Applied Mathematics & Information Sciences **3**(2) (2009), 159–176
– An International Journal

Final Outcome of an Epidemic in Two Interacting Populations

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Received March 28, 2008; Revised August 9, 2008

we consider a stochastic model for the spread of an epidemic in a closed population consisting of two groups, in which infectives cannot change their group, but are able to infect outside it. Using the matrix-geometric method we obtain a recursive relationship for the Laplace transform of the joint distribution of the number of susceptibles and infectives in the two groups. We also derive the distribution of the total observed size of the epidemic as well as its duration in the case of a general infection mechanism.

Keywords: Epidemic model, matrix-geometric, final size, duration of the epidemic, number of new cases.

1 Introduction

We consider a stochastic model for an epidemic taking place in a heterogeneous population consisting of two groups. The infection can be transmitted both within and between the groups. From the standpoint of the infection mechanism our model is a special generalization of a model considered by Gani and Yakowitz [11] in the case of a closed population. Similar models have also been studied by Bailey [2, Chapter 11] and O'Neill [16], who derived a class of results for the probability of ultimate extinction. Here we use a matrix-geometric method (cf. Neuts [14]) similar to that of Booth [7] to obtain the distribution of the total number of infections that occur in the entire population. The use of the matrix-geometric method in the study of epidemics was pioneered by Gani and Purdue [10].

The first author acknowledge the financial support of the Swedish Institute.

The paper is structured as follows. We describe the model in section 2. In Section 3 we account for the matrix-geometric method and in Section 4 we show a recursive relationship for the Laplace transform of the joint distribution of some quantities of interest. The distribution of total size is discussed in Section 5 while Section 6 is devoted to the distribution of the duration of the epidemic as well as the expected number of new cases in the two groups. Finally in Section 7 we present a simple numerical example. Some of the derivations call for tedious algebraic manipulations that are presented in the Appendix.

2 The Model

In what follows we consider a model for the spread of an epidemic in a closed population consisting of two groups of individuals G_1 and G_2. The following notation is used throughout the article. $X_i(t)$ and $Y_i(t)$ stand for the numbers of susceptibles and infectives at time t for the ith group with $(X_1(0), X_2(0), Y_1(0), Y_2(0)) = (n_1, n_2, a_1, a_2)$. In each group the rate of infection is related to the number of susceptibles and infectives in the two groups. Infective in group G_i, $i = 1, 2$, are removed at rate $\mu_i \geq 0$ so that the epidemic process is completely determined by $\{(X_1(t), X_2(t), Y_1(t), Y_2(t)); t \geq 0\}$. This process is supposed to be a continuous-time Markov chain on the state space

$$S = \{(x, y, u, v); 0 \leq x \leq n_1,\ 0 \leq y \leq n_2,\ 0 \leq u \leq N_1^x,\ 0 \leq v \leq N_2^y\},$$

where $N_1^x = n_1 + a_1 - x$ and $N_2^y = n_2 + a_2 - y$, with the following transitions and associated probabilities for a time increment $(t, t+h)$

Transition	Probability
$(X_1, X_2, Y_1, Y_2) \to (X_1 - 1, X_2, Y_1 + 1, Y_2)$	$f_{X_1X_2Y_1Y_2, X_1-1X_2Y_1+1Y_2} h + o(h)$
$(X_1, X_2, Y_1, Y_2) \to (X_1, X_2 - 1, Y_1, Y_2 + 1)$	$f_{X_1X_2Y_1Y_2, X_1X_2-1Y_1Y_2+1} h + o(h)$
$(X_1, X_2, Y_1, Y_2) \to (X_1, X_2, Y_1 - 1, Y_2)$	$\mu_1 Y_1 h + o(h)$
$(X_1, X_2, Y_1, Y_2) \to (X_1, X_2, Y_1, Y_2 - 1)$	$\mu_2 Y_2 h + o(h)$
No change	$1 + f_{X_1X_2Y_1Y_2} h + o(h)$

where

$$f_{X_1X_2Y_1Y_2} = -(f_{X_1X_2Y_1Y_2, X_1-1X_2Y_1+1Y_2} + f_{X_1X_2Y_1Y_2, X_1X_2-1Y_1Y_2+1} + \mu_1 Y_1 + \mu_2 Y_2)$$

with conventions that $f_{ijlr, i'j'l'r'} = 0$ if $(i, j, l, r) \notin S$ or $(i', j', l', r') \notin S$, and $f_{ij00, i-1j10} = f_{ij00, ij-101} = 0$. When defining these rates we have tried to use a quite general infection mechanism. Due to technical reasons we were not able to allow the same level of generality for the removal rates. Let

$$P_{ijlr}(t) = P(X_1(t) = i, X_2(t) = j, Y_1(t) = l, Y_2(t) = r) \text{ for } t \geq 0.$$

Then the forward Kolmogorov equations take the form

$$\frac{\partial P_{ijlr}(t)}{\partial t} = f_{ijlr}P_{ijlr}(t) + \mu_1(l+1)P_{ijl+1r}(t) + \mu_2(r+1)P_{ijlr+1}(t) + f_{i+1jl-1r,ijlr}P_{i+1jl-1r}(t) + f_{ij+1lr-1,ijlr}P_{ij+1lr-1}(t), \quad (2.1)$$

with the conventions that $P_{ijlr}(t) \equiv 0$ if $(i,j,l,r) \notin S$ and $P_{n_1n_2a_1a_2}(0) = 1$.

3 The Matrix-Geometric Method

For the type of model in which we are interested, the standard probability generating function methods are ineffective, as was shown by Bailey [2, Chapter 11]. However, the Kolmogorov equations can be solved recursively using the matrix-geometric method.

For $i = 0,\ldots,n_1$, $j = 0,\ldots,n_2$ and $l = 0,\ldots,N_1^i$, let A_{ij}^l, B_{ij}^{l+1}, D_{i+1j}^{l-1} and H_{ij}^l be respectively the diagonal matrices with rth diagonal element equal to $f_{ijlr}, \mu_1(l+1), f_{i+1jl-1r,ijlr}$ and $f_{ij+1lr-1,ijlr}$, $r = 0,\ldots,N_2^j$, and let C_{ij}^l be the matrix of the same dimension with the $(r, r+1)$-th entries equal to $\mu_2(r+1)$, $r = 0,\ldots,N_2^j - 1$, and all others equal to 0. In addition we take

$$P_{ij}^l(t) = \left(P_{ijl0}(t), P_{ijl1}(t), \ldots P_{ijlN_2^j-1}(t), P_{ijlN_2^j}(t)\right)^T.$$

Equations (2.1) take now the form

$$\frac{\partial P_{ij}^l(t)}{\partial t} = A_{ij}^l P_{ij}^l(t) + C_{ij}^l P^l ij(t) + D_{i+1j}^{l-1} P_{i+1j}^{l-1}(t) + H_{ij}^l P_{ij+1}^{*l}(t) + B_{ij}^{l+1} P_{ij}^{l+1}(t), \quad (3.1)$$

where $P_{ij}^{*l}(t) = (0, (P_{ij}^l(t))^T)^T$. Furthermore we introduce the column vectors

$$P_{ij}(t) = \left((P_{ij}^0(t))^T, \ldots, (P_{ij}^l(t))^T, \ldots, (P_{ij}^{N_1^i}(t))^T\right)^T,$$

the block matrices

$$D_{i+1j} = \mathrm{diag}(D_{i+1j}^l, 0 \le l \le N_1^{i+1}), \quad H_{ij} = \mathrm{diag}(H_{ij}^l, 0 \le l \le N_1^i)$$

and a matrix F_{ij} whose (l,l)-th block equals

$$A_{ij}^l + C_{ij}^l,\ l = 0,\ldots,N_1^i,$$

$(l, l+1)$-th block is equal to

$$B_{ij}^{l+1},\ l = 0,\ldots,N_1^{i+1},$$

and all other blocks are equal zero.

For each matrix A of order $(N_1^{i+p}+1)(N_2^{j+q}+1)$, $0 \leq p \leq n_1-i$ and $0 \leq q \leq n_2-j$, we define an augmented matrix

$$A(p,q) = \begin{pmatrix} \Theta_{ij}^{pq} & 0 \\ 0 & A \end{pmatrix},$$

where Θ_{ij}^{pq} is the zero matrix of order $q(N_1^i+1)+p(N_2^j+1)-pq$, and for each vector $U(t)$ of dimension $(N_1^{i+p}+1)(N_2^{j+q}+1)$ we also define

$$U(t,p,q) = \left((\theta_{ij}^{pq})^T, U^T(t)\right)^T,$$

where θ_{ij}^{pq} is the $q(N_1^i+1)+p(N_2^j+1)-pq$ zero column vector and

$$P_{ij+1}^*(t) = \left((P_{ij+1}^{*0}(t))^T, \ldots, (P_{ij+1}^{*N_1^i}(t))^T\right)^T.$$

With above notations equation (3.1) leads to

$$\frac{\partial P_{ij}(t)}{\partial t} = F_{ij}P_{ij}(t) + D_{i+1j}(1,0)P_{i+1j}(t,1,0) + H_{ij}P_{ij+1}^*(t). \tag{3.2}$$

To obtain an appropriate form for the above equations which can help us to solve (2.1) we investigate the possible relationship between $P_{ij}^*(t)$ and $P_{ij}(t,0,1)$. For this let T_{ij} be the matrix of rank $(N_1^i+1)(N_2^j+1)$, where

$$(T_{ij})_{mn} = \begin{cases} 1 & \text{if } m = r(N_2^j+1)+k \text{ and } n = N_1^i + rN_2^j + k, \\ & \text{with } 0 \leq r \leq N_j^1, 1 \leq k \leq N_2^j \\ 0 & \text{otherwise.} \end{cases}$$

By rearrangement we have $P_{ij+1}^*(t) = T_{ij}P_{ij+1}(t,0,1)$ and by substitution into (3.2) we obtain

$$\frac{\partial P_{ij}(t)}{\partial t} = F_{ij}P_{ij}(t) + D_{i+1j}(1,0)P_{i+1j}(t,1,0) + H_{ij}T_{ij}P_{ij+1}(t,0,1) \tag{3.3}$$

for $0 \leq i \leq n_1$ and $0 \leq j \leq n_2$.

The limiting distribution of the process can now be studied using Laplace transforms

$$\hat{P}_{ij}(v) = \int_0^{+\infty} e^{-vt} P_{ij}(t)\, dt.$$

Equation (3.3) becomes

$$\hat{P}_{n_1n_2}(v) = (vI_{n_1n_2} - F_{n_1n_2})^{-1}E \tag{3.4}$$

and

$$\hat{P}_{ij}(v) = (vI_{ij} - F_{ij})^{-1}D_{i+1j}(1,0)\hat{P}_{i+1j}(v,1,0) + (vI_{ij} - F_{ij})^{-1}H_{ij}T_{ij}\hat{P}_{ij+1}(v,0,1) \tag{3.5}$$

for $0 \leq i \leq n_1, 0 \leq j \leq n_2$, and $i+j \neq n_1+n_2$, where $E = P_{n_1n_2}(0) = (0,\ldots,0,1)^T$ and I_{ij} denotes the identity matrix of order $(N_1^i+1)(N_2^j+1)$.

4 The Solution

First we determine the Laplace transforms of the probabilities $P_{ijlr}(t)$. It can be shown that $F_{ij}(v) = (vI_{ij} - F_{ij})^{-1}$ has the form

$$F_{ij}(v) = \begin{pmatrix} F_{ij}^{00}(v) & F_{ij}^{01}(v) & . & . & . & F_{ij}^{0h}(v) & . & F_{ij}^{0N_1^i}(v) \\ 0 & F_{ij}^{11}(v) & . & . & . & . & . & . \\ 0 & 0 & . & . & . & . & . & . \\ . & . & . & F_{ij}^{ll}(v) & . & F_{ij}^{lh}(v) & . & F_{ij}^{lN_1^i}(v) \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ 0 & . & . & . & . & . & 0 & F_{ij}^{N_1^i N_1^i}(v) \end{pmatrix},$$

where for $0 \leq l \leq h \leq N_1^i$, $F_{ij}^{lh}(v)$ is a block of rank $N_2^j + 1$. Moreover it can be verified (cf. the Appendix) that

$$[F_{ij}^{lh}(v)]_{rs} = \begin{cases} C_{ij}(v, l, h, r, s) & \text{if } 0 \leq r \leq s \leq N_2^j \\ 0 & \text{otherwise,} \end{cases} \tag{4.1}$$

where

$$C_{ij}(v, l, h, r, s) = \mu_1^{h-l} \mu_2^{s-r} \frac{h!s!}{l!r!} \sum_{I \in D_{rs}^{h-l}} \prod_{k=0}^{h-l} \prod_{q=i_k}^{i_{k+1}} f(v, \mu_1, \mu_2, i, j, l+k, q) \tag{4.2}$$

and $f(v, \mu_1, \mu_2, i, j, l, r) = (v + \mu_1 l + \mu_2 r + f_{ijlr,i-1jl+1r} + f_{ijlr,ij-1lr+1})^{-1}$, $i_0 = r$, $i_{h-l+1} = s$ and

$$D_{rs}^{h-l} = \begin{cases} \{(i_1, i_2, \ldots, i_{h-l} \leq s) \ / \ r \leq i_1 \leq i_2 \leq \ldots \leq i_{h-l} \leq s\} & \text{if } l < h \\ \emptyset & \text{if } l = h \end{cases} \tag{4.3}$$

with the conventions that

$$\prod_{p \in B} A_p = 1 \ \text{ and } \ \sum_B 1 = 1 \text{ if } B = \emptyset \ \text{ and } \ A_p > 0. \tag{4.4}$$

The quantities $C_{ij}(v, l, h, r, s)$ can be calculated (cf. the Appendix) using the following recursive relationship for $0 \leq l \leq h \leq N_1^i$:

$$C_{ij}(v, l, l, r, s) = \mu_2^{s-r} \frac{s!}{r!} \prod_{q=r}^{s} f(v, \mu_1, \mu_2, i, j, l, q) \tag{4.5}$$

and

$$C_{ij}(v, l, h, r, s) = \mu_1 h \sum_{p=r}^{s} \mu_2^{s-p} \frac{s!}{p!} \prod_{q=p}^{s} f(v, \mu_1, \mu_2, i, j, h, q) C_{ij}(v, l, h-1, p, s). \tag{4.6}$$

For $m, n = 0, \ldots, (N_1^i + 1)(N_2^j + 1) - 1$, let l, h and r, s be, respectively, the quotients and remainders of the division of m and n by $N_2^j + 1$. We have (cf. the Appendix)

$$[F_{ij}(v)D_{i+1j}(1,0)]_{mn} = \begin{cases} C_{ij}(v,l,h,r,s)\, f_{i+1jh-1s,ijhs} & \text{if } r \leq s \text{ and } l \leq h,\ h \geq 1 \\ 0 & \text{otherwise.} \end{cases} \tag{4.7}$$

Similarly, if l, h and $r, s-1$ are, respectively, the quotients and remainders of the division of m by $N_2^j + 1$ and $n - N_1^i - 1$ by N_2^j, then

$$[F_{ij}(v)H_{ij}T_{ij}]_{mn} = \begin{cases} C_{ij}(v,l,h,r,s)\, f_{ij+1hs-1,ijhs} & \text{if } r \leq s,\ s \geq 1 \text{ and } l \leq h \\ 0 & \text{otherwise.} \end{cases} \tag{4.8}$$

Since $\hat{P}_{ijlr}(v)$ and $\hat{P}_{i+1jl-1s}(v)$ correspond, respectively, to the $(l(N_2^j+1)+r)$-th elements of the vectors $\hat{P}_{ij}(v)$ and $\hat{P}_{i+1j}(v,1,0)$ while $\hat{P}_{ij+1lr-1}(v)$ correspond to the $(N_1^i + lN_2^j + r)$-th element of $\hat{P}_{i+1j}(v,0,1)$, then using (3.4)-(4.1) and the previous result we find

$$\hat{P}_{n_1n_2lr}(v) = C_{n_1n_2}(v,l,a_1,r,a_2), \tag{4.9}$$

$$\begin{cases} \hat{P}_{in_2lr}(v) = \displaystyle\sum_{\substack{l\leq h\leq N_1^i, h\geq 1 \\ r\leq s\leq a_2}} C_{in_2}(v,l,h,r,s)\, f_{i+1n_2h-1s,in_2hs}\hat{P}_{i+1n_2h-1s}(v) \\ \hat{P}_{n_1jlr}(v) = \displaystyle\sum_{\substack{l\leq h\leq a_1 \\ r\leq s\leq N_2^j, s\geq 1}} C_{n_1j}(v,l,h,r,s)\, f_{n_1j+1hs-1,n_1jhs}\hat{P}_{n_1j+1h-1s}(v) \end{cases} \tag{4.10}$$

for $i = 0, \ldots, n_1 - 1$ and $j = 0, \ldots, n_2 - 1$, and

$$\begin{aligned} \hat{P}_{in_2-jlr}(v) = & \sum_{\substack{l\leq h\leq N_1^i, h\geq 1 \\ r\leq s\leq a_2+j}} C_{in_2-j}(v,l,h,r,s)\, f_{i+1n_2-jh-1s,in_2-jhs}\hat{P}_{i+1n_2-jh-1s}(v) \\ & + \sum_{\substack{l\leq h\leq N_1^i \\ r\leq s\leq a_2+j, s\geq 1}} C_{in_2-j}(v,l,h,r,s)\, f_{in_2-j+1hs-1,in_2-jhs}\hat{P}_{in_2-j+1hs-1}(v) \end{aligned} \tag{4.12}$$

for $i = 0, \ldots, n_1 - 1$ and $j = 1, \ldots, n_2$. From $(4.9) - (4.11)$ we conclude that the Laplace transforms can be solved recursively.

5 The Total Size

The asymptotic behaviour of the process $\{(X_1(t), X_2(t), Y_1(t), Y_2(t)); t \geq 0\}$ can be described using (4.8)–(4.11), (4.1) and the identity $\lim_{t\to\infty} P_{ijlr}(t) = \lim_{v\to 0}(v\hat{P}_{ijlr}(v))$. The epidemic ends as soon as the numbers of infectives in both groups become zero. Let π_{ij} denote the probability that exactly i and j of initially susceptible individuals ultimately escape the epidemic in G_1 and G_2, respectively. In order to determine this probability, it is necessary to calculate the limit, $\lim_{v\to 0}(vC_{ij}(v,0,h,0,s)) = C_{ij}(h,s)$. We show (cf. the appendix) that such a limit exists and is

$$C_{ij}(0,s) = \mu_2^s s! \prod_{q=1}^{s} f(\mu_1,\mu_2,i,j,0,q) \tag{5.1}$$

for $h = 0$ and

$$C_{ij}(h,s) = \mu_1^h \mu_2^s h! s! \sum_{0\le p\le s} \left\{ \prod_{q=1}^{p} f(\mu_1,\mu_2,i,j,0,q) \sum_{I_p \in B_{ps}^{h-1}} \prod_{k=0}^{h-1} \prod_{q=i_{p_k}}^{i_{p_{k+1}}} f(\mu_1,\mu_2,i,j,k,q) \right\} \tag{5.2}$$

for $h > 0$, where

$$B_{ps}^{h-1} = \begin{cases} \{(i_{p_1},\ldots,i_{p_{(h-1)}}) \,/\, p \le i_{p_1} \le \cdots \le i_{p_{(h-1)}} \le s\} & \text{if } h > 1 \\ \emptyset & \text{if } h = 1, \end{cases}$$

$f(\mu_1,\mu_2,i,j,l,r) = (\mu_1 l + \mu_2 r + f_{ijlr,i-1jl+1r} + f_{ijlr,ij-1lr+1})^{-1}$, $i_{p_0} = p$ and $i_{p_h} = s$. Finally (4.9) implies that

$$\pi_{n_1 n_2} = \lim_{v\to 0} v\hat{P}_{n_1 n_2 00}(v) = C_{n_1,n_2}(a_1,a_2).$$

Similarly from (4.10) and (4.11), respectively, it can be shown that for $i = 0,\ldots,n_1-1$ and $j = 0,\ldots,n_2-1$

$$\begin{aligned}
\pi_{n_1 j} &= \sum_{h=1}^{a_1}\sum_{s=1}^{N_2^j} C_{n_1 j}(h,s) f_{n_1 j+1hs-1,n_1 jhs} \hat{P}_{n_1 j+1hs-1}(0) \\
&\quad + \sum_{s=2}^{N_2^j} C_{n_1 j}(0,s) f_{n_1 j+10s-1,n_1 j0s} \hat{P}_{n_1 j+10s-1}(0), \\
\pi_{in_2} &= \sum_{s=1}^{a_2}\sum_{h=1}^{N_1^i} C_{in_2}(h,s) f_{i+1n_2h-1s,in_2hs} \hat{P}_{i+1n_2h-1s}(0) \\
&\quad + \sum_{h=2}^{N_1^i} C_{in_2}(h,0) f_{i+1n_2h-10,in_2h0} \hat{P}_{i+1n_2h-10}(0)
\end{aligned}$$

and

$$\begin{aligned}
\pi_{ij} &= \sum_{h=1}^{N_1^i}\sum_{s=1}^{N_2^j} C_{ij}(h,s)[f_{ij+1hs-1,ijhs}\hat{P}_{ij+1hs-1}(0) + f_{i+1jh-1s,ijhs}\hat{P}_{i+1jh-1s}(0)] \\
&\quad + \sum_{s=2}^{N_2^j} C_{ij}(0,s) f_{ij+10s-1,ij0s}\hat{P}_{ij+10s-1}(0) + \sum_{h=2}^{N_1^i} C_{ij}(h,0) f_{i+1jh-10,ijh0}\hat{P}_{ij+1h0}(0).
\end{aligned}$$

These probabilities can be determined using (5.1) and (5.2) and by means of the recursive equations (4.9)–(4.11).

6 Duration of the Epidemic and Number of Cases

Let

$$T_{n_1n_2a_1a_2} = \inf\{t \geq 0 : Y_2(t) = Y_1(t) = 0\}$$

be the duration of the epidemic defined as the duration of the time between the start of the epidemic and the moment at which the number of infectives becomes zero. If we suppose that $E(T_{n_1n_2a_1a_2}) < +\infty$, from (15)–(17) we can calculate the mean duration easily using the following fact:

$$\mathbf{E}(T_{n_1n_2a_1a_2}) = \int_0^\infty \text{Pr}(T_{n_1n_2a_1a_2} > t)dt.$$

Then

$$\mathbf{E}(T_{n_1n_2a_1a_2}) = -\frac{d\,(v\psi(v))}{dv}\;|_{v=0},$$

where

$$\psi(v) = \int_0^\infty e^{-tv}\text{Pr}(T_{n_1n_2a_1a_2} \leq t)dt = \sum_{\substack{0\leq i\leq n_1\\ 0\leq j\leq n_2}} \hat{P}_{ij00}(v).$$

Furthermore we can use the numerical inversion algorithm proved by Abate and Whitt [1] for Laplace transforms of the probabilities $P_{ijlr}(t)$ for $t > 0$ to derive the cumulative distribution of $T_{n_1n_2a_1a_2}$ as well as the expectation of new cases in each of the groups $n_1 - X_1(t)$ and $n_2 - X_2(t)$ using respectively the following expressions:

$$\text{Pr}(T_{n_1n_2a_1a_2} \leq t) = \sum_{\substack{0\leq i\leq n_1\\ 0\leq j\leq n_2}} P_{ij00}(t),$$

$$\mathbf{E}(X_1(t)) = \sum_{\substack{0\leq i\leq n_1\\ 0\leq j\leq n_2}} \sum_{\substack{0\leq l\leq n_1+a1-i\\ 0\leq r\leq n_2+a2-j}} iP_{ijlr}(t)$$

and

$$\mathbf{E}(X_2(t)) = \sum_{\substack{0\leq i\leq n_1\\ 0\leq j\leq n_2}} \sum_{\substack{0\leq l\leq n_1+a1-i\\ 0\leq r\leq n_2+a2-j}} jP_{ijlr}(t).$$

7 An Example and Remarks

The vast majority of papers on stochastic epidemical models with two groups consider a general model (see Daley and Gani [8] and Gani and Yakowitz [11]) in which the infections in the first and second group occur respectively at rates $X_1(\beta_{11}Y_1+\beta_{12}Y_2)$ and $X_2(\beta_{21}Y_1+\beta_{22}Y_2)$, where for $r, s = 1, 2$, β_{rs} is the pairwise rate for a susceptible from group r to be infected by an infective in group s. This model is of limited direct use in modelling fatal diseases such AIDS for which the infection mechanism is more complex and removal of an infective result in its death. Hence in a single population epidemic, if there are X susceptibles and Y infectives at a given time, then the probability that an individual chosen uniformly at random from the population is susceptible is given by $X=(X+Y)$, leading to overall infection rate of $XY=(X+Y)$ (see, e.g., [3]). For the heterogeneous population with two groups, the probability that a susceptible individual is chosen randomly from the group $r=1;2$ is given by $X_r(X_r+Y_r)^{1}$. Thus it would be reasonable if the standard infection rate is replaced by $X_1(X_1+Y_1)^{1}(\ _{11}Y_1+\ _{12}Y_2)$ and $X_2(X_2+Y_2)^{1}(\ _{21}Y_1+\ _{22}Y_2)$; where the parameter $_{ij}$ is defined as in Hyman and others [12] and Sani and others [18], with slight generalization, as the product of the contact rate and the probability $_{rs}$ that the successive number of contacts between a susceptible in group r and infective in group s lead to infection with $_{r1}+\ _{r2}=1$. Using the methods presented in this paper it is straightforward to obtain numerical results.

Figure 7.1: Joint (left picture) and marginal (right picture) distribution of the final sizes for $_{11}=0{:}4$, $_{12}=0{:}3$, $_{21}=4$ and $_{22}=2$

Figures 7.1-7.3 illustrate some results using the initial conditions $_1=\ _2=1$; $n_1=n_2=100$ and $a_1=0$ and $a_2=1$: The Figures in the left have different $(\ _{11};\ _{12};\ _{21};\ _{22})$ values, illustrating the simultaneous distribution of the total sizes in the two populations while the Figures in the right concern the same cases as the Figures in the left and illustrate the marginal distribution of the total sizes in group 1 ( ) and group 2 ( ).

For Figure 7.1 we note that $_{11}+\ _{21}\ \ _{22}+\ _{12}\ \ 1$. This implies that the first group acts as an important source of infection for the population as a whole, but that susceptibles in this group have few contacts with infectives in both groups $(\ _{12}\ \ 1;\ _{11}\ \ 1)$ so that infections transmitted to group 1, whether from 1 or 2, tend to die out quickly. This is, however, compensated since the parameters of the second group are above the threshold. On the other hand for Figure 7.2 we have $_{22}\ \ 1$, $_{12}\ \ 1$, $_{22}+\ _{12}\ \ 1$ and $_{11}+\ _{21}\ \ 1$ so the parameters of the first group are below the threshold while the parameters of the second group are above it and therefore the major part of the probability is concentrated between $X_2(1\ )=0$ and $X_2(1\ )=100$, illustrating the fact that the first group is relatively inactive, whereas the epidemic is major in the second group with high activity. In the case of Figure 7.3 all parameters have low values so the epidemic as a whole dies out

where I_j denotes the identity matrix of rank $N_2^j + 1$.

By using the matrices defined in Section 2 we take

$$B_{ij} = \text{diag}(B_{ij}^l, 0 \le l \le N_1^i)$$

and

$$Z_{ij} = \text{diag}(Z_{ij}^l, 0 \le l \le N_1^i),$$

where

$$Z_{ij}^l = \overline{C}_{ij}^l + D_{ij}^l - C_{ij}^l + \overline{H}_{ij}^l = (I_j - \triangle_j)\overline{C}_{ij}^l + D_{ij}^l + \overline{H}_{ij}^l$$

and the last equation is true because $C_{ij}^l = \triangle_j \overline{C}_{ij}^l$.

Since $A_{ij}^l = -B_{ij}^l - \overline{H}_{ij}^l - \overline{C}_{ij}^l - D_{ij}^l$, then $vI_{ij} - F_{ij} = vI_{ij} + Z_{ij} + B_{ij} - \triangle_{ij} B_{ij}$ and it follows that

$$\begin{aligned} F_{ij}(v) &= [vI_{ij} + Z_{ij} + B_{ij} - \triangle_{ij} B_{ij}]^{-1} \\ &= [(vI_{ij} + Z_{ij} + B_{ij})(I_{ij} - (vI_{ij} + Z_{ij} + B_{ij})^{-1}) \triangle_{ij} B_{ij}]^{-1} \\ &= [I_{ij} - (vI_{ij} + Z_{ij} + B_{ij})^{-1} \triangle_{ij} B_{ij}]^{-1} (vI_{ij} + Z_{ij} + B_{ij})^{-1}. \end{aligned} \tag{A.1}$$

The off-diagonal form of $\triangle_{ij}$ and the upper triangular form of $M_{ij}(v) = (tIij + Z_{ij} + B_{ij})^{-1}$ imply that $(M_{ij}(v) \triangle_{ij} B_{ij})^l \equiv 0$ for all integers $l > N_1^i$. Hence

$$[I_{ij} - M_{ij}(v) \triangle_{ij} B_{ij}]^{-1} = \sum_{l=0}^{N_1^i} [M_{ij}(v) \triangle_{ij} B_{ij}]^l = R_{ij}(v).$$

Let R_{ij}^{lh} and $M_{ij}^{lh}(v)$ be, respectively, the (l,h)th blocks of the matrices $R_{ij}(v)$ and $M_{ij}(v)$ of ranks $N_2^j + 1$. Since for $k = 0, \ldots, N_1^i$ the (l,h)th block of $[M_{ij}(v) \triangle_{ij} B_{ij}]^k$ is equal to

$$M_{ij}^{ll}(v) B_{ij}^{l+1} M_{ij}^{l+1l+1}(v) B_{ij}^{l+2} \ldots M_{ij}^{l+k-1l+k-1}(v) B_{ij}^{l+k} \text{ if } h = l + k \text{ and } 0 \text{ otherwise,}$$

then for $0 \le l \le h \le N_1^i$

$$R_{ij}^{lh}(v) = \prod_{k=l}^{h-1} M_{ij}^{kk}(v) B_{ij}^{k+1} = \prod_{k=l}^{h-1} (M_{ij}(v) \triangle_{ij} B_{ij})^{kk+1}. \tag{A.2}$$

The diagonal form by blocks of Z_{ij} implies that $M_{ij}^{lh}(v) = 0$ if $l \ne h$. Thus for each $l, h = 0, \ldots, N_1^i$

$$F_{ij}^{lh}(v) = \sum_{k=0}^{N_1^i} R_{ij}^{lk}(v) M_{ij}^{kh}(v) = \begin{cases} R_{ij}^{lh}(v) M_{ij}^{hh}(v) & \text{if } l \le h \\ 0 & \text{otherwise.} \end{cases} \tag{A.3}$$

Now for $l = 0, \ldots, N_1^i$ we have

$$M_{ij}^{ll}(v) = [(vI_{ij} + Z_{ij} + B_{ij})^{-1}]^{ll} = (vI_{ij}^{ll} + Z_{ij}^l + B_{ij}^l)^{-1}$$

$$= [vI_j + \overline{C}^l_{ij} + D^l_{ij} + B^l_{ij} + \overline{H}^l_{ij} - \triangle_j \overline{C}^l_{ij}]^{-1}$$
$$= [I_j - Y^l_{ij}(v)]^{-1}(vI_j + \overline{C}^l_{ij} + D^l_{ij} + B^l_{ij} + \overline{H}^l_{ij})^{-1},$$

where $Y^l_{ij}(v) = (vI_j + \overline{C}^l_{ij} + D^l_{ij} + B^l_{ij} + \overline{H}^l_{ij})^{-1} \triangle_j \overline{C}^l_{ij}$. However,

$$vI_j + \overline{C}^l_{ij} + D^l_{ij} + B^l_{ij} + \overline{H}^l_{ij} = \mathrm{diag}(v+\mu_2 r+\mu_1 l+f_{ijlr,i-1jl+1r}+f_{ijlr,ij-1lr+1}, 0 \le r \le N_2^j),$$

hence

$$(vI_j + \overline{C}^l_{ij} + D^l_{ij} + B^l_{ij} + \overline{H}^l_{ij})^{-1}$$
$$= \mathrm{diag}([v + \mu_2 r + \mu_1 l + f_{ijlr,i-1jl+1r} + f_{ijlr,ij-1lr+1}]^{-1}, 0 \le r \le N_2^j)$$

The off-diagonal form of $\triangle_j \overline{C}^l_{ij}$ implies that $[Y^l_{ij}(v)]^r \equiv 0$ for all integers $r > N_2^j$. Hence using the same technique as above we obtain for $r \le s$

$$[I_j - Y^l_{ij}(v)]^{-1}_{rs} = \mu_2^{s-r} \frac{s!}{r!} \prod_{q=r}^{s-1} (v + \mu_2 q + \mu_1 l + f_{ijlq,i-1jl+1q} + f_{ijlq,ij-1lq+1})^{-1}$$

with all other elements being equal to zero. It follows that
(A.4)

$$[M^{ll}_{ij}]_{rs} = \begin{cases} \mu_2^{s-r} \frac{s!}{r!} \prod_{k=r}^{s} (v+\mu_2 q+\mu_1 l+f_{ijlq,i-1jl+1q}+f_{ijlq,ij-1lq+1})^{-1} & \text{if } r \le s \\ 0 & \text{otherwise.} \end{cases}$$

Hence from $(A.2)$ we deduce that

$$[R^{lh}_{ij}(v)]_{rs} = \left[\prod_{k=l}^{h-1} M^{kk}_{ij}(v) B^{k+1}_{ij} \right]_{rs}$$
$$= \sum_{i_1=0}^{N_1^i} \sum_{i_2=0}^{N_1^i} \cdots \sum_{i_{h-l-1}=0}^{N_1^i} [M^{ll}_{ij}(v) B^{l+1}_{ij}]_{ri_1} \cdots [M^{h-1h}_{ij}(v) B^{h-1}_{ij}]_{i_{h-l-1}s}$$
$$= \sum_{i_1=r}^{s} \sum_{i_2=i_1}^{s} \cdots \sum_{i_{h-l-1}=i_{h-l-2}}^{s} [M^{ll}_{ij}(v)]_{ri_1} [B^{l+1}_{ij}]_{i_1 i_1} \cdots [M^{h-1h}_{ij}(v)]_{i_{h-l-1}s} [B^{h-1}_{ij}]_{ss}$$
$$= \sum_{i_1=r}^{s} \sum_{i_2=i_1}^{s} \cdots \sum_{i_{h-l-1}=i_{h-l-2}}^{s} \mu_1^{h-l} \mu_2^{s-r} \frac{h!s!}{l!r!} \prod_{k=0}^{h-l} \prod_{p=i_k}^{i_{k+1}} f(v, \mu_1, \mu_2, i, j, l+k, p),$$

where $i_0 = r$ and $i_{h-l} = s$.

Finally by substituting (A.4) and the above equation in $(A.3)$ we obtain, if $r \le s$,

$$[F^{lh}_{ij}(v)]_{rs} = \sum_{k=0}^{s} [R^{lh}_{ij}(v)]_{rk} [M^{hh}_{ij}]_{ks}$$
$$= \sum_{k=r}^{s} [R^{lh}_{ij}(v)]_{rk} \mu_2^{s-k} \frac{s!}{k!}$$

$$\times \prod_{p=k}^{s} (v + \mu_2 p + \mu_1 h + f_{ijhp,i-1jh+11p} + f_{ijhp,ij-1hp+1})^{-1}$$

$$(A.5) \qquad = \sum_{r \le i_1 \le i_2 \le \cdots \le i_{h-l} \le s} \mu_1^{h-l} \mu_2^{s-r} \frac{h!s!}{l!r!} \times \prod_{k=0}^{h-l} \prod_{p=i_k}^{i_{k+1}} f(v, \mu_1, \mu_2, i, j, l+k, p),$$

where $i_0 = r$ and $i_{h-l+1} = s$ ▫

Proof of (4.6)

From (A.2) and (A.3) we have

$$\begin{aligned}
[F_{ij}^{lh}]_{rs} &= \left[\prod_{k=l}^{h-1} (M_{ij}(v) \triangle_{ij} B_{ij})^{kk+1} M_{ij}^{hh}(v) \right]_{rs} \\
&= \left[\prod_{k=l}^{h-2} (M_{ij}(v) \triangle_{ij} B_{ij})^{kk+1} (M_{ij}(v) \triangle_{ij} B_{ij})^{h-1h} M_{ij}^{hh}(v) \right]_{rs} \\
&= \left[\prod_{k=l}^{h-1} (M_{ij}(v) \triangle_{ij} B_{ij})^{kk+1} M_{ij}^{h-1h-1}(v) B_{ij}^{hh} M_{ij}^{hh}(v) \right]_{rs} \\
&= \sum_{p=r}^{s} \left[\prod_{k=l}^{h-1} (M_{ij}(v) \triangle_{ij} B_{ij})^{kk+1} M_{ij}^{h-1h-1}(v) B_{ij}^{hh} \right]_{rp} \left[B_{ij}^{hh} M_{ij}^{hh}(v) \right]_{ps} \\
&= \sum_{p=r}^{s} \left[M_{ij}(v)^{hh} \right]_{ps} \left[B_{ij}^{hh} \right]_{pp} \left[F_{ij}^{lh-1}(v) \right]_{rp}.
\end{aligned}$$

(A.4) and (4.1) complete the proof. ▫

Proof of (4.7)

For $m, n = 0, \ldots, (N_1^i + 1)(N_2^j + 1) - 1$ let l, h and r, s be, respectively, the quotient and remainder of the Euclidean division of m, n by $N_2^j + 1$. We have

$$\begin{aligned}
[D_{i+1j}(1,0)]_{mn} &= [D_{i+1j}(1,0)]_{l(N_1^i+1)+r, h(N_2^j+j)+s} \\
&= [D_{i+1j}(1,0)]_{rs}^{lh} \\
&= \begin{cases} [D_{i+1j}^{l-1}]_{rs} & \text{if } l = h \text{ and } l \ge 1 \\ [\Theta_{ij}^{10}]_{rs} & \text{if } l = h = 0 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

because

$$D_{i+1j}(1,0) = \begin{pmatrix} \Theta_{ij}^{10} & 0 \\ 0 & D_{i+1j} \end{pmatrix}$$

so that

$$[D_{i+1j}(1,0)]_{mn} = \begin{cases} f_{i+1jl-1r,ijlr} & \text{if } l = h \text{ and } r = s \\ 0 & \text{otherwise .} \end{cases} \qquad (*)$$

We also have

$$[F_{ij}(v)]_{mn} = [F_{ij}(v)]_{l(N_2^j+1)+r,h(N_2^j+1)+s} = \begin{cases} [F_{ij}^{lh}(v)]_{rs} & \text{if } l \le h \\ 0 & \text{otherwise.} \end{cases} \qquad (**)$$

Using (*) and (**) we obtain

$$\begin{aligned} [F_{ij}(v)D_{i+1j}(1,0)]_{mn} &= [F_{ij}(v)D_{i+1j}(1,0)]_{l(N_2^j+1)+r,h(N_2^j+1)+s} \\ &= [F_{ij}(v)D_{i+1j}(1,0)]_{rs}^{lh} \\ &= [F_{ij}^{lh}(v)[D_{i+1j}(1,0)]^{hh}]_{rs} \\ &= \begin{cases} [F_{ij}^{lh}(v)D_{i+1j}^{h-1}]_{rs} & \text{if } h \ge 1 \text{ and } l = h \\ [F_{ij}^{l0}(v)\Theta_{ij}^{10}]_{rs} & \text{if } h = l = 0 \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} [F_{ij}^{lh}(v)]_{rs}[D_{i+1j}^{h-1}]_{ss} & \text{if } h \ge 1 \text{ and } l = h \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} C_{ij}(t,l,h,r,s)f_{i+1jh-1s,ijls} & \text{if } l \le h, \text{ and } h \ge 1 \\ & 0 \le r \le s \le N_2^j \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

□

Proof of (4.8)

As before we let l, h and $r, s-1$ be, respectively, the quotients and remainders of the Euclidean division of m by $N_2^j + 1$ and n by $N_2^j + 1$,

$$\begin{aligned} [H_{ij}]_{mn} &= [H_{ij}^{lh}]_{l(N_2^j+1)+r,h(N_2^j+1)+s} = [H_{ij}^{lh}]_{rs} = \begin{cases} [H_{ij}^{l}]_{rs} & \text{if } l = h \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} f_{ij+1lr-1,ijlr} & \text{if } m = n = l(N_2^j+1) + r \text{ , } 1 \le r \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Now, if l, h and $r, s-1$ are, respectively, the quotients and remainders of the Euclidean division of m by $N_2^j + 1$ and $n - N_1^i - 1$ by N_2^j, we have

$$\begin{aligned} [H_{ij}T_{ij}]_{mn} &= \sum_{k=0}^{(N_1^i+1)(N_2^j+1)-1} [H_{ij}]_{mk}[T_{ij}]_{kn} \\ &= [H_{ij}]_{mm}[T_{ij}]_{mn} \\ &= \begin{cases} f_{ij+1lr-1,ijlr} & \text{if } m = l(N_2^j+1) + r, n = N_1^i + lN_2^j + r \text{ and } 1 \le r \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Finally we obtain

$$[F_{ij}(v)H_{ij}T_{ij}]_{mn} = \begin{cases} [F_{ij}(v)]_{l(N_2^j+1)+r,h(N_2^j+1)+s}\, f_{ij+1hs-1,ijhs} & \\ \qquad \text{if } m = l(N_2^j+1)+r, & \\ \qquad n = N_1^i + hN_2^j + s & \\ \qquad \text{and } s \geq 1 & \\ 0 \qquad \text{otherwise} & \end{cases}$$

$$= \begin{cases} [F_{ij}^{lh}(v)]_{rs} f_{ij+1hs-1,ijhs} & \text{if } m = l(N_2^j+1)+r, \\ & n = N_1^i + hN_2^j + s \\ & \text{and } s \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} C_{ij}(t,l,h,r,s) f_{ij+1hs-1,ijhs} & \text{if } m = l(N_2^j+1)+r \\ & \text{and } n = N_1^i + hN_2^j + r \\ & \text{with } r \leq s, s \geq 1 \\ 0 & \text{otherwise.} \end{cases}$$

□

Proof of (4.1) *and* (5.1)

Let $h > l$ and $s \geq r$. From (A.2) and (A.3) we have

$$[F_{ij}^{lh}(v)]_{rs} = \left[\left(\prod_{k=l}^{h-1} (M_{ij}(v) \triangle_{ij} B_{ij})^{kk+1}\right) M_{ij}^{hh}(v)\right]_{rs}$$

$$= \sum_{q=r}^{s} \sum_{p=r}^{q} [(M_{ij}(v) \triangle_{ij} B_{ij})^{ll+1}]_{rp} \left[\prod_{k=l+1}^{h-1} (M_{ij}(v) \triangle_{ij} B_{ij})^{kk+1}\right]_{pq} [M_{ij}^{hh}(v)]_{qs}$$

$$= \sum_{q=r}^{s} [(M_{ij}(v) \triangle_{ij} B_{ij})^{ll+1}]_{rr} \left[\prod_{k=l+1}^{h-1} (M_{ij}(v) \triangle_{ij} B_{ij})^{kk+1}\right]_{rq} [M_{ij}^{hh}(v)]_{qs}$$

$$+ \sum_{q=r+1}^{s} \sum_{p=r+1}^{q} [(M_{ij}(v) \triangle_{ij} B_{ij})^{ll+1}]_{rp} \left[\prod_{k=l+1}^{h-1} (M_{ij}(v) \triangle_{ij} B_{ij})^{kk+1}\right]_{pq} [M_{ij}^{hh}(v)]_{qs}$$

$$= \sum_{q=r}^{s} [(M_{ij}(v) \triangle_{ij} B_{ij})^{ll+1}]_{rr} \left[\prod_{k=l+1}^{h-1} (M_{ij}(v) \triangle_{ij} B_{ij})^{kk+1})\right]_{rq} [M_{ij}^{hh}(v)]_{qs}$$

$$+ \sum_{p=r+1}^{s} [(M_{ij}(v) \triangle_{ij} B_{ij})^{ll+1}]_{rp} \sum_{q=p}^{s} \left[\prod_{k=l+1}^{h-l} (M_{ij}(v) \triangle_{ij} B_{ij})^{kk+1}\right]_{pq} [M_{ij}^{hh}(v)]_{qs},$$

but, if $r \leq s$, we have by using (A.4) that

$$[(M_{ij}(v) \triangle_{ij} B_{ij})^{ll+1}]_{rp} = \mu_1(l+1)\mu_2^{s-r} \frac{s!}{r!} \prod_{q=r}^{s} [t + \mu_1 l + \mu_2 q + f_{ijlq,i-1jl+1q} + f_{ijlq,ij-1lq+1}]^{-1}.$$

Hence, if $(l, r) = (0, 0)$ and $h > 0$, we obtain

$$\text{(A.6)}\quad [F_{ij}^{0h}(v)]_{0s}$$
$$= \frac{\mu_1}{t}\left\{[F_{ij}^{1h}(v)]_{0s} + \sum_{p=1}^{s} \mu_2^p p! \prod_{k=1}^{p} (v + \mu_2 k + f_{ijlk,i-1jl+1k} + f_{ijlk,ij-1lk+1})^{-1} [F_{ij}^{1h}]_{ps}\right\}$$
$$= \frac{\mu_1}{t}\left\{\sum_{p=0}^{s} \mu_2^p p! \prod_{k=1}^{p} (v + \mu_2 k + f_{ijlk,i-1jl+1k} + f_{ijlk,ij-1lk+1})^{-1} [F_{ij}^{1h}(v)]_{ps}\right\}.$$

In addition we see from $(\text{A}.5)$ that $\lim_{t\to 0}[F_{ij}^{lh}(v)]_{rs}$ exists if $(l, r) \neq (0, 0)$. Therefore using the second and third members in $(\text{A}.6)$ it can be shown that $\lim_{t\to 0}(tC_{ij}(t, 0, h, 0, s))$ exists and is equal to (5.2). Finally (5.1) is easily obtained by passing to the limit in (4.2) when $h = r = 0$.

Acknowledgments

The final version of this work was written while H. El Maroufy was visiting the Department of Mathematical Sciences at Chalmers University of Technology and Göteborg University. He thanks the faculty of that Department. The referees are kindly acknowledged for very interesting remarks and suggestions.

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