
For $(\varphi_0,\varphi_1)\in G$, we consider the semilinear system

$$\begin{cases} \dfrac{\partial^2\varphi(x,t)}{\partial t^2} + A\varphi(x,t) = \mathcal{N}\varphi(x,t) & \text{in } Q \\ \varphi(x,0)=\varphi^0(x),\ \dfrac{\partial\varphi(x,0)}{\partial t} = \varphi^1(x) & \text{in } \Omega \\ \dfrac{\partial\varphi(\xi,t)}{\partial\nu_A} = 0 & \text{on } \Sigma \end{cases} \quad (8)$$

which can be decomposed as follows:

$$\begin{cases} \dfrac{\partial^2\varphi_1(x,t)}{\partial t^2} = -A\varphi_1(x,t) & \text{in } Q \\ \varphi_1(x,0)=\varphi^0(x),\ \dfrac{\partial\varphi_1(x,0)}{\partial t} = \varphi^1(x) & \text{in } \Omega \\ \dfrac{\partial\varphi_1(\xi,t)}{\partial\nu_A} = 0 & \text{on } \Sigma \end{cases} \quad (9)$$

and

$$\begin{cases} \dfrac{\partial^2\theta(x,t)}{\partial t^2} = -A\theta(x,t) + \mathcal{N}(\theta(x,t)+\varphi_1(x,t)) & \text{in } Q \\ \theta(x,0)=0,\ \dfrac{\partial\theta(x,0)}{\partial t} = 0 & \text{in } \Omega \\ \dfrac{\partial\theta(\xi,t)}{\partial\nu_A} = 0 & \text{on } \Sigma \end{cases} \quad (10)$$

The linear system (9) has a unique solution $\varphi\in C(0,T;H^2(\Omega))\cap C^1(0,T;H^1(\Omega))\cap C^2(0,T;L^2(\Omega))$ (see [6]) and the function

$$\tilde\varphi_0 \in G \longmapsto \|\tilde\varphi_0\|_G = \left[\int_0^T \langle\varphi_1(t),f\rangle^2_{L^2(D)}dt\right]^{\frac12}$$

induce a semi-norm on G, with $\tilde\varphi_0 := (\varphi_0,\varphi_1)$, we denote by G the completion of G. If the system (9) is weakly observable in ω_r, then the semi-norm defines a norm on G (see [9]).
We define the auxiliary of the system (8) by

$$\begin{cases} \dfrac{\partial^2\tilde\psi(x,t)}{\partial t^2} = A^*\tilde\psi(x,t) + \mathcal{N}\tilde\psi(x,t) - \langle\varphi_1(t),f\rangle_{L^2(D)}(\chi_D)f(x) & \text{in } Q \\ \tilde\psi(x,T)=0,\ \dfrac{\partial\tilde\psi(x,T)}{\partial t} = 0 & \text{in } \Omega \\ \tilde\psi(\xi,t)=0 & \text{on } \Sigma \end{cases} \quad (11)$$

This allows to consider the application

$$\begin{aligned} \mu : G &\longrightarrow G^* \\ \tilde\varphi_0 &\longmapsto P\tilde\psi(0) \end{aligned}$$

where P is the projection on G^* and we decompose $\tilde\psi$ as follows :

$$\tilde\psi = \psi_0 + \psi_1$$

with ψ_0 and ψ_1 are solutions of the following systems

$$\begin{cases} \dfrac{\partial^2\psi_0(x,t)}{\partial t^2} = A^*\psi_0(x,t) - \langle\varphi_1(t),f\rangle_{L^2(D)}(\chi_D)f(x) & \text{in } Q \\ \psi_0(x,T)=0,\ \dfrac{\partial\psi_0(x,T)}{\partial t} = 0 & \text{in } \Omega \\ \psi_0(\xi,t)=0 & \text{on } \Sigma \end{cases} \quad (12)$$

and

$$\begin{cases} \dfrac{\partial^2\psi_1}{\partial t^2} = A^*\psi_1 + \mathcal{N}(\psi_0+\psi_1) - \langle\theta(.),f\rangle_{L^2(D)}(\chi_D)f(.) & \text{in } Q \\ \psi_1(x,T)=0,\ \dfrac{\partial\psi_1(x,T)}{\partial t} = 0 & \text{in } \Omega \\ \psi_1(\xi,t)=0 & \text{on } \Sigma \end{cases} \quad (13)$$

which allows to consider the following operator

$$\mu(\tilde\varphi_0) = P\psi_0(0) + P\psi_1(0).$$

We define the operator Λ from G to G^* as follows

$$\Lambda\tilde\varphi_0 = P\psi_0(0)$$

Then we have

$$\mu(\tilde\varphi_0) = \Lambda\tilde\varphi + K\tilde\varphi$$

where K is a nonlinear operator given by

$$\begin{aligned} K : G &\longrightarrow G^* \\ \tilde\varphi_0 &\longmapsto P\psi_1(0) \end{aligned}$$

We suppose that the linear part of the system (8) is regionally weakly observable on Γ, then Λ is invertible, and finally we obtain

$$\tilde\varphi_0 = \Lambda^{-1}P\tilde\psi(0) - \Lambda^{-1}K\tilde\varphi_0$$

We consider the following system

$$\begin{cases} \dfrac{\partial^2\bar\psi(x,t)}{\partial t^2} = A^*\bar\psi(x,t) + \mathcal{N}\bar\psi(x,t) - z(t)(\chi_D)f(x) & \text{in } Q \\ \bar\psi(x,T)=0,\ \dfrac{\partial\bar\psi(x,T)}{\partial t} = 0 & \text{in } \Omega \\ \bar\psi(\xi,t)=0 & \text{on } \Sigma \end{cases} \quad (14)$$

If $\tilde\varphi_0$ is chosen such that $\tilde\psi(0) = \bar\psi(0)$ in ω_r then the system (14) can be seen as the adjoint of the system (1) thus the problem of the observability amounts to solving the equation

$$\tilde\varphi_0 = \Theta(\tilde\varphi_0) \quad (15)$$

where $\Theta(\tilde\varphi_0)$ is the solution of the equation

$$\Lambda\Theta(\tilde\varphi_0) = P\bar\psi(0) - K(\varphi_0) \quad (16)$$

thus we obtain the following results

Proposition 4 If the system (9) augmented with the output equation (2) is regionally weakly observable in ω_r and there exists $c>0$ such that $\|\mathcal{N}(x)\| \leq c\|x\|$, then the equation (16) admits a unique fixed point.

Proof
Step 1: We consider $p>0$ and $B_p = B(0,p)\times B(0,p)$, we have

$$K(B_p) = \{P(\psi_1(T)) \mid (\varphi_0,\varphi_1)\in B_p\}$$

Let's consider

$$\tilde{B}_p = \{P\psi_1(t) \mid (\varphi_0,\varphi_1)\in B_p\ ,\ t\in[0,T]\}$$

We have $K(B_p)\subset \tilde{B}_p$, then it's sufficient to show that $\tilde{B}_p$ is relatively compact. We have $\psi_1(.)$ is a solution of (13).

Let's consider $\bar{\psi}_1 = \begin{pmatrix} \psi_1 \\ \dfrac{\partial \psi_1}{\partial t}\end{pmatrix}$.

Then we obtain

$$\begin{cases} \dfrac{\bar{\psi}_1(x,t)}{\partial t} = \bar{A}^*\bar{\psi}_1(x,t) + \begin{bmatrix} 0 \\ \mathcal{N}(\psi_0(x,t)+\psi_1(x,t)) - \langle\theta(t),f\rangle_{\chi_D} f(x)\end{bmatrix} & \text{in } Q \\ \bar{\psi}_1(x,T) = 0 & \text{in } \Omega \end{cases} \tag{17}$$

Without loss of generality we denote $\psi_1(t) := \psi_1(x,t)$ and $r_m = 1$, then we obtain

$$\psi_1(t) = \int_T^t \sum_m \frac{1}{\sqrt{-\lambda_m}}[\mathcal{N}(\psi_0+\psi_1) \\ -\langle\theta(t-\tau),f\rangle_{\chi_D} f(x), w_m]\sin(\sqrt{-\lambda_m}(t-\tau))d\tau$$

In the other hand we have :

$$\|\psi_1(t)\|^2 = \sum_{i=1}^{\infty} |\langle\psi_1(t),w_i\rangle|^2$$

Without loss of generality we denote: $\psi_0(t) := \psi_0(x,t)$, $\theta_1(t) := \theta_1(x,t)$ and $\varphi_0(t) := \varphi_0(x,t)$.
Then

$$\begin{aligned} &\|\psi_1(t)\|^2 \\ &= \sum_{i=1}^{\infty}|\langle\int_T^t \sum_{m=1}^{\infty}\frac{1}{\sqrt{-\lambda_m}}\mathcal{N}(\psi_0(t-\tau)+\psi_1(t-\tau)) - \langle\theta(t-\tau),f\rangle_{\chi_D} f(x),w_i\rangle \\ &\sin(\sqrt{-\lambda_m}(t-\tau))d\tau|^2 \\ &\leq \frac{1}{\pi^2}\left(\int_t^T \|\mathcal{N}(\psi_0(t-\tau)+\psi_1(t-\tau))\| + \|\theta(t-\tau)\|\|f\|^2 d\tau\right)^2 \\ &\leq \frac{1}{\pi^2}\left(\int_t^T c\|(\psi_0(t-\tau)\|+\|\psi_1(t-\tau))\|) + \|\theta(t-\tau)\|\|f\|^2 d\tau\right)^2 \end{aligned}$$

and ψ_0 is a solution of (12), then

$$\begin{aligned}\|\psi_0(t)\|^2 &= \sum_{i=1}^{\infty}|\langle\int_t^T \sum_{m=1}^{\infty}\frac{1}{\sqrt{-\lambda_m}}\langle\langle\varphi_1(t-\tau),f\rangle_{\chi_D} f, w_i\rangle \sin(\sqrt{-\lambda_m}(t-\tau))d\tau|^2 \\ &\leq \frac{1}{\pi^2}\left(\int_t^T \|\varphi_1(t-\tau)\|\|f\|^2 d\tau\right)^2\end{aligned}$$

since θ is a solution of (10), we have

$$\begin{aligned}\|\theta(t)\|^2 &= \sum_{i=1}^{\infty}|\langle\int_0^t \sum_{m=1}^{\infty}\frac{1}{\sqrt{-\lambda_m}}\langle\mathcal{N}(\theta(t-\tau)+\varphi_1(t-\tau)),w_i\rangle \\ &\sin(\sqrt{-\lambda_m}(t-\tau))d\tau|^2 \\ &\leq \frac{1}{\pi^2}\left(\int_0^t c(\|\theta(t-\tau)\|+\varphi_1(t-\tau)\|)\right)^2\end{aligned}$$

φ_1 is a solution of (9) then we have

$$\begin{aligned}\|\varphi_1(t)\|^2 &= \sum_{i=1}^{\infty}|\langle\varphi_0,w_i\rangle + \frac{1}{\pi^2}\langle\varphi_1,w_i\rangle|^2 \\ &\leq 2\|\varphi_0\|^2 + \frac{2}{\pi^2}\|\varphi_1\|^2 := R\end{aligned}$$

Using Gronwall theorem we have

$$\|\theta\| \leq (\frac{1}{\pi}tcR)\exp(\frac{c}{\pi}t)$$

Then we obtain

$$\|\psi_1(t)\| \leq \frac{1}{\pi^2}cTR\|f\|^2\left(1+\exp(\frac{cT}{\pi})\right) + \frac{c}{\pi}\int_t^T \|\psi_1(t-\tau)\|d\tau$$

Using Gronwall theorem, we obtain

$$\|\psi_1(t)\| \leq \frac{1}{\pi^2}cTR\|f\|^2\left(1+\exp(\frac{cT}{\pi})\right)\exp(\frac{cT}{\pi})$$

and then $\tilde{B}_p$ is uniformly bounded. We show that $\tilde{B}_p$ is equicontinuous, indeed, we obtain

$$\begin{aligned}&\|\psi_1(t_2)-\psi_1(t_1)\|^2 \\ &\leq \frac{1}{\pi^2}|\int_{t_1}^T c(\|\psi_0(t_1-\tau)\|+\|\psi_1(t_1-\tau)\|)+\|\theta(t_1-\tau)\|\|f\|^2 d\tau \\ &+\int_{t_2}^T c(\|\psi_0(t_2-\tau)\|+\|\psi_1(t_2-\tau)\|)+\|\theta(t_2-\tau)\|\|f\|^2 d\tau\end{aligned}$$

Therefore $\Theta : B_p \longrightarrow G^*$ is compact.
Step 2 The map Θ applies B_p to B_p. System (9) is regionally weakly observable in ω_r, then $\Lambda^{-1}P$ is bounded and we have

$$\|\Theta(\varphi_0)\| \leq \|\Lambda^{-1}P(\bar{\psi}(0)\| + \|\psi_1(0))\|,$$

then using Schauder theorem, the operator Θ admits a unique fixed point, this achieve the proof.

Then we obtain the following algorithm
Algorithm :
Step 1: The initial state y_0, the initial speed y_1, the region ω_r, the domain D, the repartition function f and the accuracy threshold ε.
Step 2: Repeat

–Resolution of (9) and obtention of φ_1.
–Resolution of (10) and obtention of θ.
–Resolution of (12) and obtention of ψ_0.
–Resolution of (13) and obtention of ψ_1.
–Obtention of $\Theta(\tilde{\varphi}_0)$.
–Resolution of $\tilde{\varphi}_0 = \Theta(\tilde{\varphi}_0)$ and obtention of $\tilde{\varphi}_0$.

Until $\|\tilde{\varphi}_0 - \Theta(\tilde{\varphi}_0)\| \leq \varepsilon$.
Step 3: The solution $\tilde{\varphi}_0$ corresponds to the regional state to be observed in the subregion ω_r and then y_0 is obtained as a restriction of $\tilde{\varphi}_0$ on Γ.

4 Simulations

We present a numerical example illustrating the previous algorithm. The obtained results depends on the considered region and the localization of the sensors. Let's consider the system defined for $\Omega =]0,1[\times]0,1[$ by the following equation

$$\begin{cases} \dfrac{\partial^2 y(x,t)}{\partial t^2} = \sum_{i=1}^{2} \dfrac{\partial^2 y(x,t)}{\partial {x_i}^2} \\ + \sum_{k,l=0}^{\infty} |\langle y(t),\varphi_{kl}\rangle| \langle y(t),\varphi_{kl}\rangle \varphi_{kl}(x) \ \text{ in } \Omega\times]0,T[\\ y(x,0) = y^0(x),\ \dfrac{\partial y(x,0)}{\partial t} = y^1(x) \quad \text{in } \Omega \\ \dfrac{\partial y(\xi,t)}{\partial \nu} = 0 \qquad\qquad \text{on } \partial\Omega\times]0,T[\end{cases} \tag{18}$$

with $x = (x_1,x_2)$ and $(\varphi_{kl})_{kl}$ is a complete family of $H^1(\Omega)$. The system (18) is augmented with the output equation described by a pointwise sensor located in (b_1,b_2) where $b_1 = 0.36$, $b_2 = 0.98$ and $T = 8$.

$$z(t) = y(b_1,b_2,t),\ t \in]0,T[\tag{19}$$

We consider the subregion $\omega_r =]0,0.88[\times]0,1[$ and

$$y_0(x_1,x_2) = \alpha\cos(3\pi x)\cos(3\pi y)$$

$$y_1(x_1,x_2) = \beta\cos(3\pi x)\cos(3\pi y)3\pi$$

is the observed initial state, $\Gamma = \{0\}\times[0,1]$, with α and β are chosen for numerical reasons. Using the previous algorithm, we obtain the following results :

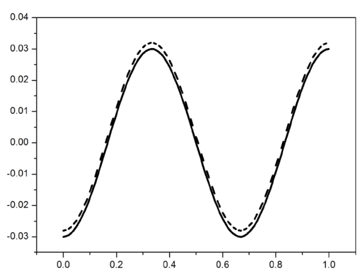

Fig. 1. Initial state (continuous line) and estimated state (discrete line) in Γ.

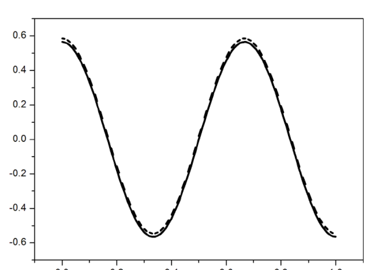

Fig. 2. Initial speed (continuous line) and estimated speed (discrete line) in Γ.

Table 1: State and speed errors in different subregions ω_r.

Subregion ω_r	State error	Speed error
$]0,1[\times]0,1[$	6.75×10^{-3}	1.74×10^{-1}
$]0,22[\times]0,1[$	5.49×10^{-3}	1.58×10^{-2}
$]0,52[\times]0,1[$	3.31×10^{-3}	1.36×10^{-2}
$]0,88[\times]0,1[$	4.07×10^{-4}	1.35×10^{-3}

We note that the initial estimated state (resp. speed) is very close to the initial exact state (resp. speed), which shows the effectiveness of the considered approach .
The initial state (resp. initial speed) is obtained with the reconstruction error

$$\|y_0 - y_{oe}\|^2 = 4.07\times10^{-4}$$

$$(resp. \|y_1 - y_{oe1}\|^2 = 1.35\times10^{-4})$$

where y_{oe} (resp. y_{oe1}) is the obtained state (resp. speed) by the previous algorithm.
The following table shows how the state (resp. the speed) error grows with respect to the subregion area.

5 Conclusion

The regional boundary observability for distributed hyperbolic semilinear systems is considered. The regional internal and boundary observability of linear systems was explored to solve the problems related to semilinear one which constitutes a natural extension. We explored Hilbert Uniqueness reconstruction approach which use the fixed point techniques leading to an algorithm which is implemented numerically. Many questions remain open, this is the case of the study of the boundary observability with the sectorial approach and the regional gradient observability of semilinear hyperbolic systems. This questions are under consideration and the results will appear in a separate papers.

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