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On the Fine Spectrum of the Generalized Difference Operator $\Delta_{a,b}$ over the Sequence Space l_p, $(1 < p < \infty)$

Saad R. El-Shabrawy [1]
Baku State University, Faculty of Mech. & Math., Z. Khalilov Str., 23, AZ 1148, Baku, Azerbaijan
Email: srshabrawy@yahoo.com

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Abstract: The generalized difference operator $\Delta_{a,b}$ on the sequence space l_p is defined by $\Delta_{a,b}x = \Delta_{a,b}(x_k) = (a_k x_k + b_{k-1}x_{k-1})_{k=0}^{\infty}$ with $x_{-1} = 0$, where (a_k) and (b_k) are two convergent sequences of nonzero real numbers satisfying certain conditions. It is the purpose of this paper to completely determine the spectrum, the point spectrum, the residual spectrum and the continuous spectrum of the operator $\Delta_{a,b}$ on the sequence space l_p, where $1 < p < \infty$.

Keywords: Spectrum of an operator, Generalized difference operator, Sequence spaces.

1 Introduction

Several authors have studied the spectrum and fine spectrum of linear operators defined by some particular limitation matrices over some sequence spaces. We summarize the knowledge in the existing literature concerned with the spectrum and the fine spectrum. The fine spectrum of the difference operator Δ over the sequence spaces c_0 and c has been studied by Altay and Başar [9]. Akhmedov and Başar [1,2] have studied the fine spectrum of the difference operator Δ over the sequence spaces l_p and bv_p, where $1 \le p < \infty$. Note that the sequence space bv_p was studied by Başar and Altay [12] and Akhmedov and Başar [2]. Malafosse [17] has studied the spectrum of the difference operator Δ over the space s_r, where s_r denotes the Banach space of all sequences $x = (x_k)$ normed by

$$\|x\|_{s_r} = \sup_{k \in \mathbb{N}} \frac{|x_k|}{r^k}, \qquad (r > 0).$$

The fine spectrum of the Zweier matrix operator Z^s over the sequence spaces l_1 and bv has been examined by Altay and Karakuş [11]. The fine spectrum of the generalized difference operator $B(r,s)$ over the sequence spaces c_0 and c has been studied by Altay and Başar [10]. Also, the fine spectrum of the operator $B(r,s)$ over the sequence spaces l_p and bv_p, where $1 < p < \infty$ has been determined by Bilgiç and Furkan [13]. The fine spectrum of the generalized difference operator

[1] Permanent Address: Mansoura University, Damietta Branch, Faculty of Science, Mathematics Department, New Damietta, Egypt.

$B(r,s,t)$ over the sequence spaces c_0 and c has been studied by Furkan et al. [15]. Also, the fine spectrum of the operator $B(r,s,t)$ over the sequence spaces l_p and bv_p, where $1<p<\infty$ has been determined by Furkan et al. [16]. The fine spectrum of the operator Δ_v over the sequence spaces c_0 and l_1 has been studied by Srivastava and Kumar [19,20]. In [7], Akhmedov and El-Shabrawy have revised some results which have been given in [20]. The fine spectrum of the operator Δ_v over the sequence space c has been studied by Akhmedov and El-Shabrawy [6]. Also, the fine spectrum of the operator Δ_v over the sequence space l_p, where $1<p<\infty$ has been determined by El-Shabrawy [14]. Recently, Akhmedov and El-Shabrawy [5,8] have modified the operator Δ_v and have studied the fine spectrum of the modified operator Δ_v over some sequence spaces. Panigrahi and Srivastava [18] have studied the fine spectrum of the generalized second order difference operator Δ_{uv}^2 over the sequence space c_0. The fine spectrum of the generalized difference operator $\Delta_{a,b}$ over the sequence spaces c_0 and c has been studied by Akhmedov and El-Shabrawy [3,4].

We begin this paper by presenting some basic concepts of spectral theory concerning the spectrum and fine spectrum of linear operators in normed spaces. In Section 3, we completely determine the fine spectrum of the operator $\Delta_{a,b}$ on the sequence space l_p, where $1<p<\infty$.

2 Preliminaries, Background and Notation

By w, we shall denote the space of all real or complex valued sequences. Any vector subspace of w is called a *sequence space*. We shall write l_∞, c, c_0 and bv for the spaces of all bounded, convergent, null and bounded variation sequences, respectively. Also by l_1, l_p and bv_p we denote the spaces of all absolutely summable sequences, p-absolutely summable sequences and p-bounded variation sequences, respectively.

A triangle is a lower triangular matrix with all of the principal diagonal elements nonzero. Let λ and μ be two sequence spaces and $A=(a_{nk})$ be an infinite matrix of real or complex numbers a_{nk}, where $n,\ k\in\mathbb{N}=\{0,1,2,...\}$. Then, we say that A defines a matrix mapping from λ into μ, and we denote it by $A:\lambda\to\mu$ if for every sequence $x=(x_k)\in\lambda$, the sequence $Ax=\{(Ax)_n\}$, the A-transform of x, is in μ, where

$$(Ax)_n=\sum_k a_{nk}x_k, \qquad (n\in\mathbb{N}). \tag{2.1}$$

For simplicity in notation, here and in what follows, the summation without limits runs from 0 to ∞. By (λ,μ), we denote the class of all matrices A such that $A:\lambda\to\mu$. Thus, $A\in(\lambda,\mu)$ if and only if the series on the right side of (2.1) converges for each $n\in\mathbb{N}$ and every $x\in\lambda$, and we have $Ax=\{(Ax)_n\}_{n\in\mathbb{N}}\in\mu$ for all $x\in\lambda$. We use the convention that any term with negative subscript is equal to naught.

We recall some basic concepts of spectral theory which are needed for our investigation [see 21, pp. 370-372].

Let $X\neq\{\theta\}$ be a complex normed space and $T:D(T)\to X$ be a linear operator with domain $D(T)\subseteq X$. With T we associate the operator

$$T_\lambda = T-\lambda I,$$

where λ is a complex number and I is the identity operator on $D(T)$. If T_λ has an inverse which is linear, we denote it by T_λ^{-1}, that is

$$T_\lambda^{-1}=(T-\lambda I)^{-1},$$

and call it the *resolvent operator* of *T*.

Many properties of T_λ and T_λ^{-1} depend on λ, and spectral theory is concerned with those properties. For instance, we shall be interested in the set of all λ in the complex plane such that T_λ^{-1} exists. The boundedness of T_λ^{-1} is another property that will be essential. We shall also ask for what λ's the domain of T_λ^{-1} is dense in *X*, to name just a few aspects.

Definition 2.1. *Let $X \neq \{\theta\}$ be a complex normed space and $T : D(T) \to X$ be a linear operator with domain $D(T) \subseteq X$. A regular value λ of T is a complex number such that*

(R1) T_λ^{-1} exists,

(R2) T_λ^{-1} is bounded,

(R3) T_λ^{-1} is defined on a set which is dense in X.

The *resolvent set* of *T*, denoted by $\rho(T,X)$, is the set of all regular values λ of *T*. Its complement $\sigma(T,X) = \mathbb{C} \setminus \rho(T,X)$ in the complex plane $\mathbb{C}$ is called the *spectrum* of *T*. Furthermore, the spectrum $\sigma(T,X)$ is partitioned into three disjoint sets as follows:

The *point* (*discrete*) *spectrum* $\sigma_p(T,X)$ is the set such that T_λ^{-1} does not exist. Any such $\lambda \in \sigma_p(T,X)$ is called an *eigenvalue* of *T*.

The *continuous spectrum* $\sigma_c(T,X)$ is the set such that T_λ^{-1} exists and satisfies (R3) but not (R2), that is, T_λ^{-1} is unbounded.

The *residual spectrum* $\sigma_r(T,X)$ is the set such that T_λ^{-1} exists (and may be bounded or not) but does not satisfy (R3), that is, the domain of T_λ^{-1} is not dense in *X*.

Hence if $(T - \lambda I)x = \theta$ for some $x \neq \theta$, then $\lambda \in \sigma_p(T,X)$, by definition, that is, λ is an eigenvalue of *T*. The vector *x* is then called an *eigenvector* of *T* corresponding to the eigenvalue λ.

Let *X* be a Banach space and $T : X \to X$ be a bounded linear operator. By *R*(*T*), we denote the range of *T*, i.e.,

$$R(T) = \{y \in X : y = Tx,\ x \in X\}.$$

By *B*(*X*), we denote the set of all bounded linear operators on *X* into itself. If $T \in B(X)$, then the adjoint T^* of *T* is a bounded linear operator on the dual X^* of *X* defined by $(T^*f)(x) = f(Tx)$ for all $f \in X^*$ and $x \in X$.

Now, we may give:

Lemma 2.1 *[22]. The matrix $A = (a_{nk})$ gives rise to a bounded linear operator $T \in B(l_1)$ from l_1 to itself if and only if the supremum of l_1 norms of the columns of A is bounded.*

Lemma 2.2 *[22]. The matrix $A = (a_{nk})$ gives rise to a bounded linear operator $T \in B(l_\infty)$ from l_∞ to itself if and only if the supremum of l_1 norms of the rows of A is bounded.*

Lemma 2.3 *[23]. Let $1 < p < \infty$ and suppose $A \in (l_\infty, l_\infty) \cap (l_1, l_1)$. Then $A \in (l_p, l_p)$.*

3 The Fine Spectrum of the Operator $\Delta_{a,b}$ on the Sequence Space l_p, $(1 < p < \infty)$

In this section we consider the operator $\Delta_{a,b}$ which is represented by the lower triangular double-band matrix

$$\Delta_{a,b} = \begin{pmatrix} a_0 & 0 & 0 & \cdots \\ b_0 & a_1 & 0 & \cdots \\ 0 & b_1 & a_2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \qquad (3.1)$$

We assume here and hereafter that the sequences (a_k) and (b_k) are two convergent sequences of nonzero real numbers satisfying

$$\lim_{k\to\infty} a_k = a > 0, \qquad (3.2)$$

$$\lim_{k\to\infty} b_k = b;\ |b| = a \qquad (3.3)$$

and

$$\sup_k a_k \le a,\quad b_k^2 \le a_k^2,\quad \text{for all } k\in\mathbb{N} \qquad (3.4)$$

It should be noted that the class of the operator $\Delta_{a,b}$ in this form includes the modified operator Δ_v of [5]. In this section, we completely characterize the sets $\sigma(\Delta_{a,b}, l_p)$, $\sigma_p(\Delta_{a,b}, l_p)$, $\sigma_r(\Delta_{a,b}, l_p)$ and $\sigma_c(\Delta_{a,b}, l_p)$. We begin with a theorem concerning the bounded linearity of the operator $\Delta_{a,b}$ acting on the sequence space l_p, where $1 < p < \infty$.

Theorem 3.1. *The generalized difference operator* $\Delta_{a,b} : l_p \to l_p$ *is a bounded linear operator and*

$$\sup_k \left(|a_k|^p + |b_k|^p\right)^{\frac{1}{p}} \le \left\|\Delta_{a,b}\right\|_{l_p} \le \sup_k |a_k| + \sup_k |b_k|.$$

Proof. The proof is obvious and so is omitted. □

Theorem 3.2. *Let* $D = \{\lambda\in\mathbb{C} : |\lambda - a| \le a\}$ *and* $E = \{a_k : k\in\mathbb{N}, |a_k - a| > a\}$. *Then*

$$\sigma(\Delta_{a,b}, l_p) = D\cup E.$$

Proof. Let $\lambda\notin D\cup E$. Then $|\lambda - a| > a$ and $\lambda \ne a_k$, for all $k\in\mathbb{N}$. So, $\Delta_{a,b} - \lambda I$ is triangle and hence $(\Delta_{a,b} - \lambda I)^{-1}$ exists. We can calculate that

$$(\Delta_{a,b} - \lambda I)^{-1} = \begin{pmatrix} \frac{1}{(a_0-\lambda)} & 0 & 0 & \cdots \\ \frac{-b_0}{(a_0-\lambda)(a_1-\lambda)} & \frac{1}{(a_1-\lambda)} & 0 & \cdots \\ \frac{b_0 b_1}{(a_0-\lambda)(a_1-\lambda)(a_2-\lambda)} & \frac{-b_1}{(a_1-\lambda)(a_2-\lambda)} & \frac{1}{(a_2-\lambda)} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Therefore, the supremum of the l_1 norms of the columns of $(\Delta_{a,b} - \lambda I)^{-1}$ is $\sup_k R_k$, where

$$R_k = \frac{1}{|a_k - \lambda|} + \frac{|b_k|}{|a_k - \lambda||a_{k+1} - \lambda|} + \frac{|b_k||b_{k+1}|}{|a_k - \lambda||a_{k+1} - \lambda||a_{k+2} - \lambda|} + \dots,\quad k\in\mathbb{N}.$$

Since $\lim_{k\to\infty}\left|\frac{b_k}{a_{k+1} - \lambda}\right| = \left|\frac{b}{a-\lambda}\right| < 1$, then there exist $k_0\in\mathbb{N}$ and a real number $q_0 < 1$ such that $\left|\frac{b_k}{a_{k+1} - \lambda}\right| < q_0$ for all $k \ge k_0$. Then, for all $k \ge k_0 + 1$,

$$R_k \le \frac{1}{|a_k - \lambda|}\left[1 + q_0 + q_0^2 + \dots\right].$$

But, there exist $k_1\in\mathbb{N}$ and a real number $q_1 < \frac{1}{a}$ such that $\frac{1}{|a_k - \lambda|} < q_1$ for all $k \ge k_1$. Then,

$$R_k \le \frac{q_1}{1 - q_0},$$

for all $k > \max\{k_0, k_1\}$. Therefore $\sup_k R_k < \infty$. This shows that $(\Delta_{a,b} - \lambda I)^{-1} \in (l_1, l_1)$. Similarly, we can show that $(\Delta_{a,b} - \lambda I)^{-1} \in (l_\infty, l_\infty)$. By Lemma 2.3, we have $(\Delta_{a,b} - \lambda I)^{-1} \in (l_p, l_p)$. Thus $\sigma(\Delta_{a,b}, l_p) \subseteq D\cup E$.

Conversely, suppose that $\lambda\notin\sigma(\Delta_{a,b}, l_p)$. Then $(\Delta_{a,b} - \lambda I)^{-1} \in B(l_p)$. Since the $(\Delta_{a,b} - \lambda I)^{-1}$ transform of the unit sequence $e = (1, 0, 0, \dots)$ is in l_p, we have $\lim_{k\to\infty}\left|\frac{b_k}{a_{k+1} - \lambda}\right|^p = \left|\frac{b}{a-\lambda}\right|^p \le 1$ and $\lambda \ne a_k$, for all $k\in\mathbb{N}$. Then

$$\{\lambda\in\mathbb{C} : |\lambda - a| < a\} \subseteq \sigma(\Delta_{a,b}, l_p)$$

and

$$\{a_k : k \in \mathbb{N}\} \subseteq \sigma(\Delta_{a,b}, l_p).$$

But, $\sigma(\Delta_{a,b}, l_p)$ is compact set, and so it is closed. Then

$$D = \{\lambda \in \mathbb{C} : |\lambda - a| \le a\} \subseteq \sigma(\Delta_{a,b}, l_p)$$

and

$$E = \{a_k : k \in \mathbb{N}, |a_k - a| > a\} \subseteq \sigma(\Delta_{a,b}, l_p).$$

This completes the proof. □

Theorem 3.3. $\sigma_p(\Delta_{a,b}, l_p) = E$

Proof. Suppose $\Delta_{a,b}x = \lambda x$ for $x \ne \theta = (0, 0, 0, ...)$ in l_p. Then by solving the system of equations

$$\left.\begin{aligned} a_0 x_0 &= \lambda x_0 \\ b_0 x_0 + a_1 x_1 &= \lambda x_1 \\ b_1 x_1 + a_2 x_2 &= \lambda x_2 \\ &\vdots \end{aligned}\right\}$$

we obtain

$$(a_0 - \lambda)x_0 = 0 \quad \text{and} \quad b_k x_k + (a_{k+1} - \lambda)x_{k+1} = 0,$$

for all $k \in \mathbb{N}$. Hence, for all $\lambda \notin \{a_k : k \in \mathbb{N}\}$, we have $x_k = 0$ for all $k \in \mathbb{N}$, which contradicts our assumption. So, $\lambda \notin \sigma_p(\Delta_{a,b}, l_p)$. Also, we can prove that $a \notin \sigma_p(\Delta_{a,b}, l_p)$. Thus

$$\sigma_p(\Delta_{a,b}, l_p) \subseteq \{a_k : k \in \mathbb{N}\} \setminus \{a\}.$$

Now, we will prove that

$$\lambda \in \sigma_p(\Delta_{a,b}, l_p) \text{ if and only if } \lambda \in E.$$

If $\lambda \in \sigma_p(\Delta_{a,b}, l_p)$, then $\lambda = a_j \ne a$ for some $j \in \mathbb{N}$ and there exists $x \in l_p$, $x \ne \theta$ such that $\Delta_{a,b}x = a_j x$. Then

$$\lim_{k\to\infty} \left|\frac{x_{k+1}}{x_k}\right|^p = \left|\frac{a}{a - a_j}\right|^p \le 1$$

Since $\left|\frac{a}{a - a_j}\right| \ne 1$, then $\lambda = a_j \in E$. Thus

$$\sigma_p(\Delta_{a,b}, l_p) \subseteq E.$$

Conversely, let $\lambda \in E$. Then there exists $i \in \mathbb{N}$ such that $\lambda = a_i \ne a$, and so, we can take $x \ne \theta$ with $\Delta_{a,b}x = a_i x$ and

$$\lim_{k\to\infty} \left|\frac{x_{k+1}}{x_k}\right|^p = \left|\frac{a}{a - a_i}\right|^p < 1,$$

that is $x \in l_p$. Thus $E \subseteq \sigma_p(\Delta_{a,b}, l_p)$. This completes the proof. □

we give the following lemma which is required in the proof of the next theorem.

Lemma 3.1. *Let* $1 < q < \infty$. *If*

$$\lambda \in \{\lambda \in \mathbb{C} : |\lambda - a| = a\},$$

then the series

$$\sum_{k=1}^{\infty} \left|\frac{(a_0 - \lambda)(a_1 - \lambda) \ldots (a_{k-1} - \lambda)}{b_0 b_1 \ldots b_{k-1}}\right|^q$$

is not convergent.

Proof. Let $\lambda = \lambda_1 + i\lambda_2 \in \mathbb{C}$ such that $|\lambda - a| = a$. Then

$$|\lambda|^2 = \lambda_1^2 + \lambda_2^2 = 2\lambda_1 a.$$

Also, for all $k \in \mathbb{N}$, we have

$$\begin{aligned} |a_k - \lambda|^2 &= (a_k - \lambda_1)^2 + \lambda_2^2 \\ &= a_k^2 + (\lambda_1^2 + \lambda_2^2) - 2\lambda_1 a_k \\ &= a_k^2 + 2\lambda_1(a - a_k) \\ &\ge b_k^2 + 2\lambda_1(a - a_k) \\ &\ge b_k^2. \end{aligned}$$

Therefore

$$\left|\frac{a_k - \lambda}{b_k}\right| \ge 1, \text{ for all } k \in \mathbb{N}.$$

This completes the proof. □

If $T : l_p \to l_p$, where $1 < p < \infty$, is a bounded linear operator with matrix *A*, then it is known that the adjoint operator $T^* : l_p^* \to l_p^*$ is defined by the transpose of the matrix *A*. it is well-known that the dual space l_p^* of l_p is isomorphic to l_q, where $p^{-1} + q^{-1} = 1$.

Theorem 3.4. $\sigma_p(\Delta_{a,b}^*, l_p^*) = \{\lambda \in \mathbb{C} : |\lambda - a| < a\} \cup E$.

Proof. Suppose that $\Delta_{a,b}^* f = \lambda f$ for $f = (f_0, f_1, f_2, ...) \neq \theta$ in $l_p^* \cong l_q$, where $p^{-1} + q^{-1} = 1$. Then, by solving the system of equations

$$\begin{gathered} a_0 f_0 + b_0 f_1 = \lambda f_0, \\ a_1 f_1 + b_1 f_2 = \lambda f_1, \\ \vdots \\ a_k f_k + b_k f_{k+1} = \lambda f_k, \\ \vdots \end{gathered}$$

we obtain that

$$f_{k+1} = \frac{\lambda - a_k}{b_k} f_k, \; k \in \mathbb{N}.$$

Therefore we must take $f_0 \neq 0$, since otherwise we would have $f = \theta$.

It is clear that, for all $k \in \mathbb{N}$, the vector $f = (f_0, f_1, ..., f_k, 0, 0, ...)$ is an eigenvector of the operator $\Delta_{a,b}^*$ corresponding to the eigenvalue $\lambda = a_k$, where $f_0 \neq 0$ and $f_n = \frac{\lambda - a_{n-1}}{b_{n-1}} f_{n-1}$, for all $1 \leq n \leq k$. Then $\{a_k : k \in \mathbb{N}\} \subseteq \sigma_p(\Delta_{a,b}^*, l_p^*)$. Also, if $\lambda \neq a_k$ for all $k \in \mathbb{N}$, then $f_k \neq 0$ for all $k \in \mathbb{N}$, and $\sum_k |f_k|^q < \infty$ if $\lim_{k\to\infty} \left|\frac{f_{k+1}}{f_k}\right|^q = \left|\frac{\lambda - a}{a}\right|^q < 1$. Thus

$$\{\lambda \in \mathbb{C} : |\lambda - a| < a\} \cup E \subseteq \sigma_p(\Delta_{a,b}^*, l_p^*).$$

Conversely, if $\lambda \in \sigma_p(\Delta_{a,b}^*, l_p^*)$, then there exists $f = (f_0, f_1, f_2, ...) \neq \theta$ in $l_p^* \cong l_q$ and $\Delta_{a,b}^* f = \lambda f$. Then, $f_{k+1} = \frac{\lambda - a_k}{b_k} f_k$, $k \in \mathbb{N}$ and $\sum_k |f_k|^q < \infty$. Therefore

$$\lim_{k\to\infty} \left|\frac{f_{k+1}}{f_k}\right|^q = \left|\frac{\lambda - a}{a}\right|^q < 1 \quad \text{or} \quad \lambda \in \{a_k : k \in \mathbb{N}\}.$$

(note that $|\lambda - a| = a$ contradicts with $\sum_k |f_k|^q < \infty$ by using Lemma 3.1). This completes the proof. □

Lemma 3.2 *[24]. T has a dense range if and only if T^* is one to one.*

Theorem 3.5. $\sigma_r(\Delta_{a,b}, l_p) = \sigma_p(\Delta_{a,b}^*, l_p^*) \setminus \sigma_p(\Delta_{a,b}, l_p)$.

Proof. For $\lambda \in \sigma_p(\Delta_{a,b}^*, l_p^*) \setminus \sigma_p(\Delta_{a,b}, l_p)$, the operator $\Delta_{a,b} - \lambda I$ is one to one and hence has an inverse. But $\Delta_{a,b}^* - \lambda I$ is not one to one. Now, Lemma 3.2 yields the fact that the range of the operator $\Delta_{a,b} - \lambda I$ is not dense in l_p. This implies that $\lambda \in \sigma_r(\Delta_{a,b}, l_p)$. □

Theorem 3.6. $\sigma_r(\Delta_{a,b}, l_p) = \{\lambda \in \mathbb{C} : |\lambda - a| < a\}$.

Proof. The proof follows from Theorems 3.3, 3.4 and 3.5. □

Theorem 3.7. $\sigma_c(\Delta_{a,b}, l_p) = \{\lambda \in \mathbb{C} : |\lambda - a| = a\}$.

Proof. Since $\sigma(\Delta_{a,b}, l_p)$ is the disjoint union of the parts $\sigma_p(\Delta_{a,b}, l_p)$, $\sigma_r(\Delta_{a,b}, l_p)$ and $\sigma_c(\Delta_{a,b}, l_p)$ we must have

$$\sigma_c(\Delta_{a,b}, l_p) = \{\lambda \in \mathbb{C} : |\lambda - a| = a\}. \; □$$

Combining Theorems 3.1, 3.2, 3.3, 3.4, 3.6 and 3.7, we can have the following main theorem:

Theorem 3.8. *The operator $\Delta_{a,b} : l_p \to l_p$ is a bounded linear operator and*

1. $\sup_k \left(|a_k|^p + |b_k|^p \right)^{\frac{1}{p}} \le \|\Delta_{a,b}\|_{l_p} \le \sup_k |a_k| + \sup_k |b_k|$.
2. $\sigma(\Delta_{a,b}, l_p) = D \cup E$.
3. $\sigma_p(\Delta_{a,b}, l_p) = E$.
4. $\sigma_p(\Delta^*_{a,b}, l^*_p) = \{\lambda \in \mathbb{C} : |\lambda - a| < a\} \cup E$.
5. $\sigma_r(\Delta_{a,b}, l_p) = \{\lambda \in \mathbb{C} : |\lambda - a| < a\}$.
6. $\sigma_c(\Delta_{a,b}, l_p) = \{\lambda \in \mathbb{C} : |\lambda - a| = a\}$.

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References

[1] A.M. Akhmedov and F. Başar, *On the fine spectra of the difference operator Δ over the sequence space l_p*, $(1 \le p < \infty)$, Demonstratio Math. **39** (3) (2006) 585–595.

[2] A.M. Akhmedov and F. Başar, *The fine spectra of the difference operator Δ over the sequence space bv_p*, $(1 \le p < \infty)$, Acta Math. Sin. (Engl. Ser.) **23** (10) (2007) 1757–1768.

[3] A.M. Akhmedov and S.R. El-Shabrawy, *On the fine spectrum of the operator $\Delta_{a,b}$ over the sequence space c*, Comput. Math. Appl. **61** (2011) 2994–3002.

[4] A.M. Akhmedov and S.R. El-Shabrawy, *On the spectrum of the generalized difference operator $\Delta_{a,b}$ over the sequence space c_0*, Baku Univ. News J., Phys. Math. Sci. Ser. **4** (2010) 12–21.

[5] A.M. Akhmedov and S.R. El-Shabrawy, *On the fine spectrum of the operator Δ_v over the sequence spaces c and l_p*, $(1 < p < \infty)$, Appl. Math. Inf. Sci. **5** (3) (2011) 635–654.

[6] A.M. Akhmedov and S.R. El-Shabrawy, *On the spectrum of the generalized lower triangle double-band matrix Δ_a over the sequence space c*, Al-Azhar Univ. Eng. J., JAUES (special issue), Dec. 2010, **5** (9) 54–63.

[7] A.M. Akhmedov and S.R. El-Shabrawy, *Notes on the spectrum of lower triangular double-band matrices*, Thai J. Math., to appear.

[8] A.M. Akhmedov and S.R. El-Shabrawy, *Some spectral properties of the generalized difference operator Δ_v*, European J. Pure Appl. Math., to appear.

[9] B. Altay and F. Başar, *On the fine spectrum of the difference operator Δ on c_0 and c*, Inform. Sci. **168** (2004) 217-224.

[10] B. Altay and F. Başar, *On the fine spectrum of the generalized difference operator $B(r,s)$ over the sequence spaces c_0 and c*, Int. J. Math. Math. Sci. **18** (2005) 3005-3013.

[11] B. Altay and M. Karakuş, *On the spectrum and the fine spectrum of the Zweier matrix as an operator on some sequence spaces*, Thai J. Math. **3** (2) (2005) 153-162.

[12] F. Başar and B. Altay, *On the space of sequences of p-bounded variation and related matrix mappings*, Ukrainian Math. J. **55** (1) (2003) 136-147.

[13] H. Bilgiç and H. Furkan, *On the fine spectrum of the generalized difference operator $B(r,s)$ over the sequence spaces l_p and bv_p*, $(1 < p < \infty)$, Nonlinear Anal. **68** (2008) 499-506.

[14] S.R. El-Shabrawy, *On the spectrum of the operator Δ_v over the space l_p*, $(1 < p < \infty)$, Baku Univ. News J., Phys. Math. Sci. Ser. **3** (2011) 55–64.

[15] H. Furkan, H. Bilgiç and B. Altay, *On the fine spectrum of the operator $B(r,s,t)$ over c_0 and c*, Comput. Math. Appl. **53** (2007) 989-998.

[16] H. Furkan, H. Bilgiç and F. Başar, *On the fine spectrum of the operator $B(r,s,t)$ over the sequence spaces l_p and bv_p*, $(1 < p < \infty)$, Comput. Math. Appl. **60** (2010) 2141-2152.

[17] B. de Malafosse, *Properties of some sets of sequences and application to the spaces of bounded difference sequences of order μ*, Hokkaido Math. J. **31** (2002) 283–299.

[18] B.L. Panigrahi and P.D. Srivastava, *Spectrum and fine spectrum of generalized second order difference operator Δ^2_{uv} on sequence space c_0*, Thai J. Math. **9** (1) (2011) 57–74.

[19] P.D. Srivastava and S. Kumar, *On the fine spectrum of the generalized difference operator Δ_v over the sequence space c_0*, Commun. Math. Anal. **6** (1)

(2009) 8-21.
[20] P.D. Srivastava and S. Kumar, *Fine spectrum of the generalized difference operator* Δ_v *on sequence space* l_1, Thai J. Math. **8** (2) (2010) 221–233.
[21] E. Kreyszig, *Introductory Functional Analysis with Applications*, John Wiley & Sons Inc., New York, Chichester, Brisbane, Toronto, 1978.
[22] B. Choudhary and S. Nanda, *Functional Analysis with Applications*, John Wiley & Sons Inc., New York, Chichester, Brisbane, Toronto, Singapore, 1989.
[23] I.J. Maddox, *Elements of Functional Analysis*, Cambridge Univ. Press, London, 1970.
[24] S. Goldberg, *Unbounded Linear Operators*, Dover Publications, Inc., New York, 1985.

Saad R. El-Shabrawy has obtained his B.Sc. and M.Sc. from Mansoura University (Damietta Branch), Faculty of Science, New Damietta, Egypt, in 2002 and 2007, respectively. He is working toward the Ph.D. at Baku State University, Mechanics and Mathematics Faculty, Baku, Azerbaijan. His research interests include Functional Analysis and Topology.