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The Semi normed space defined by χ sequences

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Abstract: In this paper we introduce the sequence spaces $\chi(p,\sigma,q,s)$, $\Lambda(p,\sigma,q,s)$ and define a semi normed space (X,q) semi normed by q. We study some properties of these sequence spaces and obtain some inclusion relations.

Keywords: Chi sequence, Analytic sequence, Invariant mean, Semi norm

1 Introduction

A complex sequence, whose kth term is x_k , is denoted by $\{x_k\}$ or simply x. Let ϕ be the set of all finite sequences. A sequence $x = \{x_k\}$ is said to be anlaytic if $\sup_k (|x_k|)^{\frac{1}{k}} < \infty$. The vector space of all analytic sequences will be denoted by Λ. A sequence x is called chi sequence if $\lim_{k\to\infty} (k!|x_k|)^{\frac{1}{k}} = 0$. The vector space of all chi sequences will be denoted by χ. Let σ be a one-one mapping of the set of positive integers into itself such that $\sigma^m(n) = \sigma(\sigma^{m-1}(n)), m = 1,2,3,\ldots$.

A continuous linear functional ϕ on Λ is said to be an invariant mean or a σ-mean if and only if (1) $\phi(x) \geq 0$ when the sequence $x = (x_n)$ has $x_n \geq 0$ for all n (2) $\phi(e) = 1$ where $e = (1,1,1,\ldots)$ and (3) $\phi(\{x_{\sigma}(n)\}) = \phi(\{x_n\})$ for all $x \in \Lambda$. For certain kinds of mappings σ, every invariant mean ϕ extends the limit functional on the space C of all real convergent sequences in the sense that $\phi(x) = \lim x$ for all $x \in C$. Consequently $C \subset V_\sigma$, where V_σ is the set of analytic sequences all of those σ-means are equal.

If $x = (x_n)$, set $Tx = (Tx)^{1/n} = (x_{\sigma}(n))$. It can be shown that

$$V_\sigma = x = (x_n) : \lim_{m\to\infty} t_{mn}(x_n)^{1/n} = L \text{ uniformly in } n, \quad (1)$$

$$L = \sigma - \lim_{n\to\infty} (x_n)^{1/n} \text{ where}$$

$$t_{mn}(x) = \frac{(x_n + Tx_n + \ldots + T^m x_n)^{1/n}}{m+1} \quad (2)$$

Given a sequence $x = \{x_k\}$ its nth section is the sequence $x^{(n)} = \{x_1, x_2, \ldots, x_n, 0, 0, \ldots\}$, $\delta^{(n)} = (0,0,\ldots,1,0,0,\ldots)$, 1 in the nth place and zeros elsewhere. An FK-space (Frechet coordinate space) is a Frechet space which is made up of numerical sequences and has the property that the coordinate functionals $p_k(x) = x_k$ $(k = 1,2,\ldots)$ are continuous.

2 Definitions and Preliminaries

Definition 1. *The space consisting of all those sequences x in w such that $(k!|x_k|)^{\frac{1}{k}} \to 0$ as $k \to \infty$ is denoted by χ. In other words $(k!|x_k|)^{1/k}$ is a null sequence χ is called the space of chi sequences. The space χ is a metric space with the metric $d(x,y) = \{\sup_k (k!|x_k - y_k|)^{\frac{1}{k}}, k = 1,2,3,\ldots\}$ for all $x = \{x_k\}$ and $y = \{y_k\}$ in χ.*

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Definition 2. *The space consisting of all those sequence x in w such that $(\sup_k(|x_k|)^{\frac{1}{k}}) < \infty$ is denoted by Λ. In other words $(\sup_k(|x_k|)^{\frac{1}{k}})$ is a bounded sequence.*

Definition 3. *Let p,q be semi norms on a vector space X. Then p is said to be stronger than q if whenever (x_n) is a sequence such that $p(x_n) \to 0$, then also $q(x_n) \to 0$. If each is stronger than the other, then p and q are said to be equivalent.*

Lemma 1. *Let p and q be semi norms on a linear space X. Then p is stronger than q if and only if there exists a constant M such that $q(x) \le Mp(x)$ for all $x \in X$.*

Definition 4. *A sequence space E is said to be solid or normal if $(\alpha_k x_k) \in E$ whenever $(x_k) \in E$ and for all sequences of scalars (α_k) with $|\alpha_k| \le 1$, for all $k \in N$.*

Definition 5. *A sequence space E is said to be monotone if it contains the canonical pre-images of all its step spaces.*

Remark. From the above two definitions, it is clear that a sequence space E is solid implies that E is monotone.

Definition 6. *A sequence E is said to be convergence free if $(y_k) \in E$ whenever $(x_k) \in E$ and $x_k = 0$ implies that $y_k = 0$.*

Let $p = (p_k)$ be a sequence of positive real numbers with $0 < p_k < \sup p_k = G$. Let $D = \max(1, 2^{G-1})$. Then for $a_k, b_k \in C$, the set of complex numbers for all $k \in N$ we have

$$|a_k + b_k|^{1/k} \le D\left\{|a_k|^{1/k} + |b_k|^{1/k}\right\} \tag{3}$$

Let (X,q) be a semi normed space over the field C of complex numbers with the semi norm q. The symbol $\Lambda(X)$ denotes the space of all analytic sequences defined over X. We define the following sequence spaces:

$$\Lambda(p,\sigma,q,s) = \left\{x \in \Lambda(X) : \sup_{n,k} k^{-s}\left[q\left(|x_{\sigma^k(n)}|^{1/k}\right)\right]^{p_k} < \infty \right.$$
$$\text{uniformly in } n \ge 0,$$
$$\left. s \ge 0\right\}$$

$$\chi(p,\sigma,q,s) = \left\{x \in \chi(X) : k^{-s}\left[q\left(|x_{\sigma^k(n)}|^{1/k}\right)\right]^{p_k} \to 0, \right.$$
$$\text{as } k \to \infty$$
$$\text{uniformly in } n \ge 0,$$
$$\left. s \ge 0\right\}$$

3 Main Results

Theorem 1. *$\chi(p,\sigma,q,s)$ is a linear space over the set of complex numbers.*

Proof. It is routine verification. Therefore the proof is omitted.

Theorem 2. *$\chi(p,\sigma,q,s)$ is a paranormed space with*

$$g^*(x) = \left\{\sup_{k \ge 1} k^{-s}\left[q\left(\sigma^k(n)!|x_{\sigma^k(n)}|\right)^{\frac{1}{k}}\right], \text{ uniformly in } n > 0\right\}$$

where $H = \max\left(1, \sup_k p_k\right)$.

Proof. Clearly $g(x) = g(-x)$ and $g(\theta) = 0$, where θ is the zero sequence. It can be easily verified that $g(x+y) \le g(x) + g(y)$. Next $x \to \theta$, λ fixed implies $g(\lambda x) \to 0$. Also $x \to \theta$ and $\lambda \to 0$ imply $g(\lambda x) \to 0$. The case $\lambda \to 0$ and x fixed implies that $g(\lambda x) \to 0$ follows from the following expressions.

$$g(\lambda x) = \left\{\sup_{k \ge 1} k^{-s}\left[q\left(|x_{\sigma^k(n)}|^{1/k}\right)\right] \text{ uniformly in } n,m \in N\right\}$$

$$g(\lambda x) = \left\{\left(|\lambda|^{1/k} r\right)^{p_m/H} : \sup_{k \ge 1} k^{-s}\left[q\left(\sigma^k(n)!|x_{\sigma^k(n)}|\right)^{1/k}\right],\right.$$
$$\left. r > 0, \text{ uniformly in } n,m \in N\right\}.$$

where $r = \frac{1}{|\lambda|^{1/k}}$. Hence $\chi(p,\sigma,q,s)$ is a paranormed space. This completes the proof.

Theorem 3. $\chi(p,\sigma,q,s) \cap \Lambda(p,\sigma,q,s) \subseteq \chi(p,\sigma,q,s)$.

Proof. It is routine verification. Therefore the proof is omitted.

Theorem 4. $\chi(p,\sigma,q,s) \subset \Lambda(p,\sigma,q,s)$.

Proof. It is routine verification. Therefore the proof is omitted.

Remark. [(i)] Let q_1 and q_2 be two semi norms on X, we have
$\chi(p,\sigma,q_1,s) \bigcap \chi(p,\sigma,q_2,s) \subseteq \chi(p,\sigma,q_1+q_2,s)$;
2. If q_1 is stronger than q_2, then $\chi(p,\sigma,q_1,s) \subseteq \chi(p,\sigma,q_2,s)$;
3. If q_1 is equivalent to q_2, then $\chi(p,\sigma,q_1,s) = \chi(p,\sigma,q_1,s)$.

Theorem 5. *[(i)]*
Let $0 \le p_k \le r_k$ and $\left\{\frac{r_k}{p_k}\right\}$ be bounded. Then $\chi(r,\sigma,q,s) \subset \chi(p,\sigma,q,s)$;
2. $s_1 \le s_2$ implies $\chi(p,\sigma,q,s_1) \subset \chi(p,\sigma,q,s_2)$.

Proof(Proof of (i)).

$$\text{Let } x \in \chi(r,\sigma,q,s) \tag{4}$$

$$k^{-s}\left[q\left(\sigma^k(n)!|x_{\sigma^k(n)}|\right)^{\frac{1}{k}}\right]^{r_k} \to 0 \text{ as } k \to \infty \tag{5}$$

Let $t_k = k^{-s}\left[q\left(\sigma^k(n)!|x_{\sigma^k(n)}|\right)^{\frac{1}{k}}\right]^{r_k} \to 0$ and $\lambda_k = \frac{p_k}{r_k}$. Since $p_k \le r_k$, we have $0 \le \lambda_k \le 1$. Take $0 < \lambda > \lambda_k$. Define $u_k = t_k$ $(t_k \ge 1)$; $u_k = 0$ $(t_k < 1)$; and

$v_k = 0$ $(t_k \geq 1)$; $v_k = t_k$ $(t_k < 1)$; $t_k = u_k + v_k$ $t_k^{\lambda_k} + v_k^{\lambda_k}$. Now it follows that

$$u_k^{\lambda_k} \leq t_k \quad \text{and} \quad v_k^{\lambda_k} \leq v_k^{\lambda} \tag{6}$$

(i.e.) $t_k^{\lambda_k} \leq t_k + v_k^{\lambda}$ by (6)

$$k^{-s}\left[q\left(\sigma^k(n)!|x_{\sigma^k(n)}|\right)^{1/k}\right]^{\lambda_k} \leq$$
$$k^{-s}\left[q\left(\sigma^k(n)!|x_{\sigma^k(n)}|\right)^{1/k}\right]^{r_k}$$
$$k^{-s}\left[q\left(\sigma^k(n)!|x_{\sigma^k(n)}|\right)^{1/k}\right]^{p_k/r_k} \leq$$
$$k^{-s}\left[q\left(\sigma^k(n)!|x_{\sigma^k(n)}|\right)^{1/k}\right]^{r_k}$$
$$k^{-s}\left[q\left(\sigma^k(n)!|x_{\sigma^k(n)}|\right)^{1/k}\right]^{p_k} \leq$$
$$k^{-s}\left[q\left(\sigma^k(n)!|x_{\sigma^k(n)}|\right)^{1/k}\right]^{r_k}.$$

But $k^{-s}\left[q\left(\sigma^k(n)!|x_{\sigma^k(n)}|\right)^{1/k}\right]^{r_k} \to 0$ as $k \to \infty$ by (5).

$$k^{-s}\left[q\left(\sigma^k(n)!|x_{\sigma^k(n)}|\right)^{1/k}\right]^{p_k} \to 0 \text{ as } k \to \infty.$$

Hence

$$x \in \chi(p,\sigma,q,s) \tag{7}$$

From (4) and (7) we get $\chi(r,\sigma,q,s) \subset \chi(p,\sigma,q,s)$. Hence the proof.

Proof(Proof of (ii)). It is routine verification. Therefore the proof is omitted.

Theorem 6. *The space $\chi(p,\sigma,q,s)$ is solid and as such is monotone.*

Proof. Let $(x_k) \in \chi(p,\sigma,q,s)$ and (α_k) be a sequence of scalars such that $|\alpha_k| \leq 1$ for all $k \in N$. Then $k^{-s}\left[q\left(\sigma^k(n)!|\alpha_k x_{\sigma^k(n)}|\right)^{1/k}\right]^{p_k} \leq k^{-s}\left[q\left(\sigma^k(n)!|x_{\sigma^k(n)}|\right)^{1/k}\right]^{p_k}$ for all $k \in N$. $\left[q\left(\sigma^k(n)!|\alpha_k x_{\sigma^k(n)}|\right)^{1/k}\right]^{p_k} \leq \left[q\left(\sigma^k(n)!|\alpha_k x_{\sigma^k(n)}|\right)^{1/k}\right]^{p_k}$ for all $k \in N$. This completes the proof.

Theorem 7. *The space $\chi(p,\sigma,q,s)$ are not convergence free in general.*

Proof. The proof follows from the following example.

Example 1. Let s= 0; $p_k = 1$ for k even and $p_k = 2$ for k odd. Let $X = C$, $q(x) = |x|$ and $\sigma(n) = n+1$ for all $n \in N$. Then we have $\sigma^2(n) = \sigma(\sigma(n)) = \sigma(n+1) = (n+1)+1 = n+2$ and $\sigma^3(n) = \sigma(\sigma^2(n)) = \sigma(n+2) = (n+2)+1 = n+3$. Therefore, $\sigma^k(n) = (n+k)$ for all $n,k \in N$. Consider the sequences (x_k) and (y_k) defined as $x_k = \left(\frac{1}{k}\right)^k \times \frac{1}{k!}$ and $(y_k) = k^k \times \frac{1}{k!}$ for all $k \in N$. (i.e.) $|x_k|^{1/k} = \frac{1}{k} \times \frac{1}{k!}$ and $|y_k|^{1/k} = \frac{1}{k} \times \frac{1}{k!}$ for all $k \in N$.

Hence $|\left(\frac{1}{(n+k)}\right)^{n+k}|^{p_k} \to 0$ as $k \to \infty$. Therefore $(x_k) \in \chi(p,\sigma)$. But $|\left(\frac{1}{(n+k)}\right)^{n+k}|^{p_k} \to 0$ as $k \to \infty$. Hence $(y_k) \notin \chi(p,\sigma)$. Hence the space $\chi(p,\sigma,q,s)$ are not convergence free in general. This completes the proof. □

References

[1] Y. Altin and M. Et, Generalized difference sequences spaces defined by a modulus function in a locally convex space, Soochow J. Math., **31**, 233–243 (2005).
[2] H. I. Brown, The summability field of a perfect *l–l* method of summation, J. Anal. Math., **20**, 281–287 (1967).
[3] R. Colak, M. Et, and E. Malkowsky, Some topics of sequence spaces, Lecture Notes in Mathematics, Firat University Press, Elazig, Turkey, (2004).
[4] C. Goffman and G. Pedrick, First Course in Functional Analysis, Prentice Hall India, New Delhi, (1974).
[5] P. K. Kamthan and M. Gupta, Sequence spaces and Series. Lecture Notes in Pure and Applied Mathematics, 65 Marcel Dekker, Inc., New York, (1981).
[6] I. J. Maddox, Elements of Functional Analysis, Cambridge Univ. Press, (1970).
[7] I. J. Maddox, Sequence spaces defined by a modulus, Math. Proc., Cambridge Philos. Soc., **100**, 161–166 (1986).
[8] H. Nakano, "Concave modulars", Journal of the Mathematical Society of Japan, **5**, 29–49 (1953).
[9] W. H. Ruckle, FK spaces in which the sequence of coordinate vectors is bounded, Canad. J. Math., **25**, 973–978 (1973).
[10] S. M. Sirajudeen, Matrix Transformation of Co(P), $l_\infty(P)$, l^P and l into χ, Indian J. Pure Appl. Math., **12**, 1106–1113 (1981).
[11] B. C. Tripathy, S. Mahanta and M. Et, On a class of generalized difference sequence space defined by modulus function, Hakkaido Math. Jour., **XXXIV**, 667–677 (2005).
[12] A. Wilansky, Functional Analysis, Blaisdell Publishing Company, New York, (1964).
[13] A. Wilansky, Summability through Functional Analysis, North Holland Mathematics Studies, North-Holland Publishing, Amsterdam, **85**, (1984).