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Detour Saturated Graph

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Abstract: A connected graph G of order p is detour saturated if and only if the detour distance between every two distinct vertices in G is $p-1$. The detour saturated graphs of order p and of minimum density are characterized in this paper. Also, detour saturated graphs are obtained from two disjoint connected graphs by using joint and Cartesian product operations.

Keywords: detour distance, connected graph, detour saturated graph.

1 Introduction

By a graph $G=(V,E)$ we mean a finite undirected graph without loops and multiple edges. The prder and size of G is denoted by p and q, respectively. for basic definitions and terminologies we refer to [1,3].

For any two vertices u and v in a connected graph G, the **detour distance**, denoted by $D(u,v)$, is the length of a longest $u-v$ path in G. A $u-v$ path of length $D(u,v)$ is called a $u-v$ **detour**. It is known [4] that the detour distance is a metric on the vertex set V. The **detour eccentricity** $e_D(v)$ of a vertex v in G is defined by $e_D(v)=max\{D(u,v):u\in V\}$.

A connected graph G is called **detour self-centered** [2], if and only if, $e_D(u)=e_D(v)$ for every two vertices u and v in G. The **detour index** $dd(G)$ of a graph G is the sum of all detour distances in G, that is

$$dd(G)=\sum_{\{u,v\}} D(u,v),$$

where the summation is taken over all unordered pairs u,v of vertices in G. If $p\geq 3$, then

$$1\leq D(u,v)\leq p-1,$$

for every two distinct vertices u and v in G.

Thus, it is clear that

$$\frac{1}{2}p(p-1)\leq dd(G)\leq \frac{1}{2}p(p-1)^2.$$

If $D(u,v)=p-1$ for every pair of distinct vertices u and v in G, then

$$dd(G)=\frac{1}{2}p(p-1)^2.$$

such graphs are called **detour saturated** [6]. For example, K_p is a detour saturated graph, and $K_{r,s}$, $r,s\geq 2$, is not. We have the following simple statement.

Observation 1. A connected graph G is detour saturated if and only if G is a Hamiltonian-connected graph.■

Of course, every detour saturated graph is detour self-centered, but the converse is not necessarily true. For example, $K_{r,r}, r\geq 2$ and $C_p, p\geq 4$ are detour self-centered graphs, but they are not detour saturated.

2 Detour Saturated Graphs of Minimum density

Let G be a detour saturated graph of order $p\geq 4$, then G does not contain a vertex of degree one and contains no bridge. Also, G does not contain a vertex of degree two as shown in the following statement. **Proposition 1**.If G is a detour saturated graph of order $p\geq 4$, then its minimum degree $\delta(G)\geq 3$.

proof. Assume, to the contrary, that there is a vertex v of degree 2 in G; and let u_1 and u_2 be the vertices adjacent to v. Then it is clear that any u_1-u_2 detour of length 3 or

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more does not contain vertex v. Thus, $D(u_1, u_2) \leq p-2$, which is a contradiction.■

From Proposition 1, we notice that the only detour saturated graph of minimum degree 2 is K_3.

In fact, there are many non-isomorphic graphs of the same order that have the same detour index. For example, the graphs G_1 and G_2 in Fig. 1 are not detour saturated, but have the same detour index.

Fig. 1: $dd(G_1) = dd(G_2) = 28$.

Also, there are non-isomorphic detour saturated graphs of the **same order** but different sizes. For example, the graphs H_1 and H_2 in Fig. 2, are detour saturated of order 8 and size 14 and 12, respectively.

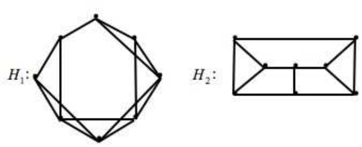

Fig. 2: Saturated graphs of order 8 and different sizes.

It may be of interest to ask what is the minimum number of edges in a detour saturated graph of detour p. It is convenient to cast this question in terms of the minimum density at which a graph G is detour saturated, where the **density** of G may be defined as the ratio of the size q to the order p, namely $\frac{q}{p}$ (see[5, p. 989]).

Consider the graphs G_1 and G_2 depicted in Fig. 3, which have density $\frac{3}{2}$; the graph G_1 is detour saturated while G_2 is not. This observation is an indicative of the fact that density is a **global** property of a graph, while detour saturation depends on the details of the connectivity and, hence, has **local** characteristics.

From Proposition 1, we have next observation.

Observation 2. The density of every detour saturated graph of order $p \geq 4$, is not less than $\frac{3}{2}$.■

Of course, if G is a cubic graph, then its order is even and its density is $\frac{3}{2}$. Randic, et al, [6], showed that the

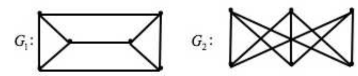

Fig. 3: Graphs of density $\frac{3}{2}$

cubic graph of order $p \geq 8$, shown in Fig. 4, is detour saturated. thus, the minimum density for every detour saturated graph of **even** order, $p \geq 4$, is $\frac{3}{2}$. Thus, it remains to find the minimum density of a detour saturated graph of **odd** order.

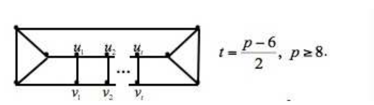

Fig. 4: A detour saturated graph of minimum density $\frac{3}{2}$

Proposition 2. Let G be a (p,q) detour saturated graph of odd order $p \geq 5$, then its density $\frac{q}{p}$ is not less than $\frac{3}{2} + \frac{1}{2p}$.

Proof. Every graph of odd order must contain at least one even vertex. Since G is detour saturated, then it must contain an even vertex of degree ≥ 4 (by Proposition 1). therefore,

$$2q \geq 4 + 3(p-1).$$

. Thus,

$$\frac{q}{p} \geq \frac{3}{2} + \frac{1}{2p}.$$

■

The graph G_1 in Fig. 5, is detour saturated of order 5 and size 8, thus its density is $\frac{3}{2} + +\frac{1}{10}$. And, the graph G_2 in Fig. 2.5, is also detour saturated of order 7 and size 11, thus its density is $\frac{3}{2} + +\frac{1}{14}$. Therefore, both G_1 and G_2 are detour saturated of minimum density.

For odd $p \geq 9$, one may check that the graph shown in

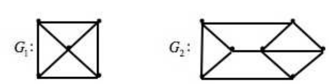

Fig. 5: Detour saturated graphs of minimum $\frac{q}{p}$.

Fig. 6 is detour saturated with density $\frac{q}{p} = \frac{3}{2} + \frac{1}{2p}$. Therefore, it is of minimum density.

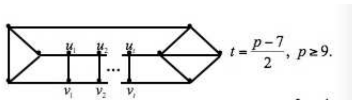

Fig. 6: A detour saturated graph of odd order p and minimum density $\frac{3}{2}+\frac{1}{2p}$.

From our previous discussions, we obtain the next theorem:

Theorem 3. The minimum density of detour saturated graphs of order $p \geq 4$, is $\frac{3}{2}$ if p is even, and $\frac{3}{2}+\frac{1}{2p}$ if p is odd.■

Corollary 4. Let G be a (p,q) Hamiltonian-connected graph, $p \geq 4$. Then G has minimum density $\frac{q}{p}$ if and only if, it is cubic or has exactly one vertex of degree 4 and all other vertices of degree 3.

proof. Since G is Hamiltonian-connected then G is detour saturated. Thus by Theorem 3, the minimum density $\frac{q}{p}$ is $\frac{3}{2}$ if p is even, and it is $\frac{3}{2}+\frac{1}{2p}$ if p is odd. Thus, G is cubic when p is even (by Proposition 1), and for odd p, G has exactly one vertex of degree 4 and all other vertices of degree 3.■

Now, we give some other results on detour saturated graphs obtained by using operations, namely joint and Cartesian product.

Proposition 5. Let G_1 and G_2be disjoint connected graphs of order p_1 and p_2, respectively, such that $p_1 \leq p_2$. If $\delta(G_2) \geq \lceil \frac{p_2-p_1+1}{2} \rceil$, then G_1+G_2 is detour saturated.

proof. Let v be any vertex of $G=G_1+G_2$. Then, if v is a vertex in G_1, then

$$deg_G v \geq p_2+1 > \frac{p+1}{2},$$

where $p=p_1+p_2$.

If v is a vertex of G_2, then

$$deg_G v \geq p_1+\delta(G_2) \geq p_1+\lceil \frac{p_2-p_1+1}{2} \rceil \geq \frac{p+1}{2}.$$

Thus, by Corollary 6.8b in [3, p. 190], G is a Hamiltonian-connected graph, and so it is detour saturated.■

Corollary 6. If G_1 and G_2 are disjoint connected graphs of the same order, then G_1+G_2 is detour saturated.■

Proposition 7. If G is a detour saturated graph, then $G+K_1$ is detour saturated.

Proof. It is obvious.■

The converse of this proposition is not true, for example, the wheel $W_n, n \geq 5$, is detour saturated but $C_{n-1}+K=W_n$, and C_{n-1} is not detour saturated.

Proposition 8. If G is a detour saturated graph of order $p \geq 3$, then $G \times K_2$ is a detour saturated graph.

Proof. Let G_1 and G_2 be the two copies of G in $G \times K_2$. Let u_1 and v_1 be any two vertices in G_1, and let P be a u_1-v_1 detour in G_1. Moreover, let w_1 be the last vertex before v_1 on P. There is in G_2, a detour Q from w_2 to v_2, where w_2 and v_2 are the vertices in G_2 corresponding to w_1 and v_1, respectively. Then, the path in $G \times K_2$ constructed from $P-v_1$, edge w_1w_2, Q and edge v_2v_1 in a u_1-v_1 detour of length $2p-1$. Thus, there is a detour between any two vertices of G_1 (or G_2) in $G \times K_2$, of length $2p-1$.

Now, let u_1 be any vertex in G_1 and v_2 be any vertex in G_2. Let x_2 be any vertex in G_2 other than u_2, v_2, and let x_1 be the vertex in G_1 corresponding to x_2. Then, there is u_1-v_2 detour in $G \times K_2$ of length $2p_1$, constructed from a u_1-x_1 detour in G_1, followed by the edge x_1x_2, then an x_2-v_2 detour in G_2.■

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