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Some Generalizations of Pochhammer's Symbol and their Associated Families of Hypergeometric Functions and Hypergeometric Polynomials

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Abstract: In 2012, H. M. Srivastava *et al.* [37] introduced and studied a number of interesting fundamental properties and characteristics of a family of potentially useful incomplete hypergeometric functions. The definitions of these incomplete hypergeometric functions were based essentially upon some generalization of the Pochhammer symbol by mean of the incomplete gamma functions $\gamma(s,x)$ and $\Gamma(s,x)$. Our principal objective in this article is to present a systematic investigation of several further properties of these incomplete hypergeometric functions and some general classes of the incomplete hypergeometric polynomials which are associated with them. Various (known or new) special cases and consequences of the results presented in this article are considered. Several other generalizations of the Pochhammer symbol and their associated families of hypergeometric functions and hypergeometric polynomials are also briefly pointed out.

Keywords: Gamma function; Incomplete Gamma functions; Incomplete Pochhammer symbols; Incomplete hypergeometric functions and polynomials; Generating functions; Lagrange expansion theorem; Srivastava-Buschman generating function; Combinatorial identities; Gould's identity; Reduction formulas; Chu-Vandermonde summation formula.
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1 Introduction and Definitions

Throughout this presentation, we shall (as usual) denote by $\mathbb{R}$ and $\mathbb{C}$ the sets of real and complex numbers, respectively. In terms of the familiar (Euler's) Gamma function $\Gamma(z)$ which is defined, for $z \in \mathbb{C} \setminus \mathbb{Z}_0^-$, by

$$\Gamma(z) = \begin{cases} \displaystyle\int_0^\infty e^{-t}\, t^{z-1}\, dt & \big(\Re(z) > 0\big) \\ \dfrac{\Gamma(z+n)}{\prod_{j=0}^{n-1}(z+j)} & \big(z \in \mathbb{C}\setminus\mathbb{Z}_0^-;\ n \in \mathbb{N}\big), \end{cases} \tag{1}$$

$$\big(\mathbb{Z}_0^- := \mathbb{Z}^- \cup \{0\};\ \mathbb{Z}^- := \{-1,-2,-3,\cdots\};\ \mathbb{N} := \{1,2,3,\cdots\}\big),$$

a generalized binomial coefficient $\binom{\lambda}{\mu}$ may be defined (for real or complex parameters λ and μ) by

$$\binom{\lambda}{\mu} := \frac{\Gamma(\lambda+1)}{\Gamma(\mu+1)\Gamma(\lambda-\mu+1)} =: \binom{\lambda}{\lambda-\mu} \qquad (\lambda,\mu \in \mathbb{C}), \tag{2}$$

so that, in the special case when

$$\mu = n \qquad (n \in \mathbb{N}_0;\ \mathbb{N}_0 := \mathbb{N} \cup \{0\}),$$

we have

$$\binom{\lambda}{n} = \frac{\lambda(\lambda-1)\cdots(\lambda-n+1)}{n!} = \frac{(-1)^n\,(-\lambda)_n}{n!} \qquad (n \in \mathbb{N}_0), \tag{3}$$

where $(\lambda)_\nu$ $(\lambda,\nu \in \mathbb{C})$ denotes the Pochhammer symbol given, in general, by

$$(\lambda)_\nu := \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} = \begin{cases} 1 & (\nu = 0;\ \lambda \in \mathbb{C}\setminus\{0\}) \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (\nu \in \mathbb{N};\ \lambda \in \mathbb{C}), \end{cases} \tag{4}$$

it being assumed *conventionally* that $(0)_0 := 1$ and understood *tacitly* that the Γ-quotient exists (see, for details, [42, p. 21 *et seq.*]).

The so-called *incomplete Gamma functions* $\gamma(s,x)$ and $\Gamma(s,x)$ defined, respectively, by

$$\gamma(s,x) := \int_0^x t^{s-1} e^{-t} dt \qquad (\Re(s) > 0;\ x \geqq 0) \tag{5}$$

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and

$$\Gamma(s,x) := \int_x^\infty t^{s-1}e^{-t}dt \qquad (x \geqq 0;\ \Re(s) > 0 \ \text{ when } \ x = 0), \tag{6}$$

are known to satisfy the following decomposition formula:

$$\gamma(s,x) + \Gamma(s,x) = \Gamma(s) \qquad (\Re(s) > 0). \tag{7}$$

The function $\Gamma(z)$, and its incomplete versions $\gamma(s,x)$ and $\Gamma(s,x)$, play important rôles in the study of the analytic solutions of a variety of problems in diverse areas of science and engineering (see, for example, [1], [7], [6], [12], [15], [17], [21], [23], [24], [38], [39], [40], [41], [51], [52] and [53]; see also [37] and the references cited therein).

In a recent paper, the following family of generalized incomplete hypergeometric functions was introduced and studied systematically by Srivastava *et al.* [37, p. 675, Equations (4.1) and (4.2)]:

$${}_p\gamma_q\left[\begin{array}{r}(a_1,x),a_2,\cdots,a_p;\\ \\ b_1,\cdots,b_q;\end{array} z\right] := \sum_{n=0}^{\infty} \frac{(a_1;x)_n(a_2)_n\cdots(a_p)_n}{(b_1)_n\cdots(b_q)_n}\frac{z^n}{n!} \tag{8}$$

and

$${}_p\Gamma_q\left[\begin{array}{r}(a_1,x),a_2,\cdots,a_p;\\ \\ b_1,\cdots,b_q;\end{array} z\right] := \sum_{n=0}^{\infty} \frac{[a_1;x]_n(a_2)_n\cdots(a_p)_n}{(b_1)_n\cdots(b_q)_n}\frac{z^n}{n!}, \tag{9}$$

where, in terms of the incomplete Gamma functions $\gamma(s,x)$ and $\Gamma(s,x)$ defined by (5) and (6), the *incomplete* Pochhammer symbols

$$(\lambda;x)_\nu \qquad \text{and} \qquad [\lambda;x]_\nu \qquad (\lambda,\nu \in \mathbb{C};\ x \geqq 0)$$

are defined as follows:

$$(\lambda;x)_\nu := \frac{\gamma(\lambda+\nu,x)}{\Gamma(\lambda)} \qquad (\lambda,\nu \in \mathbb{C};\ x \geqq 0) \tag{10}$$

and

$$[\lambda;x]_\nu := \frac{\Gamma(\lambda+\nu,x)}{\Gamma(\lambda)} \qquad (\lambda,\nu \in \mathbb{C};\ x \geqq 0). \tag{11}$$

so that, obviously, these incomplete Pochhammer symbols $(\lambda;x)_\nu$ and $[\lambda;x]_\nu$ satisfy the following decomposition relation:

$$(\lambda;x)_\nu + [\lambda;x]_\nu = (\lambda)_\nu \qquad (\lambda,\nu \in \mathbb{C};\ x \geqq 0), \tag{12}$$

where $(\lambda)_\nu$ is the Pochhammer symbol given by (4).

Remark 1. The argument $x \geqq 0$ in the definitions (5) and (6), (8) and (9), (10) and (11), and elsewhere in this paper, is *independent* of the argument $z \in \mathbb{C}$ which occurs in the definitions (1), (8) and (9), and also in the results presented in this paper.

As already pointed out by Srivastava *et al.* [37, p. 675, Remark 7], since

$$|(\lambda;x)_n| \leqq |(\lambda)_n| \qquad \text{and} \qquad |[\lambda;x]_n| \leqq |(\lambda)_n| \qquad (n \in \mathbb{N}_0;\ \lambda \in \mathbb{C};\ x \geqq 0), \tag{13}$$

the precise (*sufficient*) conditions under which the infinite series in the definitions (8) and (9) would converge absolutely can be derived from those that are well-documented in the case of the generalized hypergeometric function ${}_pF_q$ $(p,q \in \mathbb{N}_0)$ (see, for details, [29, pp. 72–73] and [40, p. 20]; see also [8], [3], [20] and [30]). Indeed, in their special case when $x = 0$, both ${}_p\gamma_q$ $(p,q \in \mathbb{N}_0)$ and ${}_p\Gamma_q$ $(p,q \in \mathbb{N}_0)$ would reduce immediately to the extensively-investigated generalized hypergeometric function ${}_pF_q$ $(p,q \in \mathbb{N}_0)$. Furthermore, as an immediate consequence of the definitions (8) and (9), we have the following decomposition formula:

$${}_p\gamma_q\left[\begin{array}{r}(a_1,x),a_2,\cdots,a_p;\\ \\ b_1,\cdots,b_q;\end{array} z\right] + {}_p\Gamma_q\left[\begin{array}{r}(a_1,x),a_2,\cdots,a_p;\\ \\ b_1,\cdots,b_q;\end{array} z\right]$$
$$= {}_pF_q\left[\begin{array}{r}a_1,\cdots,a_p;\\ \\ b_1,\cdots,b_q;\end{array} z\right] \tag{14}$$

in terms of the familiar generalized hypergeometric function ${}_pF_q$ $(p,q \in \mathbb{N}_0)$.

Motivated essentially by the demonstrated potential for applications of the generalized incomplete hypergeometric functions ${}_p\gamma_q$ and ${}_p\Gamma_q$ in many diverse areas of mathematical, physical, engineering and statistical sciences (see, for details, [37] and the references cited therein), we aim here at presenting a systematic investigation of several *further* properties of these generalized incomplete hypergeometric functions and some classes of incomplete hypergeometric polynomials associated with them. Specifically, we make use of several such combinatorial identities as Gould's identity (18) below, which stem essentially from the Lagrange expansion theorem (see, for example, [42, Chapter 7]), with a view to deriving many general families of generating functions for a certain class of incomplete hypergeometric polynomials associated with these generalized incomplete hypergeometric functions (see also some interesting recent developments [35] and [47]). Various (known or new) special cases and consequences of the results presented in this article are considered. We choose also to point out several other generalizations of the Pochhammer symbol and their associated families of hypergeometric functions and hypergeometric polynomials.

2 Generating Functions Based Upon the Lagrange Expansion Theorem and Gould's Identity

Upon suitable specialization, the Lagrange expansion theorem (see [28, p. 146, Problem 206] and [53, p. 133]; see also the Appendix for *complete* details) is known to yield each of the following combinatorial identities [28, p. 349, Problem 216]:

$$\sum_{n=0}^{\infty} \binom{\alpha+(\beta+1)n}{n} t^n = \frac{(1+\zeta)^{\alpha+1}}{1-\beta\zeta} \tag{15}$$

and [28, p. 348, Problem 212]:

$$\sum_{n=0}^{\infty} \frac{\alpha}{\alpha+(\beta+1)n} \binom{\alpha+(\beta+1)n}{n} t^n = (1+\zeta)^{\alpha}, \tag{16}$$

where α and β are complex numbers independent of n and ζ is a function of t defined *implicitly* by

$$\zeta = t(1+\zeta)^{\beta+1} \quad \text{and} \quad \zeta(0)=0. \tag{17}$$

In view of the following obvious combinatorial identity: $\binom{\alpha+(\beta+1)n}{n} = \frac{\alpha}{\alpha+(\beta+1)n}\binom{\alpha+(\beta+1)n}{n} + (\beta+1)\binom{\alpha+(\beta+1)n-1}{n-1}$, the expansion formula (15) can easily be shown to imply the expansion formula (16). In fact, it is not difficult to show that the expansion formulas (15) and (16) are equivalent (see, for details, [42, pp. 354–356]).

The following interesting generalization (and unification) of the equivalent expansion formulas (15) and (16) was given by Gould [14, p. 196, Equation (6.1)]:

$$\sum_{n=0}^{\infty} \frac{\kappa}{\kappa+(\beta+1)n} \binom{\alpha+(\beta+1)n}{n} t^n = (1+\zeta)^{\alpha} \sum_{n=0}^{\infty} (-1)^n \binom{\alpha-\kappa}{n} \binom{n+\kappa/(\beta+1)}{n}^{-1} \left(\frac{\zeta}{1+\zeta}\right)^n, \tag{18}$$

where α, β and κ are complex numbers independent of n and ζ is a function of t defined *implicitly* by (17).

For $\kappa=\alpha$, Gould's identity (18) reduces at once to the expansion formula (16). Moreover, in its limit case when $|\kappa| \to \infty$, (18) corresponds (*at least formally*) to the expansion formula (15). Thus, for *bounded* κ $(\kappa \neq \alpha)$, Gould's identity (18) may naturally be looked upon as being independent of the equivalent expansion formulas (15) and (16).

The existing literature on generating functions is abundant in results that are based essentially upon the Lagrange expansion theorem as well as the three formulas (15), (16) and (18) (see, for details, [42, Chapter 7]; see also [10], [2], [32], [33], [34], [36], [43], [46] and [54], and as well as many references to other closely-related investigations cited in each of these works). With a view to applying it to derive generating functions for a certain class of generalized incomplete hypergeometric polynomials, we recall here a general result on generating functions asserted by Lemma 1 below, known as the Srivastava-Buschman generating function, which is due to Srivastava and Buschman [36, p. 366, Theorem 3] and [42, p. 373, Theorem 9]).

Lemma 1. *Corresponding to the power series $\Lambda(z)$ given by*

$$\Lambda(z) = \sum_{n=0}^{\infty} \Omega_n z^n \qquad (\Omega_0 \neq 0), \tag{19}$$

let the polynomial system $S_{n,N}^{(\alpha,\beta)}(\lambda;z)$ be defined by

$$S_{n,N}^{(\alpha,\beta)}(\lambda;z) = \sum_{k=0}^{[n/N]} \frac{(-n)_{Nk}\left(\alpha+(\beta+1)n+1\right)_{\lambda k}}{(\alpha+\beta n+1)_{(\lambda+N)k}} \Omega_k z^k \qquad (\alpha,\beta,\lambda \in \mathbb{C};\ N \in \mathbb{N}), \tag{20}$$

where $[\omega]$ denotes the greatest integer in $\omega \in \mathbb{R}$. Suppose also that

$$\vartheta(n,N;\alpha,\beta,\kappa,\lambda;z) = \sum_{k=0}^{\infty} \frac{\kappa}{\kappa+(\beta+1)Nk} \binom{\alpha-\kappa+\lambda k}{n} \binom{n+Nk+\kappa/(\beta+1)}{n}^{-1} \Omega_k z^k \tag{21}$$

$$(n \in \mathbb{N}_0;\ \alpha,\beta,\kappa,\lambda \in \mathbb{C};\ N \in \mathbb{N}).$$

Then

$$\sum_{n=0}^{\infty} \frac{\kappa}{\kappa+(\beta+1)n} \binom{\alpha+(\beta+1)n}{n} S_{n,N}^{(\alpha,\beta)}(\lambda;z) t^n = (1+\zeta)^{\alpha}\, \varphi\left[z(-\zeta)^N (1+\zeta)^{\lambda}, -\frac{\zeta}{1+\zeta}\right], \tag{22}$$

where

$$\varphi[z,w] = \sum_{n=0}^{\infty} \vartheta(n,N;\alpha,\beta,\kappa,\lambda;z) w^n \tag{23}$$

where ζ is given by (17), it being assumed that both members of the generating function (22) exist.

The *main* generating functions for the aforementioned associated class of generalized incomplete hypergeometric polynomials are contained in the following theorem (see also [50]).

Theorem 1. *Let $\Delta(N;\lambda)$ denote the array of N parameters*

$$\frac{\lambda}{N}, \frac{\lambda+1}{N}, \cdots, \frac{\lambda+N-1}{N} \qquad (\lambda \in \mathbb{C};\ N \in \mathbb{N}),$$

the array $\Delta(N;\lambda)$ being empty when $N=0$. Suppose also that

$$\theta(n,N,L;\alpha,\beta,\kappa;z) = \sum_{k=0}^{\infty} \frac{\kappa}{\kappa+(\beta+1)Nk} \binom{\alpha-\kappa+Lk}{n} \binom{n+Nk+\kappa/(\beta+1)}{n}^{-1} \cdot \frac{(a_0;x)_k (a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{z^k}{k!} \tag{24}$$

$$(x \geqq 0;\ n \in \mathbb{N}_0;\ \alpha,\beta,\kappa \in \mathbb{C};\ N,L \in \mathbb{N})$$

and

$$\Theta(n,N,L;\alpha,\beta,\kappa;z)=\sum_{k=0}^{\infty}\frac{\kappa}{\kappa+(\beta+1)Nk}\binom{\alpha-\kappa+Lk}{n}\binom{n+Nk+\kappa/(\beta+1)}{n}^{-1}\cdot\frac{[a_0;x]_k(a_1)_k\cdots(a_p)_k}{(b_1)_k\cdots(b_q)_k}\frac{z^k}{k!}\tag{25}$$

$$(x\geqq 0;\ n\in\mathbb{N}_0;\ \alpha,\beta,\kappa\in\mathbb{C};\ N,L\in\mathbb{N}).$$

Then the following generating functions hold true for the associated class of generalized incomplete hypergeometric polynomials:

$$\sum_{n=0}^{\infty}\frac{\kappa}{\kappa+(\beta+1)n}\binom{\alpha+(\beta+1)n}{n}\cdot{}_{p+N+L+1}\gamma_{q+N+L}\left[\begin{array}{r}\Delta(N;-n),\Delta\left(L;\alpha+(\beta+1)n+1\right),(a_0,x),a_1,\cdots,a_p;\\ \Delta(N+L;\alpha+\beta n+1),b_1,\cdots,b_q;\end{array} z\right]t^n$$
$$=(1+\zeta)^{\alpha}\,\phi\left[\left(\frac{(N+L)^{N+L}}{N^N\,L^L}\right)z(-\zeta)^N\,(1+\zeta)^L,-\frac{\zeta}{1+\zeta}\right]\tag{26}$$

and

$$\sum_{n=0}^{\infty}\frac{\kappa}{\kappa+(\beta+1)n}\binom{\alpha+(\beta+1)n}{n}\cdot{}_{p+N+L+1}\Gamma_{q+N+L}\left[\begin{array}{r}\Delta(N;-n),\Delta\left(L;\alpha+(\beta+1)n+1\right),(a_0,x),a_1,\cdots,a_p;\\ \Delta(N+L;\alpha+\beta n+1),b_1,\cdots,b_q;\end{array} z\right]t^n$$
$$=(1+\zeta)^{\alpha}\,\Phi\left[\left(\frac{(N+L)^{N+L}}{N^N\,L^L}\right)z(-\zeta)^N\,(1+\zeta)^L,-\frac{\zeta}{1+\zeta}\right],\tag{27}$$

where

$$\phi[z,w]=\sum_{n=0}^{\infty}\theta(n,N,L;\alpha,\beta,\kappa;z)w^n\tag{28}$$

and

$$\Phi[z,w]=\sum_{n=0}^{\infty}\Theta(n,N,L;\alpha,\beta,\kappa;z)w^n,\tag{23}$$

and ζ *is given by* (17), *it being assumed that both members of the generating functions* (26) *and* (27) *exist.*

*Proof.*The assertions (26) and (27) of Theorem 1 can be proven by appealing appropriately to Lemma 1. Indeed, if in Lemma 1, we set $\lambda=L\ \ (L\in\mathbb{N})$,

$$\Omega_n=\frac{(a_0;x)_n(a_1)_n\cdots(a_p)_n}{n!\,(b_1)_n\cdots(b_q)_n}\qquad(x\geqq 0;\ n,p,q\in\mathbb{N}_0)\tag{30}$$

and

$$\Omega_n=\frac{[a_0;x]_n(a_1)_n\cdots(a_p)_n}{n!\,(b_1)_n\cdots(b_q)_n}\qquad(x\geqq 0;\ n,p,q\in\mathbb{N}_0),\tag{31}$$

and then interpret the incomplete hypergeometric polynomials resulting from (20) by means of the definitions (8) and (9), respectively, the generating functions (26) and (27) asserted by Theorem 1 would follow after series iterations and necessary simplifications.

Alternatively, of course, the assertions (26) and (27) of Theorem 1 can be derived *directly* by using Gould's identity (18) in an appropriate manner.

Since the limit cases of the generating functions (26) and (27) when $|\kappa|\to\infty$ are equivalent to the corresponding obvious special cases of the generating functions (26) and (27) when $\kappa=\alpha$, just as we observed above in connection with the three combinatorial identities (15), (16) and (18), it would suffice our purpose if we state only the limiting cases of the generating functions (26) and (27) when $|\kappa|\to\infty$ as Corollary 1 below.

Corollary 1. *Assume that* $x\geqq 0$ *and* $N,L\in\mathbb{N}$. *Then the following generating functions hold true for the associated class of generalized incomplete hypergeometric polynomials*:

$$\sum_{n=0}^{\infty}\binom{\alpha+(\beta+1)n}{n}\cdot{}_{p+N+L+1}\gamma_{q+N+L}\left[\begin{array}{r}\Delta(N;-n),\Delta\left(L;\alpha+(\beta+1)n+1\right),(a_0,x),a_1,\cdots,a_p;\\ \Delta(N+L;\alpha+\beta n+1),b_1,\cdots,b_q;\end{array} z\right]t^n$$
$$=\frac{(1+\zeta)^{\alpha+1}}{1-\beta\zeta}\,{}_{p+1}\gamma_q\left[\begin{array}{r}(a_0,x),a_1,\cdots,a_p;\\ b_1,\cdots,b_q;\end{array}\left(\frac{(N+L)^{N+L}}{N^N\,L^L}\right)z(-\zeta)^N\,(1+\zeta)^L\right]\tag{32}$$

and

$$\sum_{n=0}^{\infty}\binom{\alpha+(\beta+1)n}{n}\cdot{}_{p+N+L+1}\Gamma_{q+N+L}\left[\begin{array}{r}\Delta(N;-n),\Delta\left(L;\alpha+(\beta+1)n+1\right),(a_0,x),a_1,\cdots,a_p;\\ \Delta(N+L;\alpha+\beta n+1),b_1,\cdots,b_q;\end{array} z\right]t^n$$
$$=\frac{(1+\zeta)^{\alpha+1}}{1-\beta\zeta}\,{}_{p+1}\gamma_q\left[\begin{array}{r}(a_0,x),a_1,\cdots,a_p;\\ b_1,\cdots,b_q;\end{array}\left(\frac{(N+L)^{N+L}}{N^N\,L^L}\right)z(-\zeta)^N\,(1+\zeta)^L\right],\tag{33}$$

where ζ *is given by* (17), *it being assumed that both members of the generating functions* (32) *and* (33) *exist.*

The methodology and techniques used here and in the treatise on generating functions by Srivastava and Manocha (see, for details, [42, Chapter 7]) can be applied *mutatis mutandis* in order to obtain a remarkably large variety of generating functions for the associated class of generalized incomplete hypergeometric polynomials of the type which we have considered in the preceding section. The details involved in these derivations may be omitted here.

Various interesting special cases of the generating functions (26) and (27) asserted by Theorem 1 and the generating functions (32) and (33) asserted by Corollary 1, which would correspond to the special cases of the potentially useful Srivastava-Buschman generating function (22) asserted by Lemma 1 when (for example) $\lambda=0$ and $\lambda=-1$, can also be derived fairly easily. Thus, for instance, Theorem 1 in its *exceptional* case when $L=0$ would yield the following result.

Corollary 2. *Let* $N\in\mathbb{N}$,

$$\psi_n(\alpha,\beta,\kappa;\zeta)=\Psi_n(\alpha,\beta,\kappa;\zeta)=:\Xi_n\qquad(x\geqq 0;\ n\in\mathbb{N}_0;\ \alpha,\beta,\kappa\in\mathbb{C}),\tag{34}$$

where

$$\Xi_n := (-1)^n \binom{\alpha-\kappa}{n} \binom{n+\kappa/(\beta+1)}{n}^{-1} \left(\frac{\zeta}{1+\zeta}\right)^n \tag{35}$$

$$(x \geqq 0;\ n \in \mathbb{N}_0;\ \alpha, \beta, \kappa \in \mathbb{C}).$$

Then the following generating functions hold true for the associated class of generalized incomplete hypergeometric polynomials:

$$\sum_{n=0}^{\infty} \frac{\kappa}{\kappa+(\beta+1)n} \binom{\alpha+(\beta+1)n}{n} {}_{p+N+1}\gamma_{q+N} \left[\begin{matrix} \Delta(N;-n), (a_0,x), a_1, \cdots, a_p; \\ \Delta(N;\alpha+\beta n+1), b_1, \cdots, b_q; \end{matrix} z\right] t^n$$
$$= (1+\zeta)^{\alpha} \sum_{n=0}^{\infty} \psi_n(\alpha,\beta,\kappa;\zeta)\, {}_{p+N+1}\gamma_{q+N} \left[\begin{matrix} \Delta(N;\kappa/(\beta+1)), (a_0,x), a_1, \cdots, a_p; \\ \Delta(N;1+n+\kappa/(\beta+1)), b_1, \cdots, b_q; \end{matrix} z(-\zeta)^N\right] \tag{36}$$

and

$$\sum_{n=0}^{\infty} \frac{\kappa}{\kappa+(\beta+1)n} \binom{\alpha+(\beta+1)n}{n} {}_{p+N+1}\Gamma_{q+N} \left[\begin{matrix} \Delta(N;-n), (a_0,x), a_1, \cdots, a_p; \\ \Delta(N;\alpha+\beta n+1), b_1, \cdots, b_q; \end{matrix} z\right] t^n$$
$$= (1+\zeta)^{\alpha} \sum_{n=0}^{\infty} \Psi_n(\alpha,\beta,\kappa;\zeta)\, {}_{p+N+1}\Gamma_{q+N} \left[\begin{matrix} \Delta(N;\kappa/(\beta+1)), (a_0,x), a_1, \cdots, a_p; \\ \Delta(N;1+n+\kappa/(\beta+1)), b_1, \cdots, b_q; \end{matrix} z(-\zeta)^N\right], \tag{37}$$

where ζ *is given by* (17), *it being assumed that both members of the generating functions* (36) *and* (37) *exist.*

If, in the aforementioned exceptional case of Theorem 1 when $L = 0$, *we set* $\kappa = \alpha$ *(or, equivalently, let* $|\kappa| \to \infty$*), we arrive at the following exceptional case of Corollary 1 when* $L = 0$.

Corollary 3. *Suppose that* $x \geqq 0$ *and* $N \in \mathbb{N}$. *Then the following generating functions hold true for the associated class of generalized incomplete hypergeometric polynomials*:

$$\sum_{n=0}^{\infty} \binom{\alpha+(\beta+1)n}{n} {}_{p+N+1}\gamma_{q+N} \left[\begin{matrix} \Delta(N;-n), (a_0,x), a_1, \cdots, a_p; \\ \Delta(N;\alpha+\beta n+1), b_1, \cdots, b_q; \end{matrix} z\right] t^n$$
$$= \frac{(1+\zeta)^{\alpha+1}}{1-\beta\zeta}\, {}_{p+1}\gamma_q \left[\begin{matrix} (a_0,x), a_1, \cdots, a_p; \\ b_1, \cdots, b_q; \end{matrix} z(-\zeta)^N\right] \tag{38}$$

and

$$\sum_{n=0}^{\infty} \binom{\alpha+(\beta+1)n}{n} {}_{p+N+1}\Gamma_{q+N} \left[\begin{matrix} \Delta(N;-n), (a_0,x), a_1, \cdots, a_p; \\ \Delta(N;\alpha+\beta n+1), b_1, \cdots, b_q; \end{matrix} z\right] t^n$$
$$= \frac{(1+\zeta)^{\alpha+1}}{1-\beta\zeta}\, {}_{p+1}\Gamma_q \left[\begin{matrix} (a_0,x), a_1, \cdots, a_p; \\ b_1, \cdots, b_q; \end{matrix} z(-\zeta)^N\right], \tag{39}$$

where ζ *is given by* (17), *it being assumed that both members of the generating functions* (38) *and* (39) *exist.*

3 Further Generating Functions for the Associated Class of Generalized Incomplete Hypergeometric Polynomials

Many general families of generating functions as well as their basic (or q-) extensions for various polynomial systems in one and more variables were derived by Srivastava (see, for details, [33]; see also [42, p. 142 et seq.]). We choose to recall here one of Srivastava's results as Lemma 2 below (see [33, p. 331, Equation (2.2)] and [42, p. 144, Equation 2.6 (28)]).

Lemma 2. *Let* $\{\Theta_n\}_{n\in\mathbb{N}_0}$ *and* $\{\Phi_{n,k}\}_{n,k\in\mathbb{N}_0}$ *denote, respectively, suitably bounded single and double sequences of essentially arbitrary complex parameters. Then*

$$\sum_{n=0}^{\infty} \frac{\prod_{j=1}^{\ell}(c_j)_n}{\prod_{j=1}^{m}(d_j)_n} \frac{\Theta_n\, t^n}{n!} \sum_{k=0}^{[n/N]} \frac{(-n)_{Nk} \prod_{j=1}^{m}(1-d_j-n)_k}{\prod_{j=1}^{\ell}(1-c_j-n)_k} \frac{\Phi_{n,k}\, z^k}{k!}$$
$$= \sum_{n,k=0}^{\infty} \frac{\prod_{j=1}^{\ell}(c_j)_n}{\prod_{j=1}^{m}(d_j)_n} \Theta_{n+Nk}\, \Phi_{n+Nk,k} \frac{t^n}{n!} \frac{\left(z\left\{(-1)^{\ell-m+1}\, t\right\}^N\right)^k}{k!} \qquad (\ell, m \in \mathbb{N}_0;\ N \in \mathbb{N}), \tag{40}$$

provided that each member of (40) *exists*, $[\kappa]$ *being the greatest integer in* $\kappa \in \mathbb{R}$.

Remark 2. In results such as the generating function (40), an *empty* product is interpreted (as usual) to be 1. Thus, for example, it is always understood that $\prod_{j=1}^{\ell}(c_j)_n = 1$ when $\ell = 0$ and $\prod_{j=1}^{m}(d_j)_n = 1$ when $m = 0$.

With a view to applying Lemma 2 to a certain class of generalized incomplete hypergeometric polynomials which are associated naturally with the generalized incomplete hypergeometric functions ${}_p\gamma_q$ and ${}_p\Gamma_q$ defined by (8) and (9), respectively, we set

$$\Phi_{n,k} = \frac{\prod_{j=1}^{r}(g_j+n)_{Lk}}{\prod_{j=1}^{s}(h_j+n)_{Mk}} \frac{(a_0;x)_k (a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \qquad (x \geqq 0;\ p,q,r,s \in \mathbb{N}_0;\ L,M \in \mathbb{N})$$

and

$$\Phi_{n,k} = \frac{\prod_{j=1}^{r}(g_j+n)_{Lk}}{\prod_{j=1}^{s}(h_j+n)_{Mk}} \frac{[a_0;x]_k (a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \qquad (x \geqq 0;\ p,q,r,s \in \mathbb{N}_0;\ L,M \in \mathbb{N}).$$

For convenience, we denote the array of N parameters

$$\frac{\lambda}{N}, \frac{\lambda+1}{N}, \cdots, \frac{\lambda+N-1}{N} \qquad (\lambda \in \mathbb{C};\ N \in \mathbb{N})$$

by $\Delta(N;\lambda)$ and the array of Nr parameters $\frac{\lambda_j}{N}, \frac{\lambda_j+1}{N}, \cdots, \frac{\lambda_j+N-1}{N} \qquad (\lambda_j \in \mathbb{C};\ j = 1, \cdots, r;\ N \in \mathbb{N})$ by $\Delta(N,r;\lambda)$, the array being empty when $N = 0$ (and indeed also when $r = 0$), so that

$$\Delta(N,1;\lambda_j) = \Delta(N;\lambda_1).$$

We are thus led eventually to the following family of generating functions for the associated class of generalized incomplete hypergeometric polynomials.

Theorem 2. *Let* $\{\Theta_n\}_{n\in\mathbb{N}_0}$ *denote a suitably bounded sequence of essentially arbitrary complex parameters. Then the following generating functions hold true for the associated class of generalized incomplete hypergeometric polynomials*:

$$\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{\ell}(c_j)_n}{\prod_{j=1}^{m}(d_j)_n}\frac{\Theta_n\,t^n}{n!}\;{}_{p+Lr+(m+1)N+1}\gamma_{q+Ms+N\ell}\left[\begin{matrix}\Delta(N;-n),\Delta(L,r;g_j+n),\Delta(N,m;1-d_j-n),(a_0,x),a_1,\cdots,a_p;\\ \Delta(M,s;h_j+n),\Delta(N,\ell;1-c_j-n),b_1,\cdots,b_q;\end{matrix}\left(\frac{L^{Lr}}{M^{Ms}\,N^{(\ell-m-1)N}}\right)z\right]$$

$$=\sum_{n,k=0}^{\infty}\frac{\prod_{j=1}^{\ell}(c_j)_n}{\prod_{j=1}^{m}(d_j)_n}\Theta_{n+Nk}\frac{\prod_{j=1}^{r}(g_j+n+Nk)_{Lk}}{\prod_{j=1}^{s}(h_j+n+Nk)_{Mk}}\frac{(a_0;x)_k(a_1)_k\cdots(a_p)_k}{(b_1)_k\cdots(b_q)_k}\cdot\frac{t^n}{n!}\frac{\left(z\left\{(-1)^{\ell-m+1}\,t\right\}^N\right)^k}{k!}\qquad(x\geqq 0;\ \ell,m,p,q,r,s\in\mathbb{N}_0;\ L,M,N\in\mathbb{N})\tag{41}$$

and

$$\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{\ell}(c_j)_n}{\prod_{j=1}^{m}(d_j)_n}\frac{\Theta_n\,t^n}{n!}\;{}_{p+Lr+(m+1)N+1}\Gamma_{q+Ms+N\ell}\left[\begin{matrix}\Delta(N;-n),\Delta(L,r;g_j+n),\Delta(N,m;1-d_j-n),(a_0,x),a_1,\cdots,a_p;\\ \Delta(M,s;h_j+n),\Delta(N,\ell;1-c_j-n),b_1,\cdots,b_q;\end{matrix}\left(\frac{L^{Lr}}{M^{Ms}\,N^{(\ell-m-1)N}}\right)z\right]$$

$$=\sum_{n,k=0}^{\infty}\frac{\prod_{j=1}^{\ell}(c_j)_n}{\prod_{j=1}^{m}(d_j)_n}\Theta_{n+Nk}\frac{\prod_{j=1}^{r}(g_j+n+Nk)_{Lk}}{\prod_{j=1}^{s}(h_j+n+Nk)_{Mk}}\frac{[a_0;x]_k(a_1)_k\cdots(a_p)_k}{(b_1)_k\cdots(b_q)_k}\cdot\frac{t^n}{n!}\frac{\left(z\left\{(-1)^{\ell-m+1}\,t\right\}^N\right)^k}{k!}\qquad(x\geqq 0;\ \ell,m,p,q,r,s\in\mathbb{N}_0;\ L,M,N\in\mathbb{N}),\tag{42}$$

provided that both sides of (41) *and* (42) *exist.*

Several interesting corollaries and consequences of the generating functions (41) and (42) asserted by Theorem 2 are worthy of mention here. First of all, if we set
$\Theta_n=(\lambda)_n\quad(n\in\mathbb{N}_0)$ and $N-1=\ell=m=r=s=0$, we find from (41) and (42) that

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\;{}_{p+2}\gamma_q\left[\begin{matrix}-n,(a_0,x),a_1,\cdots,a_p;\\ b_1,\cdots,b_q;\end{matrix}\,z\right]t^n=(1-t)^{-\lambda}\;{}_{p+2}\gamma_q\left[\begin{matrix}\lambda,(a_0,x),a_1,\cdots,a_p;\\ b_1,\cdots,b_q;\end{matrix}\,-\frac{zt}{1-t}\right]\qquad(x\geqq 0;\ |t|<1;\ \lambda\in\mathbb{C})\tag{43}$$

and

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\;{}_{p+2}\Gamma_q\left[\begin{matrix}-n,(a_0,x),a_1,\cdots,a_p;\\ b_1,\cdots,b_q;\end{matrix}\,z\right]t^n=(1-t)^{-\lambda}\;{}_{p+2}\Gamma_q\left[\begin{matrix}\lambda,(a_0,x),a_1,\cdots,a_p;\\ b_1,\cdots,b_q;\end{matrix}\,-\frac{zt}{1-t}\right]\qquad(x\geqq 0;\ |t|<1;\ \lambda\in\mathbb{C}).\tag{44}$$

Secondly, upon setting $\Theta_n=(\lambda)_n\quad(n\in\mathbb{N}_0)$ and $N-1=\ell=m=r-1=s=0\quad(g_1=\lambda)$, the generating functions (41) and (42) yield the following special cases:

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\;{}_{p+3}\gamma_q\left[\begin{matrix}-n,\lambda+n,(a_0,x),a_1,\cdots,a_p;\\ b_1,\cdots,b_q;\end{matrix}\,z\right]t^n=(1-t)^{-\lambda}\;{}_{p+3}\gamma_q\left[\begin{matrix}\Delta(2;\lambda),(a_0,x),a_1,\cdots,a_p;\\ b_1,\cdots,b_q;\end{matrix}\,-\frac{4zt}{(1-t)^2}\right]\tag{45}$$

$$(x\geqq 0;\ |t|<1;\ \lambda\in\mathbb{C})$$

and

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\;{}_{p+3}\Gamma_q\left[\begin{matrix}-n,\lambda+n,(a_0,x),a_1,\cdots,a_p;\\ b_1,\cdots,b_q;\end{matrix}\,z\right]t^n=(1-t)^{-\lambda}\;{}_{p+3}\Gamma_q\left[\begin{matrix}\Delta(2;\lambda),(a_0,x),a_1,\cdots,a_p;\\ b_1,\cdots,b_q;\end{matrix}\,-\frac{4zt}{(1-t)^2}\right]\tag{46}$$

$$(x\geqq 0;\ |t|<1;\ \lambda\in\mathbb{C}).$$

Thirdly, we set $\Theta_n=1\quad(n\in\mathbb{N}_0)$ and $N-1=\ell-1=m=r=s=0\quad(c_1=\lambda)$. Then the generating functions (41) and (42) reduce to the following forms:

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\;{}_{p+2}\gamma_{q+1}\left[\begin{matrix}-n,(a_0,x),a_1,\cdots,a_p;\\ 1-\lambda-n,b_1,\cdots,b_q;\end{matrix}\,z\right]t^n=(1-t)^{-\lambda}\;{}_{p+1}\gamma_q\left[\begin{matrix}(a_0,x),a_1,\cdots,a_p;\\ b_1,\cdots,b_q;\end{matrix}\,zt\right]\qquad(x\geqq 0;\ |t|<1;\ \lambda\in\mathbb{C})\tag{47}$$

and

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\;{}_{p+2}\Gamma_{q+1}\left[\begin{matrix}-n,(a_0,x),a_1,\cdots,a_p;\\ 1-\lambda-n,b_1,\cdots,b_q;\end{matrix}\,z\right]t^n=(1-t)^{-\lambda}\;{}_{p+1}\Gamma_q\left[\begin{matrix}(a_0,x),a_1,\cdots,a_p;\\ b_1,\cdots,b_q;\end{matrix}\,zt\right]\qquad(x\geqq 0;\ |t|<1;\ \lambda\in\mathbb{C}).\tag{48}$$

Fourthly, if we set

$\Theta_n=1\quad(n\in\mathbb{N}_0)$ and $N-1=\ell=m=r=s=0$,

then the generating functions (41) and (42) would immediately yield

$$\sum_{n=0}^{\infty}\;{}_{p+2}\gamma_q\left[\begin{matrix}-n,(a_0,x),a_1,\cdots,a_p;\\ b_1,\cdots,b_q;\end{matrix}\,z\right]\frac{t^n}{n!}=e^t\;{}_{p+1}\gamma_q\left[\begin{matrix}(a_0,x),a_1,\cdots,a_p;\\ b_1,\cdots,b_q;\end{matrix}\,-zt\right]\qquad(x\geqq 0;\ |t|<1)\tag{49}$$

and

$$\sum_{n=0}^{\infty} {}_{p+2}\Gamma_q \left[\begin{matrix} -n, (a_0,x), a_1, \cdots, a_p; \\ b_1, \cdots, b_q; \end{matrix} z \right] \frac{t^n}{n!}$$

$$= e^t \, {}_{p+1}\Gamma_q \left[\begin{matrix} (a_0,x), a_1, \cdots, a_p; \\ b_1, \cdots, b_q; \end{matrix} -zt \right] \qquad (x \geqq 0; \; |t| < 1). \tag{50}$$

Lastly, upon setting

$$\Theta_n = (\lambda)_n \quad (n \in \mathbb{N}_0) \qquad \text{and} \qquad \ell = m = r = s = 0,$$

the generating functions (41) and (42) yield the following results:

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} \, {}_{p+N+1}\gamma_q \left[\begin{matrix} \Delta(N;-n), (a_0,x), a_1, \cdots .a_p; \\ b_1, \cdots, b_q; \end{matrix} z \right] t^n$$

$$= (1-t)^{-\lambda} \, {}_{p+N+1}\gamma_q \left[\begin{matrix} \Delta(N;\lambda), (a_0,x), a_1, \cdots .a_p; \\ b_1, \cdots, b_q; \end{matrix} z \left(-\frac{t}{1-t} \right)^N \right] \tag{51}$$

$$(x \geqq 0; \; |t| < 1; \; \lambda \in \mathbb{C}; \; N \in \mathbb{N})$$

and

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} \, {}_{p+N+1}\Gamma_q \left[\begin{matrix} \Delta(N;-n), (a_0,x), a_1, \cdots .a_p; \\ b_1, \cdots, b_q; \end{matrix} z \right] t^n$$

$$= (1-t)^{-\lambda} \, {}_{p+N+1}\Gamma_q \left[\begin{matrix} \Delta(N;\lambda), (a_0,x), a_1, \cdots .a_p; \\ b_1, \cdots, b_q; \end{matrix} z \left(-\frac{t}{1-t} \right)^N \right] \tag{52}$$

$$(x \geqq 0; \; |t| < 1; \; \lambda \in \mathbb{C}; \; N \in \mathbb{N}).$$

Remark 3. The generating functions (43) and (44) are obvious consequences of the generating functions (51) and (52), respectively, in their special case when $N = 1$. Moreover, the generating functions (43) and (44) would follow also as the limit cases of the generating functions (45) and (46), respectively, if we first replace z in (45) and (46) by z/λ and then proceed to the limit when $|\lambda| \to \infty$.

Remark 4. The generating functions (49) and (50) are the limit cases of the generating functions (43) and (44), respectively, if we first replace t in (43) and (44) by t/λ and then proceed to the limit when $|\lambda| \to \infty$. Furthermore, the generating functions (49) and (50) can be deduced also as the limit cases of the generating functions (47) and (48), respectively, if we first replace t and z in (47) and (48) by t/λ and λz, respectively, and then proceed to the limit when $|\lambda| \to \infty$.

Remark 5. For $x = 0$, the generating functions (43) to (48) were given by Chaundy [11] who also gave a much more general result than the case of the generating functions (49) and (50) when $x = 0$ (see, for details, [42, Section 2.6]; see also [33]). In fact, as already observed by Srivastava [33, p. 329], the case of the generating functions (49) and (50) when $x = 0$ (see [33, p. 329, Equation (1.7)] and [42, p. 141, Equation 2.6 (19)]) is usually attributed to Rainville (*cf.*, *e.g.*, [13, p. 267, Equation 19.10 (25)]). The case of the last generating functions (51) and (52) when $x = 0$ is due to Brafman (see [9, p. 187, Equation (55)] and [42, p. 136, Equation 2.6 (2)]). See also [27] and [42, p. 178, Problem 34] for more general families of hypergeometric generating functions than the aforecited results of Brafman [9] and Chaundy [11].

4 Reducible Cases of the Generalized Incomplete Hypergeometric Functions

Our main results in this section are contained in Theorem 3 below (see also [49]).

Theorem 3. *The following reduction formulas hold true for the generalized incomplete hypergeometric functions* ${}_p\gamma_q$ *and* ${}_p\Gamma_q$ *:*

$${}_{p+1}\gamma_q \left[\begin{matrix} (a_0,x), b_1+m_1, \cdots, b_r+m_r, a_{r+1}, \cdots, a_p; \\ b_1, \cdots, b_r, b_{r+1}, \cdots, b_q; \end{matrix} z \right] = \sum_{j_1=0}^{m_1} \cdots \sum_{j_r=0}^{m_r} \Lambda(j_1, \cdots, j_r)$$

$$\cdot z^{J_r} \, {}_{p-r+1}\gamma_{q-r} \left[\begin{matrix} (a_0+J_r,x), a_{r+1}+J_r, \cdots, a_p+J_r; \\ b_{r+1}+J_r, \cdots, b_q+J_r; \end{matrix} z \right] \tag{53}$$

$(x \geqq 0; \; r \leqq \min\{p,q\}; \; p,q \in \mathbb{N}_0; \; p < q \quad \text{when} \quad z \in \mathbb{C}; \; p = q \quad \text{when} \quad |z| < 1)$

and

$${}_{p+1}\Gamma_q \left[\begin{matrix} (a_0,x), b_1+m_1, \cdots, b_r+m_r, a_{r+1}, \cdots, a_p; \\ b_1, \cdots, b_r, b_{r+1}, \cdots, b_q; \end{matrix} z \right] = \sum_{j_1=0}^{m_1} \cdots \sum_{j_r=0}^{m_r} \Lambda(j_1, \cdots, j_r)$$

$$\cdot z^{J_r} \, {}_{p-r+1}\Gamma_{q-r} \left[\begin{matrix} (a_0+J_r,x), a_{r+1}+J_r, \cdots, a_p+J_r; \\ b_{r+1}+J_r, \cdots, b_q+J_r; \end{matrix} z \right] \tag{54}$$

$(x \geqq 0; \; r \leqq \min\{p,q\}; \; p,q \in \mathbb{N}_0; \; p < q \quad \text{when} \quad z \in \mathbb{C}; \; p = q \quad \text{when} \quad |z| < 1),$

where, for convenience,

$$J_r := j_1 + \cdots + j_r$$

and

$$\Lambda(j_1, \cdots, j_r) = \binom{m_1}{j_1} \cdots \binom{m_r}{j_r} \frac{(b_2+m_2)_{J_1} \cdots (b_r+m_r)_{J_{r-1}} \, (a_{r+1})_{J_r} \cdots (a_p)_{J_r}}{(b_1)_{J_1} \cdots (b_r)_{J_r} (b_{r+1})_{J_r} \cdots (b_q)_{J_r}}.$$

Proof. Our demonstrations of the reduction formulas (53) and (54) are based upon the principle of mathematical induction on the integer $r \in \mathbb{N}$. Indeed, in its special case when $r = 1$, the reduction formula (53) can be written in the following form:

$${}_{p+1}\gamma_q \left[\begin{matrix} (a_0,x), b_1+m_1, a_2, \cdots, a_p; \\ b_1, \cdots, b_q; \end{matrix} z \right] = \sum_{j=0}^{m_1} \binom{m_1}{j} \frac{(a_2)_j \cdots (a_p)_j}{(b_1)_j \cdots (b_q)_j}$$

$$\cdot z^j \, {}_p\gamma_{q-1} \left[\begin{matrix} (a_0+j,x), a_2+j, \cdots, a_p+j; \\ b_2+j, \cdots, b_q+j; \end{matrix} z \right] \tag{55}$$

$(x \geqq 0;\ m_1, p, q \in \mathbb{N}_0;\ p < q \quad \text{when} \quad z \in \mathbb{C};\ p = q \quad \text{when} \quad |z| < 1).$

In order to prove this last reduction formula (55), we denote its right-hand side by Ω and apply the identity (3) and the definition (4). We thus find that

$$\Omega := \sum_{j=0}^{m_1} \binom{m_1}{j} \frac{(a_2)_j \cdots (a_p)_j}{(b_1)_j \cdots (b_q)_j} \cdot z^j \ {}_p\gamma_{q-1} \left[\begin{array}{r} (a_0+j,x), a_2+j, \cdots, a_p+j; \\ b_2+j, \cdots, b_q+j; \end{array} z \right]$$

$$= \sum_{n=0}^{\infty} \frac{\gamma(a_0+n,x)}{\Gamma(a_0+n)} \frac{(a_2)_n \cdots (a_p)_n}{(b_2)_n \cdots (b_q)_n} \frac{z^n}{n!} \sum_{j=0}^{\min\{m_1,n\}} \frac{(-m_1)_j (-n)_j}{j!(b_1)_j}$$

$$= \sum_{n=0}^{\infty} \frac{\gamma(a_0+n,x)}{\Gamma(a_0+n)} \frac{(b_1+m_1)_n (a_2)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{z^n}{n!}, \tag{56}$$

where we have also applied the familiar Chu-Vandermonde summation formula (see, for example, [30, p. 243, Entry (II.4)]):

$${}_2F_1 \left[\begin{array}{r} -n, b; \\ c; \end{array} 1 \right] = \frac{(c-b)_n}{(c)_n} \qquad (n \in \mathbb{N}_0;\ b \in \mathbb{C};\ c \in \mathbb{C} \setminus \mathbb{Z}_0^-), \tag{57}$$

$\mathbb{Z}_0^-$ being the set of *nonpositive* integers.

Upon interpreting the last member of (56) by means of the definitions (8) and (10), the reduction formula (55) would follow immediately. The general reduction formula (53) can then be deduced by repeated applications of (56) to itself and appealing to the principle of mathematical induction on $r \in \mathbb{N}$.

The second assertion (54) of Theorem 3 can be proven in an analogous manner by using the definitions (9) and (11) instead of the definitions (8) and (10), respectively.

Remark 6. In its special case when $x = 0$, each of the assertions (53) and (54) of Theorem 3 reduces at once to the following known reduction formula (see, for example, [16] and [32]; see also [48, p. 1080] and the references to more general results on hypergeometric reduction formulas, which are cited in [48]):

$${}_pF_q \left[\begin{array}{r} b_1+m_1, \cdots, b_r+m_r, a_{r+1}, \cdots, a_p; \\ b_1, \cdots, b_r, b_{r+1}, \cdots, b_q; \end{array} z \right]$$

$$= \sum_{j_1=0}^{m_1} \cdots \sum_{j_r=0}^{m_r} \Lambda(j_1, \cdots, j_r) z^{J_r} \ {}_{p-r}F_{q-r} \left[\begin{array}{r} a_{r+1}+J_r, \cdots, a_p+J_r; \\ b_{r+1}+J_r, \cdots, b_q+J_r; \end{array} z \right] \tag{58}$$

$(r \leqq \min\{p,q\};\ p, q \in \mathbb{N}_0;\ p < q+1 \quad \text{when} \quad z \in \mathbb{C};\ p = q+1 \quad \text{when} \quad |z| < 1),$

where, as also in (53) and (54),

$$J_r := j_1 + \cdots + j_r$$

and

$$\Lambda(j_1, \cdots, j_r) = \binom{m_1}{j_1} \cdots \binom{m_r}{j_r} \frac{(b_2+m_2)_{J_1} \cdots (b_r+m_r)_{J_{r-1}}}{(b_1)_{J_1} \cdots (b_r)_{J_r} (b_{r+1})_{J_r} \cdots (b_q)_{J_r}} (a_{r+1})_{J_r} \cdots (a_p)_{J_r}.$$

Remark 7. The general hypergeometric identity (58) was proved by Karlsson [16] and (in two markedly different *simpler* ways) by Srivastava [32]. More interestingly, various generalizations and *basic* (or q-) extensions of the hypergeometric identity (58) can be found in several sequels to the works by Karlsson [16] and Srivastava [32] (see, for example, [25]). Reference [26], on the other hand, contains *further* general results stemming from the hypergeometric identity (58) including *multivariable* generalizations. Furthermore, Karlsson's proof of the Karlsson-Minton summation formula (see, for details, [16]; see also [22] and [48, p. 1080, Equation (20)]) was based upon the hypergeometric reduction formula (58).

Remark 8. Numerous *further* corollaries and consequences of the general results asserted by Theorems 1, 2 and 3 can indeed be derived in a manner analogous to those of the specializations that we have indicated in this presentation. We choose to omit the details involved in deriving these *additional* corollaries and consequences of Theorems 1, 2 and 3.

5 Concluding Remarks and Observations

In view of the demonstrated potential for applications of the generalized incomplete hypergeometric functions ${}_p\gamma_q$ and ${}_p\Gamma_q$ in many diverse areas of mathematical, physical, engineering and statistical sciences (see, for details, [37] and the references cited therein), we have successfully presented here a systematic investigation of several *further* properties of these generalized incomplete hypergeometric functions and some classes of incomplete hypergeometric polynomials associated with them. Specifically, we make use of several such combinatorial identities as Gould's identity (18) below, which stem essentially from the Lagrange expansion theorem (see, for example, [42, Chapter 7]), with a view to deriving many general families of generating functions for a certain class of incomplete hypergeometric polynomials associated with these generalized incomplete hypergeometric functions. We have also indicated various (known or new) special cases and consequences of the results presented in this article. Here, in this concluding section, we choose to point out several other generalizations of the Pochhammer symbol and their associated families of hypergeometric functions and hypergeometric polynomials.

First of all, in the widely-scattered literature on the subject of this paper, one can find several interesting generalizations of the familiar (Euler's) gamma function $\Gamma(z)$ defined by (1), as well as the corresponding generalizations and extensions of the Beta function $B(\alpha,\beta)$, the hypergeometric functions ${}_1F_1$ and ${}_2F_1$, and the generalized hypergeometric functions ${}_pF_q$. For example, for a suitably bounded sequence $\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}$ of essentially arbitrary (real or complex) numbers, Srivastava *et al.* [44, p. 243 *et seq.*] recently considered the function $\Theta(z)$ given by

$$\Theta\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0};z\right)=\begin{cases}\sum_{\ell=0}^{\infty}\kappa_\ell\,\frac{z^\ell}{\ell!} & (|z|<R;\ R>0;\ \kappa_0:=1)\\ \mathfrak{M}_0\,z^\omega\,\exp(z)\left[1+O\left(\frac{1}{|z|}\right)\right] & (|z|\to\infty;\ \mathfrak{M}_0>0;\ \omega\in\mathbb{C})\end{cases} \tag{59}$$

for some suitable constants $\mathfrak{M}_0$ and ω depending essentially upon the sequence $\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}$.Then, in terms of the function $\Theta(z)$ defined by (59), Srivastava *et al.* [44] introduced a remarkably deep generalization of the extended Gamma function $\Gamma_\mathfrak{p}(z)$, the extended Beta function $B_\mathfrak{p}(\alpha,\beta)$ and the extended hypergeometric function $F_\mathfrak{p}(a,b;c;z)$ (see, for details, [6] *and* [5]) by

$$\Gamma_\mathfrak{p}^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}(z):=\int_0^\infty t^{z-1}\,\Theta\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0};-t-\frac{\mathfrak{p}}{t}\right)dt \tag{60}$$

$$\left(\Re(z)>0;\ \Re(\mathfrak{p})\geqq 0\right),$$

$$\mathfrak{B}_\mathfrak{p}^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}(\alpha,\beta)=\mathfrak{B}^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}(\alpha,\beta;\mathfrak{p})$$
$$:=\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,\Theta\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0};-\frac{\mathfrak{p}}{t(1-t)}\right)dt \tag{61}$$

$$\left(\min\{\Re(\alpha),\Re(\beta)\}>0;\ \Re(\mathfrak{p})\geqq 0\right)$$

and

$$\mathfrak{F}_\mathfrak{p}^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}(a,b;c;z):=\frac{1}{B(b,c-b)}\sum_{n=0}^{\infty}(a)_n\,\mathfrak{B}_\mathfrak{p}^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}(b+n,c-b)\,\frac{z^n}{n!} \tag{62}$$

$$\left(|z|<1;\ \Re(c)>\Re(b)>0;\ \Re(\mathfrak{p})\geqq 0\right),$$

provided that the defining integrals in (60), (61) and (62) exist.

Remark 9. In their special cases when $\kappa_\ell=1\quad(\ell\in\mathbb{N}_0)$, the equations (60), (61) and (62) would reduce immediately to the corresponding definitions of the gamma function $\Gamma(z)$, the Beta function $B(\alpha,\beta)$ and the hypergeometric function ${}_2F_1(a,b;c;z)$, respectively. The definition (4) of the Pochhammer symbol $(\lambda)_\nu\ \ (\lambda,\nu\in\mathbb{C})$ can thus be generalized as follows:

$$\left(\lambda;\mathfrak{p},\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)_\nu:=\frac{\Gamma_\mathfrak{p}^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}(\lambda+\nu)}{\Gamma_\mathfrak{p}^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}(\lambda)}\qquad(\lambda,\nu\in\mathbb{C}). \tag{63}$$

Based upon the definition (63), we can introduce a family of generalized hypergeometric functions given by

$${}_p\mathscr{F}_q\left[\begin{matrix}\left(a_1;\mathfrak{p},\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right),a_2,\cdots,a_p;\\ b_1,\cdots,b_q;\end{matrix}\ z\right]:=\sum_{n=0}^{\infty}\frac{\left(a_1;\mathfrak{p},\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)_n(a_2)_n\cdots(a_p)_n}{(b_1)_n\cdots(b_q)_n}\,\frac{z^n}{n!}, \tag{64}$$

provided that the series on the right-hand side converges. Indeed, whenever one or the other of the numerator parameters $a_2,\cdots,a_p$ in (64) is a *nonpositive* integer, the definition (64) will define the corresponding family of hypergeometric polynomials.

Next, in his investigation of the asymptotic expansions of a class of *branch-cut* integrals occurring in diffraction theory by means of the Wiener-Hopf technique, Kobayashi (see [18] and [19]) encountered an integral of the following type:

$$\Gamma_m(u,v):=\int_0^\infty\frac{t^{u-1}\,e^{-t}}{(v+t)^m}\,dt=v^{u-m}\int_0^\infty\frac{t^{u-1}\,e^{-vt}}{(1+t)^m}\,dt \tag{65}$$

$$\left(\min\{\Re(u),\Re(v)\}>0;\ m\in\mathbb{N}_0\right),$$

so that, in the special case when $m=0$ and $v=1$, we have

$$\Gamma_0(u,1)=\int_0^\infty t^{u-1}\,e^{-t}\,dt=:\Gamma(u)\qquad\left(\Re(u)>0\right). \tag{66}$$

Remark 10. In view of their importance and usefulness in diffraction theory and probability distributions, several extensions and generalizations of the gamma-type function $\Gamma_m(u,v)$ defined by (66) were investigated in recent years (see, for details, [45] for a systematic study of a unified presentation of all such extensions and generalizations).

Finally, by introducing a generalization of the definition (4) of the Pochhammer symbol $(\lambda)_\nu\ \ (\lambda,\nu\in\mathbb{C})$ given by

$$(\lambda;m,v)_\nu:=\frac{\Gamma_m(\lambda+\nu,v)}{\Gamma_m(\lambda,v)}\qquad\left(\lambda,\nu\in\mathbb{C};\ \Re(v)>0;\ m\in\mathbb{N}_0\right), \tag{67}$$

we can define a family of generalized hypergeometric functions as follows:

$${}_p\mathscr{G}_q\left[\begin{matrix}(a_1;m,v),a_2,\cdots,a_p;\\ b_1,\cdots,b_q;\end{matrix}\ z\right]:=\sum_{n=0}^{\infty}\frac{(a_1;m,v)_n(a_2)_n\cdots(a_p)_n}{(b_1)_n\cdots(b_q)_n}\,\frac{z^n}{n!}, \tag{68}$$

provided that each member of (68) exists. In fact, whenever one or the other of the numerator parameters $a_2,\cdots,a_p$ in (68) is a *nonpositive* integer, the definition (68) will define the corresponding family of hypergeometric polynomials.

Other (known or new) extensions of the familiar (Euler's) gamma function $\Gamma(z)$ will lead us similarly to the corresponding generalizations of the definition (4) of the Pochhammer symbol $(\lambda)_\nu\ \ (\lambda,\nu\in\mathbb{C})$ and, consequently, also to the resulting families of generalized hypergeometric functions and generalized hypergeometric polynomials. Moreover, by suitably applying the methodology and techniques described fairly adequately by Srivastava *et al.* [37] *and* in the preceding sections, one can derive analogous properties and relationships involving such families of generalized hypergeometric functions and generalized hypergeometric polynomials as those stemming from the definitions (64) and (68).

Appendix

Here, for the convenience of the interested reader, we recall the celebrated *Lagrange Expansion Theorem* (see, for details, [28, p. 146, Problem 206] and [53, p. 133]; see also [42, p. 354 *et seq.*]) and its such already used consequences as the combinatorial identities (15) and (16).

Lagrange Expansion Theorem. *Let the function* $\varphi(z)$ *be holomorphic at the point* $z = z_0$ *in the complex z-plane and let*

$$\varphi(z_0) \neq 0. \qquad (A.1)$$

Suppose also that

$$z = z_0 + w\varphi(z). \qquad (A.2)$$

Then an analytic function $f(z)$, *which is holomorphic at the point* $z = z_0$, *can be expanded in a power series in w as follows*:

$$f(z) = f(z_0) + \sum_{n=1}^{\infty} \frac{w^n}{n!} \frac{d^{n-1}}{dz^{n-1}} \left\{ f'(z)[\varphi(z)]^n \right\} \bigg|_{z=z_0} \qquad (A.3)$$

or, *equivalently*,

$$\frac{f(z)}{1 - w\varphi'(z)} = \sum_{n=0}^{\infty} \frac{w^n}{n!} \frac{d^n}{dz^n} \left\{ f(z)[\varphi(z)]^n \right\} \bigg|_{z=z_0}. \qquad (A.4)$$

For $\varphi(z) \equiv 1$, both (A.3) and (A.4) evidently yield the relatively more familiar *Taylor-Maclaurin Expansion*:

$$f(z) = \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{n!} f^{(n)}(z_0). \qquad (A.5)$$

In order to derive (for example) the *first* combinatorial identity (15), which we have already used in Section 2 of this presentation, we set

$$z_0 = 0, \quad f(z) = (1+z)^{\alpha} \quad \text{and} \quad \varphi(z) = (1+z)^{\beta+1}, \qquad (A.6)$$

and then make the following notational changes:

$$w \mapsto t \quad \text{and} \quad z \mapsto \zeta,$$

where α and β are complex numbers independent of n and ζ is a function of t defined *implicitly* by (17). Moreover, as we pointed out in Section 2, the expansion formula (15) can easily be shown to imply the expansion formula (16). In fact, the expansion formulas (15) and (16) are equivalent (see, for details, [42, pp. 354–356]).

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