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Characterization of Distributions Through Contraction and Dilation of Dual Generalized Order Statistics

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Abstract: Distributional properties of two non-adjacent dual generalized order statistics have been used to characterize distributions. Further, one sided contraction and dilation for the dual generalized order statistics are discussed and then the results are deduced for generalized order statistics, order statistics and adjacent dual generalized order statistics and generalized order statistics

Keywords: Order statistics; generalized order statistics; dual generalized order statistics; contraction; dilation. Characterization of distributions; exponential; power and Pareto distributions.

1 Introduction

Kamps [7] introduced the concept of generalized order statistics (*gos*) as follows: Let $X_1, X_2, \cdots, X_n$ be a sequence of independent and identically distributed *(iid)* random variables *(rv)* with the absolutely continuous distribution function *(df)* $F(x)$ and the probability density function *(pdf)* $f(x)$, $x \in (\alpha, \beta)$. Let $n \in N$, $n \geq 2$, $k > 0$, $\tilde{m} = (m_1, m_2, \cdots, m_{n-1}) \in \Re^{n-1}$, $M_r = \sum_{j=r}^{n-1} m_j$, such that $\gamma_r^{(n)} = k + (n-r) + M_r > 0$ for all $r \in \{1,2,\cdots,n-1\}$. If $m_1 = m_2 = \cdots = m_{n-1} = m$, then $X(r,n,m,k)$ is called the r^{th} *m-gos* and its *pdf* is given as:

$$f_{X(r,n,m,k)}(x) = \frac{C_{r-1}^{(n)}}{(r-1)!} [\bar{F}(x)]^{\gamma_r^{(n)}-1} \left[\frac{1-[\bar{F}(x)]^{m+1}}{m+1}\right]^{r-1} f(x), \qquad \alpha < x < \beta \tag{1.1}$$

Based on the generalized order statistics (*gos*), Burkschat *et al.* [5] introduced the concept of the dual generalized order statistics (*dgos*) where the *pdf* of the r^{th} *m-dgos* $X^*(r,n,m,k)$ is given as

$$f_{X^*(r,n,m,k)}(x) = \frac{C_{r-1}^{(n)}}{(r-1)!} [F(x)]^{\gamma_r^{(n)}-1} \left[\frac{1-[F(x)]^{m+1}}{m+1}\right]^{r-1} f(x), \qquad \alpha < x < \beta \tag{1.2}$$

which is obtained just by replacing

$\bar{F}(x) = 1 - F(x)$ by $F(x)$

where

$$\gamma_j^{(n)} = k + (n-j)(m+1)\,, \qquad 1 \leq j \leq n\,, n \in N \tag{1.3}$$

$$C_{r-1}^{(n)} = \prod_{i=1}^{r} \gamma_i^{(n)}, \qquad 1 \leq r \leq n \tag{1.4}$$

If support of the distribution $F(x)$ be over (α, β) , then by convention, we will write

$$X(0,n,m,k) = \alpha \quad \text{and} \quad X^*(0,n,m,k) = \beta \tag{1.5}$$

Ahsanullah [1] has characterized uniform distribution under random contraction for adjacent *dgos*. In this paper, distributional properties of the *dgos* have been used to characterize a general form of distributions for non- adjacent *dgos* under random translation, dilation and contraction, thus generalizing the results of Ahsanullah [1] and Beutner and Kamps [4]. Further, results in terms of generalized order statistics and order statistics are deduced. One may also refer to Ahsanullah [2], Arnold *et al.* [3], Castaño-Martínez *et al.* [6], Khan and Shah [8], Wesolowski and Ahsanullah [12] and Khan et al [13] for the related results.

It may be seen that if Y is a measureable function of X with the relation

$$Y = h(X)$$

then

(i) $$Y^*(r,n,m,k) = h[X^*(r,n,m,k)] \tag{1.6}$$

(ii) $$Y(r,n,m,k) = h[X(r,n,m,k)] \tag{1.7}$$

(iii) $$Y_{r:n} = h(X_{r:n}) \tag{1.8}$$

if h is an increasing function and

(i) $$Y(r,n,m,k) = h[X^*(r,n,m,k)] \tag{1.9}$$

(ii) $$Y^*(r,n,m,k) = h[X(r,n,m,k)] \tag{1.10}$$

(iii) $$Y_{n-r+1:n} = h(X_{r:n}) \tag{1.11}$$

if h is a decreasing function

where $X_{r:n}$ is the r^{th} order statistic from a sample of size n, $X(r,n,m,k)$ is the r^{th} *m-gos* and $X^*(r,n,m,k)$ is the r^{th} *m- dgos.*

We will denote

(i) $X \sim exp(\alpha)$

if X has an exponential distribution with the *df*

$$F(x) = 1 - e^{-\alpha x}, \quad 0 < x < \infty, \ \alpha > 0 \tag{1.12}$$

(ii) $X \sim Par(\alpha)$

if X has a Pareto distribution with the *df*

$$F(x) = 1 - x^{-\alpha}, \quad 1 < x < \infty, \alpha > 0 \tag{1.13}$$

(iii) $X \sim pow(\alpha)$

if X has a power function distribution with the *df*

$$F(x) = x^{\alpha}, \quad 0 < x < 1, \ \alpha > 0 \tag{1.14}$$

(iv) $X \sim genexp(\alpha)$

if X has a generalized exponential distribution with the *df*

$$F(x) = [1 - e^{-\alpha(m+1)x}]^{\frac{1}{m+1}}, \quad 0 < x < \infty, \ \alpha > 0, \ m > -1 \tag{1.15}$$

(v) $X \sim genPar(\alpha)$

if X has a generalized Pareto distribution with the *df*

$$F(x) = [1 - x^{-\alpha(m+1)}]^{\frac{1}{m+1}}, \qquad 1 < x < \infty,\ \alpha > 0,\ m > -1 \tag{1.16}$$

(vi) $X \sim genpow(\alpha)$

if X has a generalized power function distribution with the *df*

$$F(x) = 1 - [1 - x^{\alpha(m+1)}]^{\frac{1}{m+1}}, \qquad 0 < x < 1,\ \alpha > 0,\ m > -1 \tag{1.17}$$

It may further be noted that

if $\log X \sim exp(\alpha)$ then $X \sim Par(\alpha)$ (1.18)

if $-\log X \sim exp(\alpha)$ then $X \sim pow(\alpha)$ (1.19)

if $\log X \sim genexp(\alpha)$ then $X \sim genPar(\alpha)$ (1.20)

and if $-\log X \sim genexp(\alpha)$ then $X \sim genpow(\alpha)$ (1.21)

It has been assumed here throughout that the *df* is differentiable $w.r.t.$ its argument.

2 Characterizing results

Theorem 2.1 *Let $X^*(r,n,m,k)$ be the r^{th} m- dgos from a sample of size n drawn from a continuous population with the pdf $f(x)$ and the df $F(x)$ with $F(0) = 0$, then*

$$X^*(r, n_2 + j, m, k) \overset{d}{=} X^*(r, n_2, m, k) + Y_j ,$$

$$j = n_1 - n_2 - 1, n_1 - n_2;\ 1 \le r < n_2 < n_1 \tag{2.1}$$

where $Y_j \overset{d}{=} Y(j, n_1, m, k)$, the j^{th} m-gos from a sample of size n_1 drawn from an $exp(\alpha)$ distribution and is independent of $X^*(r, n_2, m, k)$ if and only if $X_1 \sim genexp(\alpha)$.

Proof. To prove the necessary part, let the moment generating function (*mgf*) of $X^*(r, n_1, m, k)$ be $M_{X^*_{(r,n_1,m,k)}}(t)$, then

$$X^*(r, n_1, m, k) \overset{d}{=} X^*(r, n_2, m, k) + Y$$

implies

$$M_{X^*_{(r,n_1,m,k)}}(t) \overset{d}{=} M_{X^*_{(r,n_2,m,k)}}(t) \cdot M_Y(t)$$

Since for the $genexp(\alpha)$ distribution.

$$M_{X^*_{(r,n_1,m,k)}}(t) = \frac{C_{r-1}^{(n_1)}}{(r-1)!\,(m+1)^r} \frac{\Gamma\left(r - \frac{t}{\alpha(m+1)}\right)\Gamma\left(\frac{\gamma_r^{(n_1)}}{(m+1)}\right)}{\Gamma\left(\frac{\gamma_r^{(n_1)}}{(m+1)} - \frac{t}{\alpha(m+1)} + r\right)}$$

Therefore,

$$M_Y(t) = \frac{M_{X^*_{(r,n_1,m,k)}}(t)}{M_{X^*_{(r,n_2,m,k)}}(t)} = \frac{C_{r-1}^{(n_1)}}{C_{r-1}^{(n_2)}} \frac{\Gamma\left(\frac{\gamma_r^{(n_2)}}{(m+1)} - \frac{t}{\alpha(m+1)} + r\right)\ \Gamma\left(\frac{\gamma_r^{(n_1)}}{(m+1)}\right)}{\Gamma\left(\frac{\gamma_r^{(n_1)}}{(m+1)} - \frac{t}{\alpha(m+1)} + r\right)\ \Gamma\left(\frac{\gamma_r^{(n_2)}}{(m+1)}\right)}$$

$$= \prod_{i=1}^{n_1-n_2}\left(1 - \frac{t}{\alpha\gamma_i^{(n_1)}}\right)^{-1}$$

as $\gamma_{r+j}^{(n_2)} = \gamma_{r+j}^{(n_1)} - (n_1 - n_2)(m+1)$ and $\gamma_{r+j}^{(n_2)} = \gamma_{n_1-n_2+j}^{(n_1)} - r(m+1)$

But this is the *mgf* of $Y(n_1 - n_2, n_1, m, k)$, the $(n_1 - n_2)^{th}$ *gos* from a sample of size n_1 drawn from $exp\ (\alpha)$ and hence the result.

For the proof of sufficiency part, we have by the convolution method

$$f_{X^*_{(r,n_1,m,k)}}(x) = \int_0^x f_{X^*_{(r,n_2,m,k)}}(y) f_{Y_{(n_1-n_2,n_1,m,k)}}(x-y)dy$$

$$= \frac{\alpha\, C_{n_1-n_2-1}^{(n_1)}}{(n_1-n_2-1)!(m+1)^{n_1-n_2-1}} \int_0^x [e^{-\alpha(x-y)}]^{\gamma_{n_1-n_2}^{(n_1)}}$$

$$\times [1 - (e^{-\alpha(x-y)})^{m+1}]^{n_1-n_2-1} f_{X^*_{(r,n_2,m,k)}}(y)dy \qquad (2.2)$$

Differentiating both the sides of (2.2) $w.r.t.x$, we get

$$\frac{d}{dx} f_{X^*_{(r,n_1,m,k)}}(x) = \frac{\alpha(m+1)(n_1-n_2-1)\, c_{n_1-n_2-1}^{(n_1)}}{(n_1-n_2-1)!(m+1)^{n_1-n_2-1}} \int_0^x \alpha[e^{-\alpha(x-y)}]^{\gamma_{n_1-n_2}^{(n_1)}+(m+1)}$$

$$\times [1 - (e^{-\alpha(x-y)})^{m+1}]^{n_1-n_2-2} f_{X^*_{(r,n_2,m,k)}}(y)dy$$

$$- \frac{\alpha\, \gamma_{n_1-n_2}^{(n_1)} c_{n_1-n_2-1}^{(n_1)}}{(n_1-n_2-1)!(m+1)^{n_1-n_2-1}} \int_0^x \alpha[e^{-\alpha(x-y)}]^{\gamma_{n_1-n_2}^{(n_1)}}$$

$$\times [1 - (e^{-\alpha(x-y)})^{m+1}]^{n_1-n_2-1} f_{X^*_{(r,n_2,m,k)}}(y)dy$$

Now since,

$$f_{X_{(n_1-n_2,n_1,m,k)}}(x) = \frac{\alpha c_{n_1-n_2-1}^{(n_1)}}{(n_1-n_2-1)!(m+1)^{n_1-n_2-1}} [e^{-\alpha x}]^{\gamma_{n_1-n_2}^{(n_1)}}$$

$$\times [1 - (e^{-\alpha x})^{m+1}]^{n_1-n_2-2} [1 - (e^{-\alpha x})^{m+1}]$$

$$= \frac{\alpha c_{n_1-n_2-1}^{(n_1)}}{(n_1-n_2-1)!(m+1)^{n_1-n_2-1}} [e^{-\alpha x}]^{\gamma_{n_1-n_2}^{(n_1)}} [1 - (e^{-\alpha x})^{m+1}]^{n_1-n_2-2}$$

$$- \frac{\alpha c_{n_1-n_2-1}^{(n_1)}}{(n_1-n_2-1)!(m+1)^{n_1-n_2-1}} [e^{-\alpha x}]^{\gamma_{n_1-n_2}^{(n_1)}+(m+1)} [1 - (e^{-\alpha x})^{m+1}]^{n_1-n_2-2}$$

implying

$$\frac{\alpha(m+1)(n_1-n_2-1)c_{n_1-n_2-1}^{(n_1)}}{(n_1-n_2-1)!(m+1)^{n_1-n_2-1}} [e^{-\alpha x}]^{\gamma_{n_1-n_2}^{(n_1)}+(m+1)} [1 - e^{-\alpha x(m+1)}]^{n_1-n_2-2}$$

$$= \frac{c_{n_1-n_2-1}^{(n_1)}}{c_{n_1-n_2-2}^{(n_1-1)}} f_{X_{(n_1-n_2-1,n_1-1,m,k)}}(x) - (m+1)(n_1-n_2-1) f_{X_{(n_1-n_2,n_1,m,k)}}(x)$$

after noting that

$$C_{n_1-n_2-1}^{(n_1)} = \gamma_1^{(n_1)}\, C_{n_1-n_2-2}^{(n_1-1)}$$

and $\frac{\gamma_r^{(n_2)}}{(m+1)} + (n_1 - n_2 - j + r) = \frac{\gamma_j^{(n_1)}}{(m+1)}$; $\gamma_{r+j}^{(n_2)} = \gamma_{n_1-n_2+r+j}^{(n_1)}$,

This leads to,

$$\frac{d}{dx} f_{X^*_{(r,n_1,m,k)}}(x) = \alpha\gamma_1^{(n_1)} [f_{X^*_{(r,n_1-1,m,k)}}(x) - f_{X^*_{(r,n_1,m,k)}}(x)] \qquad (2.3)$$

$$\text{or,} \quad f_{X^*_{(r,n_1,m,k)}}(x) = \alpha\, \gamma_1^{(n_1)} [F_{X^*_{(r,n_1-1,m,k)}}(x) - F_{X^*_{(r,n_1,m,k)}}(x)] \qquad (2.4)$$

Now, since (Kamps, [7])

$$[F_{X^*_{(r,n_1-1,m,k)}}(x) - F_{X^*_{(r,n_1,m,k)}}(x)] =$$

$$\frac{C_{r-1}^{(n_1)}}{(m+1)^r (r-1)!} \frac{(m+1)}{\gamma_1^{(n_1)}} [F(x)]^{\gamma_r^{(n_1)}-(m+1)} [1-(F(x))^{m+1}]^r \tag{2.5}$$

Therefore, in view of (1.2), (2.4) and (2.5),we have

$$\frac{(m+1)[F(x)]^m f(x)}{[1-(F(x))^{m+1}]} = \alpha(m+1)$$

implying that

$$F(x) = [1-e^{-\alpha(m+1)x}]^{\frac{1}{m+1}}$$

and hence the proof.

Corollary 2.1 *Let $X^*(r,n,m,k)$ be the r^{th} m- dgos from a sample of size n drawn from a continuous population with the pdf $f(x)$ and the df $F(x)$, then*

$$X^*(r,n_2+j,m,k) \stackrel{d}{=} X^*(r,n_2,m,k) \cdot Y_j ,$$

$$j = n_1-n_2-1, n_1-n_2 \;\; ; \;\; 1 \le r < n_2 \le n_1 \tag{2.6}$$

where $Y_j \stackrel{d}{=} Y(j,n_1,m,k)$, the j^{th} m-gos from a sample of size n_1 drawn from $Par(\alpha)$ distribution and is independent of $X^*(r,n_2,m,k)$ if and only if $X_1 \sim genPar(\alpha)$.

Proof. Here the product $X^*(r,n_2,m,k) \cdot Y_j$ in (2.6) is called random dilation of $X^*(r,n_2,m,k)$ (Beutner and Kamps, [4]). Note that

$$\log X^*(r,n_1,m,k) \stackrel{d}{=} \log X^*(r,n_2,m,k) + \log Y_{n_1-n_2}$$

implies

$$X^*(r,n_1,m,k) \stackrel{d}{=} X^*(r,n_2,m,k) \cdot Y_{n_1-n_2}$$

and the proof follows in view of (1.6), (1.7), (1.18) and (1.20).

Remark 2.1 Since order statistics is a particular case of *gos* and *dgos*, Corollary 2.1 can be deduced for order statistics as

$$X_{n_1-r+1:n_1} \stackrel{d}{=} X_{n_2-r+1:n_2} \cdot X_{n_1-n_2:n_1}$$

or, $X_{s:n_1} \stackrel{d}{=} X_{r:n_2} \cdot X_{s-r:n_1}$, $1 \le r < s < n_2 < n_1$

a characterizing result which is still an open problem Arnold *et al.* [3].

Corollary 2.2 *Let $X(r,n,m,k)$ be the r^{th} m- gos from a sample of size n drawn from a continuous population with the pdf $f(x)$ and the df $F(x)$ then*

$$X(r,n_2+j,m,k) \stackrel{d}{=} X(r,n_2,m,k) \cdot Y_j^* ,$$

$$j = n_1-n_2-1, n_1-n_2 \; ; \; 1 \le r < n_2 \le n_1 \tag{2.7}$$

where $Y_j^* \stackrel{d}{=} Y^*(j,n_1,m,k)$ is the j^{th} m-dgos from a sample of size n_1 drawn from $pow(\alpha)$ distribution and is independent of $X(r,n_2,m,k)$ if and only if $X_1 \sim genpow(\alpha)$.

Proof. Here the product $X(r,n_2,m,k)\cdot Y_j^*$ in (2.7) is called random contraction of $X(r,n_2,m,k)$ (Beutner and Kamps, [4]). It may be noted that

$$-\log X^*(r,n_1,m,k) \overset{d}{=} -\log X^*(r,n_2,m,k) - \log\ Y_{n_1-n_2}$$

implies

$$X(r,n_1,m,k) \overset{d}{=} X(r,n_2,m,k)\ \cdot\ Y^*_{n_1-n_2}$$

and the result follows with an appeal to (1.9), (1.10), (1.19) and (1.21).

Remark 2.2 At $= 1$ $(i.e.\, at\, n_1 = n_2 + 1)$, we get

$$X(r,n_1,m,k) \overset{d}{=} X(r,n_1-1,m,k)\ \cdot\ Y^*(1,n_1,m,k)$$

as obtained by Beutner and Kamps [4].

Remark 2.3 The relation at $m = 0$ is of the form

$$X_{r:n_1} \overset{d}{=} X_{r:n_2}\cdot X_{n_1:n_1} \quad , \quad 1 \le r < n_2 < n_1$$

which at $r = 1$, reduces to

$$X_{1:n_1} \overset{d}{=} X_{1:n_2}\cdot X_{n_1:n_1}$$

as discussed by Arnold *et al.* [3]

At $j = 1$ $(i.e.\, at\, n_1 = n_2 + 1)$, we have

$$X_{r:n_1} \overset{d}{=} X_{r:n_1-1}\ \cdot\ X_{n_1:n_1}$$

where $X_{n_1:n_1} \sim pow\ (\alpha n_1)$, as given by Castaño-Martínez *et al.* [6] , Wesolowski and Ahsanullah [12] and Khan and Shah [8].

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