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On Linearization by Generalized Sundman Transformations of a Class of Liénard Type Equations and Its Generalization

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Abstract: We study the linearization of a class of Liénard type nonlinear second-order ordinary differential equations from the generalized Sundman transformation viewpoint. The linearizing generalized Sundman transformation for the class of equations is constructed. The transformation is used to map the underlying class of equations into a linear second-order ordinary differential equation which is not in the Laguerre form. The general solution of this class of equations is obtained by integrating the linearized equation and applying the generalized Sundman transformation. Moreover, we apply a Riccati transformation to a general linear third-order variable coefficients ordinary differential equation to extend the underlying class of equations and we also derive the conditions of linearizability of this new class of nonlinear second-order ordinary differential equations using the generalized Sundman transformation method and obtain its general solution.

Keywords: Linearization; generalized Sundman transformation; Riccati transformation; Liénard equation.

1 Introduction

We consider the class of nonlinear second-order ordinary differential equations (ODEs)

$$y'' + (b+3ky)y' + k^2y^3 + bky^2 + \lambda y = 0, \qquad (1.1)$$

where prime denotes differentiation with respect to the independent variable x with b, k, and λ are arbitrary constants. The equations in class (1.1) are known as the Liénard type ODEs and they are used as models of physical and other phenomena in several real world applications such as in nonlinear oscillations, the stability of gaseous spheres and so on.

A particular class of the equations (1.1), when $b = \lambda = 0$, was studied in [1–3] for the exact solutions by constructing a linearizing point transformation utilizing the symmetry properties of this class. For $b = 0$, the authors in [4] have obtained the general solution of (1.1) indirectly through the use of nonlocal transformations associated with the modified Prelle-Singer method given in [5]. In [6], Bluman et al. investigated the class of equations (1.1) and have obtained the general solution of (1.1) by mapping (1.1) to the free particle equation by an explicit complex point transformation by utilizing the symmetries of the determining equations leading to the transformation. The authors in [7], constructed linearizing Riccati and Lie point transformations to transform the underlying class of equations (1.1) into linear third- and second-order ODEs, respectively. The general solution of this class of equations is then obtained by integrating the linearized equations.

In this paper, we revisit the equation (1.1) from the viewpoint of linearization by generalized Sundman transformation and derive the general solution of this class of equations. We construct a linearizing generalized Sundman transformation for (1.1) utilizing the necessary and sufficient conditions for the linearizability of a general second-order ODE (see [8] and the references therein for detail account of this approach). Here we show that the Sundman transformation approach to linearization of (1.1) is much simpler than the Lie and Riccati methods used in [7], and yields a class of

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solutions which is different from those obtained in [7]. Furthermore, we make use of a Riccati transformation of a general linear third-order variable coefficients ODE to extend the underlying class of equations (1.1), and we also derive the conditions of linearizability of this new class of nonlinear second-order ODEs using the generalized Sundman transformation method and obtain its general solution. Again we show that the extended equation of (1.1) via the Riccati transformation is simpler in form in comparison to the extended equation in [7], and can readily be linearizable by the Sundman transformation to obtain the general solution.

The outline of the paper is as follows. Section 2 contains preliminaries on the generalized Sundman transformation approach to the linearizability of a general second-order ODE. In Section 3, we present the generalized Sundman transformation which linearizes (1.1) into a linear constant coefficients second-order ODE and the general solution of (1.1). We provide in Section 4, extension of the class of equations (1.1) and the conditions that determine linearizability of this new class of equations by the generalized Sundman transformation and the general solution of this class of equations. Finally, in Section 5, concluding remarks are made.

2 Preliminaries

Here we provide some main results based on the generalized Sundman transformation which are taken from [8, 9].

A non-point transformation which is known as the generalized Sundman transformation is defined by the equations

$$u(t) = F(x,y), \quad dt = G(x,y)\,dx, \quad F_y G \neq 0. \tag{2.1}$$

The necessary form of the second-order linearizable ODE which can be transformed into a linear ODE

$$u'' + \beta u' + \alpha u = \gamma, \tag{2.2}$$

via the transformation (2.1), is given by

$$y'' + \lambda_2(x,y)y'^2 + \lambda_1(x,y)y' + \lambda_0(x,y) = 0, \tag{2.3}$$

where in (2.2), prime means differentiation with respect to t, and α, β and γ are arbitrary constants.

The sufficient conditions for the equation (2.3) to be linearizable by (2.1) for the case $\lambda_3 \neq 0$ and $\lambda_5 \neq 0$, where

$\lambda_3 = \lambda_{1y} - 2\lambda_{2x}$

$\lambda_5 = \lambda_{2xx} + \lambda_{2x}\lambda_1 + \lambda_{3x} + \lambda_1\lambda_3$

are given by

$$\lambda_{0x} = 2\lambda_0(\lambda_5 - \lambda_1\lambda_3)/\lambda_3, \tag{2.4}$$

$$\lambda_{2xxy} = -\lambda_{2xy}\lambda_1 - \lambda_{3xy} - 2\lambda_{2x}^2 - 2\lambda_{2x}\lambda_3 - \lambda_{3y}\lambda_1 + (\lambda_{3y}\lambda_5)\lambda_3^{-1}, \tag{2.5}$$

$$\lambda_{2xxx} = -\lambda_{3xx} - \lambda_{1x}\lambda_{2x} - \lambda_{1x}\lambda_3 + \lambda_{2x}\lambda_1^2 + \lambda_1^2\lambda_3 - 2\lambda_1\lambda_5 + \lambda_3^{-1}\lambda_5(\lambda_{3x} + \lambda_5), \tag{2.6}$$

$$\lambda_3\lambda_5(6\lambda_{0y}\lambda_{2x} + 2\lambda_{2xy}\lambda_0 + 4\lambda_{2x}\lambda_0\lambda_2 + 2\lambda_{3y}\lambda_0 + 4\lambda_0\lambda_2\lambda_3 + \lambda_1\lambda_5) - \lambda_3^2(6\lambda_{2x}^2\lambda_0$$

$$+12\lambda_{2x}\lambda_0\lambda_3 - 6\lambda_{0y}\lambda_5 + 6\lambda_0\lambda_3^2) - \lambda_4\lambda_5^2 - 2\lambda_5^3 = 0, \tag{2.7}$$

where in (2.7)

$\lambda_4 = 2\lambda_{0yy} - 2\lambda_{1xy} + 2\lambda_0\lambda_{2y} - \lambda_{1y}\lambda_1 + 2\lambda_{0y}\lambda_2 + \lambda_{2xx}$.

In this case, the functions F and G are obtained by solving the following equations:

$$F_x = 0, \quad F_{yy} = (G_y F_y + \lambda_2 G F_y)/G, \tag{2.8}$$

$$G_x = G(\lambda_{2xx} + \lambda_{2x}\lambda_1 + \lambda_{3x})/\lambda_3, \tag{2.9}$$

$$G_y = G\lambda_3(\lambda_{2x} + \lambda_3)/\lambda_5. \tag{2.10}$$

The constants α, β and γ in (2.2), are determined from the following equations:

$$\alpha = [-G_y\lambda_0 + G(\lambda_{0y} + \lambda_0\lambda_2)]/G^3, \tag{2.11}$$

$$\beta = (G_x + G\lambda_1)/G^2, \tag{2.12}$$

$$\gamma = (-F_y\lambda_0 + \alpha F G^2)/G^2. \tag{2.13}$$

3 Generalized Sundman transformations and general solution of (1.1)

For the equation (1.1), we have

$\lambda_0 = k^2y^3 + bky^2 + \lambda y, \ \lambda_1 = b + 3ky, \ \lambda_2 = 0,$

$\lambda_3 = 3k, \ \lambda_4 = k(b + 3ky), \ \lambda_5 = 3k(b + 3ky).$

Since $k \neq 0$, it follows that $\lambda_3 \neq 0$ and $\lambda_5 \neq 0$. One can easily check that the equations (2.4)-(2.6) are satisfied. Now, the equation (2.7) is satisfied when the following condition holds, that is,

$$b(2b^2 - 9\lambda) = 0. \tag{3.1}$$

Two cases arise.

Case 1 ($b = 0$). In this case, the equation (1.1) takes the form

$$y'' + 3kyy' + k^2y^3 + \lambda y = 0. \tag{3.2}$$

Solving the equations (2.8)-(2.10), we obtain the following simplest of the solutions for the functions F and G, namely, $F = y^2/2$ and $G = y$. Therefore, the linearizing generalized Sundman transformation of (3.2) takes the form

$$u(t) = \frac{y^2}{2}, \quad dt = y\,dx. \tag{3.3}$$

Now the equations (2.11)-(2.13) yields

$$\alpha = 2k^2, \quad \beta = 3k, \quad \gamma = 0. \tag{3.4}$$

Hence the equation (3.2) is mapped by the transformation (3.3) into the linear equation

$$u'' + 3ku' + 2k^2u = 0. \tag{3.5}$$

The general solution of (3.5) is

$$u(t) = c_1e^{-kt} + c_2e^{-2kt},$$

where c_1 and c_2 are arbitrary constants. The application of the generalized Sundman transformation (3.3) to (3.2) yields the general solution of (3.2) given by

$$y(x) = \pm\sqrt{2c_1e^{-k\phi(x)} + 2c_2e^{-2k\phi(x)}},$$

where the function $t = \phi(x)$ is a solution of the equation

$$\frac{dt}{dx} = \pm\sqrt{2c_1e^{-kt} + 2c_2e^{-2kt}}.$$

Case 2 ($b \neq 0$ and $b = \pm 3\sqrt{\lambda/2}$). By solving the equations (2.8)-(2.10) for this case, we obtain the following simplest of the solutions for the functions F and G, that is, $F = by + (3k/2)y^2$ and $G = b + 3ky$. Hence the generalized Sundman transformation which linearizes (1.1) is of the form

$$u(t) = by + \frac{3k}{2}y^2, \;\; dt = (b + 3ky)\,dx. \tag{3.6}$$

From the equations (2.11)-(2.13) one gets

$$\alpha = 2/9, \;\; \beta = 1, \;\; \gamma = 0. \tag{3.7}$$

Therefore, the transformation (3.6) maps the equation (1.1) into the linear equation

$$9u'' + 9u' + 2u = 0. \tag{3.8}$$

The general solution of (3.8) is given by

$$u(t) = c_1e^{-1/3t} + c_2e^{-2/3t},$$

where c_1 and c_2 are arbitrary constants. Applying the generalized Sundman transformation (3.6) to (1.1), we obtain the following general solution of (1.1) given by

$$y(x) = \frac{-b \pm \sqrt{b^2 + 6kc_1e^{-1/3\phi(x)} + 6kc_2e^{-2/3\phi(x)}}}{3k},$$

where the function $t = \phi(x)$ is a solution of the equation

$$\frac{dt}{dx} = \pm\sqrt{b^2 + 6kc_1e^{-1/3t} + 6kc_2e^{-2/3t}}.$$

4 Generalization of the equation (1.1)

In this section, we extend the linearizable equation (1.1). We obtain a bigger class of nonlinear second-order ODEs than (1.1) which is linearizable by a Riccati transformation into a class of linear variable coefficient third-order ODEs. We also determine the conditions of linearizability of this new class of nonlinear second-order ODEs using the generalized Sundman transformation method and obtain its general solution.

Let

$$w''' + a(v)w'' + b(v)w' + c(v)w = 0 \tag{4.1}$$

be the general linearized form of the class of variable coefficient third-order ODEs. It can be readily shown that the Riccati transformation

$$x = v, \;\; y = \frac{w'}{w}, \tag{4.2}$$

reduces the equation (4.1) into the following nonlinear second-order ODE

$$y'' + [a(x) + 3y]y' + y^3 + a(x)y^2 + b(x)y + c(x) = 0. \tag{4.3}$$

Thus the extended form of (1.1) is the class of equations (4.3) which is linearizable by the Riccati transformation (4.2).

Remark 1: One can note that $k = 1$ in (4.3). Moreover, if $a(x) = b$, $b(x) = \lambda$ and $c(x) = 0$, then (4.3) reduces to the class of equations (1.1).

4.1 Linearization of (4.3) by generalized Sundman transformations and general solution

For the equation (4.3), we have

$$\lambda_0 = y^3 + a(x)y^2 + b(x)y + c(x), \; \lambda_1 = a(x) + 3y, \; \lambda_2 = 0,$$
$$\lambda_3 = 3, \; \lambda_4 = a(x) + 3y, \; \lambda_5 = 3[a(x) + 3y].$$

We note that $\lambda_3 \neq 0$ and $\lambda_5 \neq 0$. Moreover, the equations (2.5)-(2.6) are satisfied. Now, the equation (2.4) and (2.7) are satisfied when the following conditions,

$$a(x) = c_1, \; b(x) = c_2, \; c(x) = c_3, \tag{4.4}$$

$$2c_1^3 - 9c_1c_2 + 27c_3 = 0, \tag{4.5}$$

where c_1, c_2 and c_3 are arbitrary constants, are satisfied. Therefore, the equation (4.3) is linearizable by the generalized Sundman transformations if the conditions (4.4)-(4.5) are satisfied.

By solving the equations (2.8)-(2.10), we obtain the following simplest of the solutions for the functions F and G, that is, $F = c_1y + (3/2)y^2$ and $G = c_1 + 3y$. Hence the generalized Sundman transformation which linearizes (4.3) is of the form

$$u(t) = c_1y + \frac{3}{2}y^2, \;\; dt = (c_1 + 3y)\,dx. \tag{4.6}$$

From the equations (2.11)-(2.13) we obtain

$$\alpha = 2/9, \;\; \beta = 1, \;\; \gamma = 1/27\,(2c_1^2 - 9c_2). \tag{4.7}$$

Thus the transformation (4.6) maps the equation (4.3) into the linear equation

$$9u'' + 9u' + 2u = \frac{1}{3}(2c_1^2 - 9c_2). \tag{4.8}$$

The general solution of (4.8) is given by

$$u(t) = A_1 e^{-1/3t} + A_2 e^{-2/3t} + \frac{1}{6}(2c_1^2 - 9c_2),$$

where A_1 and A_2 are arbitrary constants. Applying the generalized Sundman transformation (4.6) to (4.3) we obtain the following general solution of (4.3) given by

$$y(x) = \frac{-c_1 \pm \sqrt{3c_1^2 - 9c_2 + 6A_1 e^{-1/3\phi(x)} + 6A_2 e^{-2/3\phi(x)}}}{3},$$

where the function $t = \phi(x)$ is a solution of the equation

$$\frac{dt}{dx} = \pm\sqrt{3c_1^2 - 9c_2 + 6A_1 e^{-1/3t} + 6A_2 e^{-2/3t}}.$$

5 Concluding remarks

We studied the three-parameter class of Liénard type nonlinear ODEs (1.1) from the viewpoint of straightforward linearization by the generalized Sundman transformation. We used the generalized Sundman transformation to transform the class of equations into a linear constant coefficients second-order ODE. By solving the linearized equation and making use of the generalized Sundman transformation we obtained the general solution of this class of equations. It is always a nontrivial task of finding a linearizing Riccati transformation and to construct a linearizing point transformation utilizing the Lie point symmetries of (1.1). Thus, we have shown that the procedure of linearization of (1.1) via the generalized Sundman transformation is simpler in comparison to Lie and Riccati transformations approaches given in [7] and results in obtaining a distinct class of solutions for (1.1). Moreover, we have extended the underlying equation (1.1) by the application of a Riccati transformation to a general linear third-order variable coefficients ODE and determined the conditions of linearizability of this new class of nonlinear second-order ODEs using the generalized Sundman transformation approach. It is shown that the extended equation (4.3) is linearizable by the generalized Sundman transformation if a, b and c in (4.1) are constants and also the condition (4.5) is satisfied. Notwithstanding, unlike in the work of [7], we have shown how one can utilize the generalized Sundman transformation to find the general solution of the extended equation. We note that the application of the generalized Sundman transformation for linearization of the Linéard system and its extension has not been reported previously in literature.

Further to the results presented in this paper, as future lines of research, it will be of interest to look for the general Sundman transformations of extended classes of equations of the form (1.1) as well as to compare these with the Lie and other approaches to linearization. Moreover, it will be of benefit to consider such types of transformations for other equations from applications.

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