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Common Fixed Point Theorems in Complex Valued Metric Spaces

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Abstract: In this paper, first we prove a common fixed point theorem for a pair of weakly compatible self maps in complex valued metric space for rational inequality. Secondly, we prove common fixed point theorems for weakly compatible mappings along with (CLR_g) and E.A. properties.

Keywords: Complex valued metric space, Partial order, Weakly compatible maps, E.A. property, (CLR_g) property.

1 Introduction

In 2011, Azam et. al [5] introduced the notion of complex valued metric space which is a generalization of the classical metric space. They established some fixed point results for mappings satisfying a rational inequality. The idea of complex valued metric spaces can be exploited to define complex valued normed spaces and complex valued Hilbert spaces; additionally, it offers numerous research activities in mathematical analysis.

A complex number $z \in \mathbb{C}$ is an ordered pair of real numbers, whose first co-ordinate is called $\text{Re}(z)$ and second coordinate is called $\text{Im}(z)$. Thus a complex-valued metric d is a function from a set $X \times X$ into $\mathbb{C}$, where X is a nonempty set and $\mathbb{C}$ is the set of complex numbers.

Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order $\precsim$ on $\mathbb{C}$ as follows:
$z_1 \precsim z_2$ if and only if $\text{Re}(z_1) \leq \text{Re}(z_2)$ and $\text{Im}(z_1) \leq \text{Im}(z_2)$, that is $z_1 \precsim z_2$, if one of the following holds

(C1) $\text{Re}(z_1) = \text{Re}(z_2)$ and $\text{Im}(z_1) = \text{Im}(z_2)$;
(C2) $\text{Re}(z_1) < \text{Re}(z_2)$ and $\text{Im}(z_1) = \text{Im}(z_2)$;
(C3) $\text{Re}(z_1) = \text{Re}(z_2)$ and $\text{Im}(z_1) < \text{Im}(z_2)$;
(C4) $\text{Re}(z_1) < \text{Re}(z_2)$ and $\text{Im}(z_1) < \text{Im}(z_2)$.

In particular, we will write $z_1 \precnsim z_2$ if $z_1 \neq z_2$ and one of (C2), (C3), and (C4) is satisfied and we will write $z_1 \prec z_2$ if only (C4) is satisfied.

*Remark.*We note that the following statements hold:

(i) $a, b \in \mathbb{R}$ and $a \leq b \Rightarrow az \precsim bz$ for all $z \in \mathbb{C}$.
(ii) $0 \precsim z_1 \precnsim z_2 \Rightarrow |z_1| < |z_2|$,
(iii) $z_1 \precsim z_2$ and $z_2 \prec z_3 \Rightarrow z_1 \prec z_3$.

Definition 1.*Let X be a nonempty set. Suppose that the mapping $d : X \times X \to \mathbb{C}$ satisfies the following conditions:*

(i)$0 \precsim d(x,y)$, *for all* $x, y \in X$ *and* $d(x,y) = 0$ *if and only if* $x = y$;
(ii)$d(x,y) = d(y,x)$ *for all* $x, y \in X$;
(iii)$d(x,y) \precsim d(x,z) + d(z,y)$, *for all* $x, y, z \in X$.

Then d is called a complex valued metric on X and (X,d) is called a complex valued metric space.

*Example 1.*Let $X = \mathbb{C}$. Define the mapping $d : X \times X \to \mathbb{C}$ by

$$d(z_1, z_2) = 2i|z_1 - z_2|, \quad \text{for all } z_1, z_2 \in X.$$

Then (X,d) is a complex valued metric space.

Definition 2.*Let (X,d) be a complex valued metric space, $\{x_n\}$ be a sequence in X and $x \in X$.*

(i)*If for every $c \in \mathbb{C}$, with $0 \prec c$ there is $k \in \mathbb{N}$ such that for all $n > k$, $d(x_n, x) \prec c$, then $\{x_n\}$ is said to be convergent, $\{x_n\}$ converges to x and x is the limit point of $\{x_n\}$. We denote this by $\{x_n\} \to x$ as $n \to \infty$ or $\lim_{n\to\infty} x_n = x$.*

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(ii)*If for every* $c \in \mathbb{C}$, *with* $0 \prec c$ *there is* $k \in \mathbb{N}$ *such that for all* $n > k$, $d(x_n, x_{n+m}) \prec c$, *where* $m \in \mathbb{N}$, *then* $\{x_n\}$ *is said to be Cauchy sequence.*

(iii)*If every Cauchy sequence in* X *is convergent, then* (X,d) *is said to be a complete complex valued metric space.*

Lemma 1.*Let* (X,d) *be a complex valued metric space and let* $\{x_n\}$ *be a sequence in* X. *Then* $\{x_n\}$ *converges to* x *if and only if* $|d(x_n,x)| \to 0$ *as* $n \to \infty$.

Lemma 2.*Let* (X,d) *be a complex valued metric space and let* $\{x_n\}$ *be a sequence in* X. *Then* $\{x_n\}$ *is a Cauchy sequence if and only if* $|d(x_n,x_{n+m})| \to 0$ *as* $n \to \infty$, *where* $m \in \mathbb{N}$.

In 1996, Jungck [6] introduced the concept of weakly compatible maps as follows:

Definition 3.*Two self maps* f *and* g *are said to be weakly compatible if they commute at coincidence points.*

In 2002, Aamri et al. [1] introduced the notion of E.A. property as follows:

Definition 4.*Two self-mappings* f *and* g *of a metric space* (X,d) *are said to satisfy E.A. property if there exists a sequence* $\{x_n\}$ *in* X *such that* $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ *for some* t *in* X.

In 2011, Sintunavarat et al. [7] introduced the notion of (CLR$_g$) property as follows:

Definition 5.*Two self-mappings* f *and* g *of a metric space* (X,d) *are said to satisfy (CLR$_g$) property if there exists a sequence* $\{x_n\}$ *in* X *such that* $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = gx$ *for some* x *in* X.

In the same way, we can introduce these notions in complex valued metric spaces.

*Example 2.*Let $X = \mathbb{C}$. Define the mapping $d : X \times X \to \mathbb{C}$ by

$$d(z_1,z_2) = 2i|z_1 - z_2|, \text{ for all } z_1, z_2 \in X.$$

Then (X,d) is a complex valued metric space.
Define $f,g : X \to X$ by

$$fz = z + i \text{ and } gz = 2z, \text{ for all } z \in X.$$

Consider a sequence $\{z_n\} = \left\{ i - \frac{1}{n} \right\}, n \in \mathbb{N}$, in X, then

$$\lim_{n\to\infty} fz_n = \lim_{n\to\infty} (z_n + i) = \lim_{n\to\infty} i - \frac{1}{n} + i = 2i,$$
$$\lim_{n\to\infty} gz_n = \lim_{n\to\infty} 2z_n = \lim_{n\to\infty} 2\left(i - \frac{1}{n}\right) = 2i,$$

where $2i \in X$.
Thus, f and g satisfies E.A. property.
Also, we have

$$\lim_{n\to\infty} fz_n = \lim_{n\to\infty} gz_n = 2i = g(i), \quad \text{where } i \in X.$$

Thus, f and g satisfies (CLRg) property.
Now, we shall prove our results relaxing the condition of complex valued metric space being complete.

2 Weakly Compatible Maps

Theorem 1.*Let* f *and* g *be self maps of a complex valued metric space* (X,d) *satisfying the following:*

(2.1) $fX \subseteq gX$,

(2.2) $$d(fx,fy) \precsim Ad(gx,gy) + B\frac{d(gx,fx)d(fy,gy)}{1+d(gx,gy)} + C\frac{d(gx,fy)d(gx,gy)}{1+d(gx,gy)} + D\frac{d(gx,fx)d(gx,gy)}{1+d(gx,gy)} + E\frac{d(gx,fy)d(fy,gy)}{1+d(gx,gy)}, \text{ for all } x, y \text{ in } X,$$

where A, B, C, D *and* E *are non-negative constants with* $A+B+C+D+E<1$,

(2.3) gX *is a complete subspace of* X.

Then f *and* g *have a coincidence point.*

Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point.

Proof.Let$x_0 \in X$. From (2.1), we can construct sequences $\{x_n\}$ and $\{y_n\}$ in X by $y_n = gx_{n+1} = fx_n$, $n = 0,1,2,\ldots$.
From (2.2), we have

$$\begin{aligned}
d(y_{n+1},y_n) &= d(fx_{n+1},fx_n) \\
&= Ad(gx_{n+1},gx_n) \\
&\quad + B\frac{d(gx_{n+1},fx_{n+1})d(fx_n,gx_n)}{1+d(gx_{n+1},gx_n)} \\
&\quad + C\frac{d(gx_{n+1},fx_n)d(gx_n,gx_{n+1})}{1+d(gx_{n+1},gx_n)} \\
&\quad + D\frac{d(gx_n,fx_n)d(gx_n,gx_{n+1})}{1+d(gx_{n+1},gx_n)} \\
&\quad + E\frac{d(gx_{n+1},fx_n)d(gx_n,gx_{n+1})}{1+d(gx_{n+1},gx_n)} \\
&= Ad(y_n,y_{n-1}) + B\frac{d(y_n,y_{n+1})d(y_n,y_{n-1})}{1+d(y_n,y_{n-1})} \\
&\quad + C\frac{d(y_n,y_n)d(y_{n-1},y_n)}{1+d(y_n,y_{n-1})} \\
&\quad + D\frac{d(y_{n-1},y_n)d(y_{n-1},y_n)}{1+d(y_n,y_{n-1})} \\
&\quad + E\frac{d(y_n,y_n)d(y_n,y_{n-1})}{1+d(y_n,y_{n-1})} \\
&= Ad(y_n,y_{n-1}) + B\frac{d(y_n,y_{n+1})d(y_n,y_{n-1})}{1+d(y_n,y_{n-1})} \\
&\quad + D\frac{d(y_{n-1},y_n)d(y_{n-1},y_n)}{1+d(y_n,y_{n-1})}.
\end{aligned}$$

Thus, we have

$$|d(y_n,y_{n+1})| \le A|d(y_n,y_{n-1})| + B\frac{|d(y_n,y_{n+1})|\,|d(y_n,y_{n-1})|}{|d(y_n,y_{n-1})|} + D\frac{|d(y_{n-1},y_n)|\,|d(y_{n-1},y_n)|}{|d(y_n,y_{n-1})|}.$$

Since

$$|1+d(y_n,y_{n-1})| > |d(y_n,y_{n-1})|,$$

we have

$$(1-B)|d(y_{n+1},y_n)| \leq (A+D)|d(y_n,y_{n-1})|,$$

that is,

$$\begin{aligned}|d(y_{n+1},y_n)| &\leq \frac{A+D}{1-B}|d(y_n,y_{n-1})| \\ &= k|d(y_n,y_{n-1})|,\end{aligned}$$

where $k = \dfrac{A+D}{1-B} < 1$.

Consequently, it can be concluded that

$$\begin{aligned}d(y_n,y_{n+1}) &\precsim kd(y_{n-1},y_n) \\ &\precsim k^2 d(y_{n-2},y_{n-1}) \\ &\vdots \\ &\precsim k^n d(y_0,y_1).\end{aligned}$$

Now, for all $m > n$,

$$\begin{aligned}&d(y_m,y_n) \\ &\precsim d(y_n,y_{n+1}) + d(y_{n+1},y_{n+2}) + \cdots + d(y_m,y_{m-1}) \\ &\precsim k^n d(y_0,y_1) + k^{n+1} d(y_0,y_1) + \cdots + k^{m-1} d(y_0,y_1) \\ &\precsim \frac{k^n}{1-k} d(y_0,y_1).\end{aligned}$$

Therefore, we have

$$|d(y_m,y_n)| \leq \frac{k^n}{1-k}|d(y_0,y_1)|.$$

Hence,

$$\lim_{n\to\infty} |d(y_m,y_n)| = 0.$$

Hence, $\{y_n\}$ is a Cauchy sequence in gX. But gX is a complete subspace of X, so there is a u in gX such that $y_n \to u$ as $n \to \infty$. Let $v \in g^{-1}u$. Then $gv = u$.

Now, we shall prove that $fv = u$.

Putting $x = v$ and $y = x_{n-1}$ in (2.2), we get

$$\begin{aligned}d(fv, fx_{n-1}) \precsim\ & Ad(gv, gx_{n-1}) \\ &+ B\frac{d(gv, fv)d(fx_{n-1}, gx_{n-1})}{1+d(gv, gx_{n-1})} \\ &+ C\frac{d(gv, fx_{n-1})d(gv, gx_{n-1})}{1+d(gv, gx_{n-1})} \\ &+ D\frac{d(gv, fv)d(gv, gx_{n-1})}{1+d(gv, gx_{n-1})} \\ &+ E\frac{d(gv, fx_{n-1})d(fx_{n-1}, gx_{n-1})}{1+d(gv, gx_{n-1})}.\end{aligned}$$

Letting $n \to \infty$, we have

$$\begin{aligned}d(fv,u) \precsim\ & Ad(u,u) + B\frac{d(gv, fv)d(u,u)}{1+d(u,u)} \\ &+ C\frac{d(u,u)d(u,u)}{1+d(u,u)} + D\frac{d(gv, fv)d(u,u)}{1+d(u,u)} \\ &+ E\frac{d(u,u)d(u,u)}{1+d(u,u)},\end{aligned}$$

that is, $|d(u, fv)| \leq 0$, implies that, $fv = u$.

Thus, $fv = u = gv$, and hence v is the coincidence point of f and g.

Now, since f and g are weakly compatible, so, $u = fv = gv$, implies that, $fu = fgv = gfv = gu$.

Now, we claim that $gu = u$. Let, if possible, $gu \neq u$.

From (2.2), we have

$$\begin{aligned}d(u,gu) &= d(fv, fu) \\ &\precsim Ad(gv,gu) + B\frac{d(gv, fv)d(fu,gu)}{1+d(gv,gu)} \\ &\quad + C\frac{d(gv, fu)d(gv,gu)}{1+d(gv,gu)} + D\frac{d(gv, fv)d(gv,gu)}{1+d(gv,gu)} \\ &\quad + E\frac{d(gv, fu)d(fu,gu)}{1+d(gv,gu)} \\ &= Ad(u,gu) + C\frac{d(u,gu)d(u,gu)}{1+d(u,gu)},\end{aligned}$$

that is,

$$|d(u,gu)| \leq A|d(u,gu)| + C\frac{|d(u,gu)|\,|d(u,gu)|}{|1+d(u,gu)|},$$

Since

$$|1+d(u,gu)| > |d(u,gu)|,$$

we have

$$|d(u,gu)| \leq (A+C)|d(u,gu)|,$$

implies that, $A+C \geq 1$, a contradiction.

Hence, $gu = u = fu$.

Therefore, u is the common fixed point of f and g.

For the uniqueness, let w be another common fixed point of f and g such that $w \neq u$.

From (2.2), we have

$$\begin{aligned}d(w,u) &= d(fw, fu) \\ &\precsim Ad(gw,gu) + B\frac{d(gw, fw)d(fu,gu)}{1+d(gw,gu)} \\ &\quad + C\frac{d(gw, fu)d(gw,gu)}{1+d(gw,gu)} \\ &\quad + D\frac{d(gw, fw)d(gw,gu)}{1+d(gw,gu)} \\ &\quad + E\frac{d(gw, fu)d(fu,gu)}{1+d(gw,gu)} \\ &= Ad(w,u) + C\frac{d(w,u)d(w,u)}{1+d(w,u)},\end{aligned}$$

that is,

$$|d(w,u)| \le A|d(w,u)| + C\frac{|d(w,u)|\,|d(w,u)|}{|1+d(w,u)|}.$$

Since

$$|1+d(w,u)| > |d(w,u)|,$$

we have

$$|d(w,u)| \le (A+C)|d(w,u)|,$$

implies that, $A+C \ge 1$, a contradiction.

Hence f and g have a unique common fixed point.

Corollary 1.*Let f and g be self mappings of a complex valued metric space (X,d) satisfying (2.1), (2.3) and the following:*

(2.4) $$d(fx,fy) \precsim Ad(gx,gy) + B\frac{d(gx,fx)d(fy,gy)}{1+d(gx,gy)} + C\frac{d(gx,fy)d(gx,gy)}{1+d(gx,gy)} + D\frac{d(gx,fx)d(gx,gy)}{1+d(gx,gy)},$$

for all x, y in X, where A, B, C, D are non-negative constants with $A+B+C+D<1$.

Then f and g have a coincidence point.

Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point.

*Proof.*By putting $E=0$ in Theorem 2.1, we get the Corollary 1.

Corollary 2.*Let f and g be self mappings of a complex valued metric space (X,d) satisfying (2.1), (2.3) and the following:*

(2.5)*$d(fx,fy) \precsim Ad(gx,gy)$, for all x, y in X, where $0 \le A < 1$.*

Then f and g have a coincidence point.

Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point.

*Proof.*By putting $B=C=D=E=0$ in Theorem 2.1, we get the Corollary 2.

*Example 3.*Let $X=[0,1]$ and define $d: X\times X \to \mathbb{C}$ by $d(x,y) = i|x-y|$, for all $x,y \in X$.

Then (X,d) is a complex valued metric space.

Define the functions $f,g: X\to X$ by $fx = \frac{x}{3}$ and $gx = \frac{x}{2}$.

Clearly $fX = \left[0,\frac{1}{3}\right] \subseteq \left[0,\frac{1}{2}\right] = gX$.

Also f and g are weakly compatible.

For $A = \frac{2}{3} < 1$, we have

$$d(fx,fy) \precsim Ad(gx,gy), \quad \text{for all } x,y \in X.$$

Also 0 is the unique common fixed point of f and g.

Hence all the conditions of Corollary 2 are satisfied.

3 Weakly compatible and (CLR$_g$) properties

Theorem 2.*Let f and g be self mappings of a complex valued metric space (X,d) satisfying (2.2) and the following:*

(3.1) *f and g satisfy (CLR$_g$) property,*
(3.2) *f and g are weakly compatible.*

Then f and g have a unique common fixed point.

*Proof.*Since f and g satisfy the (CLR$_g$) property, there exists a sequence $\{x_n\}$ in X such that

(3.3) $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = gx$, for some x in X.

From (2.2), we have

$$d(fx,fx_n) \precsim Ad(gx,gx_n) + B\frac{d(gx,fx)d(fx_n,gx_n)}{1+d(gx,gx_n)} + C\frac{d(gx,fx_n)d(gx,gx_n)}{1+d(gx,gx_n)} + D\frac{d(gx,fx)d(gx,gx_n)}{1+d(gx,gx_n)} + E\frac{d(gx,fx_n)d(fx_n,gx_n)}{1+d(gx,gx_n)}.$$

Letting $n\to\infty$, we have

$$d(fx,gx) \precsim Ad(gx,gx) + B\frac{d(gx,fx)d(gx,gx)}{1+d(gx,gx)} + C\frac{d(gx,gx)d(gx,gx)}{1+d(gx,gx)} + D\frac{d(gx,fx)d(gx,gx)}{1+d(gx,gx)} + E\frac{d(gx,gx)d(gx,gx)}{1+d(gx,gx)} = 0,$$

implies that,

$$|d(fx,gx)| \le 0,$$

that is, $fx = gx$.

Now, let $u = fx = gx$. Since f and g are weakly compatible mappings, therefore, $fgx = gfx$, implies that, $fu = fgx = gfx = gu$.

Now, we claim that $gu = u$. Let, if possible, $gu \ne u$.

From (2.2), we have

$$d(u,gu) = d(fx,fu) \precsim Ad(gx,gu) + B\frac{d(gx,fx)d(fu,gu)}{1+d(gx,gu)} + C\frac{d(gx,fu)d(gx,gu)}{1+d(gx,gu)}$$

$$+D\frac{d(gx,fx)d(gx,gu)}{1+d(gx,gu)}$$
$$+E\frac{d(gx,fu)d(fu,gu)}{1+d(gx,gu)}$$
$$=Ad(u,gu)+C\frac{d(u,gu)d(u,gu)}{1+d(u,gu)},$$

that is,

$$|d(u,gu)|\leq A|d(u,gu)|+C\frac{|d(u,gu)|\,|d(u,gu)|}{1+|d(u,gu)|}.$$

Since
$$|1+d(u,gu)|>|d(u,gu)|,$$
we have
$$|d(u,gu)|\leq(A+C)|d(u,gu)|,$$
implies that, $A+C\geq 1$, a contradiction.

Hence, $gu=u=fu$.

Therefore, u is the common fixed point of f and g.

For the uniqueness, let w be another common fixed point of f and g such that $w\neq u$.

From (2.2), we have

$$d(w,u)=d(fw,fu)$$
$$\precsim Ad(gw,gu)+B\frac{d(gw,fw)d(fu,gu)}{1+d(gw,gu)}$$
$$+C\frac{d(gw,fu)d(gw,gu)}{1+d(gw,gu)}$$
$$+D\frac{d(gw,fw)d(gw,gu)}{1+d(gw,gu)}$$
$$+E\frac{d(gw,fu)d(fu,gu)}{1+d(gw,gu)}$$
$$=Ad(w,u)+C\frac{d(w,u)d(w,u)}{1+d(w,u)},$$

that is,

$$|d(w,u)|\leq A|d(w,u)|+C\frac{|d(w,u)|\,|d(w,u)|}{|1+d(w,u)|}.$$

Since
$$|1+d(w,u)|>|d(w,u)|,$$
we have
$$|d(w,u)|\leq(A+C)|d(w,u)|,$$
implies that, $A+C\geq 1$, a contradiction.

Hence f and g have a unique common fixed point.

Corollary 3.*Let f and g be self mappings of a complex valued metric space (X,d) satisfying (3.1), (3.2) and the following:*

(3.4)$d(fx,fy)\precsim Ad(gx,gy)+C\frac{d(gx,fy)d(gx,gy)}{1+d(gx,gy)}$, *for all x,y in X,*
where A and C are non-negative constants with $A+C<1$.

Then f and g have a unique common fixed point.

*Proof.*By putting $B=D=E=0$ in Theorem 2, we get the Corollary 3.

Corollary 4.*Let f and g be self mappings of a complex valued metric space (X,d) satisfying (3.1), (3.2) and the following:*

(3.5)$d(fx,fy)\precsim C\frac{d(gx,fy)d(gx,gy)}{1+d(gx,gy)}$, *for all x,y in X,*
where C is a non-negative constant with $C<1$.

Then f and g have a unique common fixed point.

*Proof.*By putting $A=0$ in Corollary 3, we get the Corollary 4.

Corollary 5.*Let f and g be self mappings of a complex valued metric space (X,d) satisfying (3.1), (3.2) and the following:*

(3.6)$d(fx,fy)\precsim Ad(gx,gy)$, *for all x,y in X,*

where A is a non-negative constant with $A<1$.

Then f and g have a unique common fixed point.

*Proof.*By putting $C=0$ in Corollary 3, we get the Corollary 5.

*Example 4.*Let $X=[0,1]$ and define $d:X\times X\to\mathbb{C}$ by $d(x,y)=i|x-y|$, for all $x,y\in X$.

Then (X,d) is a complex valued metric space.

Define the functions $f,g:X\to X$ by $fx=\frac{x}{8}$ and $gx=\frac{x}{2}$.

Clearly $fX=\left[0,\frac{1}{8}\right]\subseteq\left[0,\frac{1}{2}\right]=gX$.

Also, f and g are weakly compatible.

Consider the sequence $\{x_n\}=\left\{\frac{1}{n}\right\}$, $n\in\mathbb{N}$.

Since $\lim_{n\to\infty}fx_n=\lim_{n\to\infty}gx_n=0=g_0$, so f and g satisfy (CLR$_g$) property.

Also, for $A=\frac{1}{4}<1$, we have

$$d(fx,fy)\precsim Ad(gx,gy),\ \text{for all} x,y\in X.$$

Here 0 is the unique common fixed point of f and g.

Hence all the conditions of Corollary 5 are satisfied.

4 Weakly compatible and E.A. Properties

Theorem 3.*Let f and g be self mappings of a complex valued metric space (X,d) satisfying (2.1), (2.2), (3.2) and the following:*

(4.1)*f and g satisfy E.A. property,*
(4.2)*gX is a closed subset of X.*

Then f and g have a unique common fixed point.

*Proof.*Since f and g satisfy the E.A. property, there exists a sequence $\{x_n\}$ in X such that

(4.3)$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$, for some z in X.

Now, gX is closed subset of X, therefore $\lim_{n\to\infty} gx_n = ga$, for some a in X.

So, from (4.3), we have

$$\lim_{n\to\infty} fx_n = ga.$$

We claim that $fa = ga$.

From (2.2), we have

$$\begin{aligned} d(fa, fx_n) \precsim\ & Ad(ga,gx_n) + B\frac{d(ga, fa)d(fx_n,gx_n)}{1+d(ga,gx_n)} \\ &+ C\frac{d(ga, fx_n)d(ga,gx_n)}{1+d(ga,gx_n)} \\ &+ D\frac{d(ga, fa)d(ga,gx_n)}{1+d(ga,gx_n)} \\ &+ E\frac{d(ga, fx_n)d(fx_n,gx_n)}{1+d(ga,gx_n)}. \end{aligned}$$

Letting $n \to \infty$, we have

$$\begin{aligned} d(fa,ga) \precsim\ & Ad(ga,ga) + B\frac{d(ga, fa)d(ga,ga)}{1+d(ga,ga)} \\ &+ C\frac{d(ga,ga)d(ga,ga)}{1+d(ga,ga)} \\ &+ D\frac{d(ga, fa)d(ga,ga)}{1+d(ga,ga)} \\ &+ E\frac{d(ga,ga)d(ga,ga)}{1+d(ga,ga)} \\ =\ & 0, \end{aligned}$$

implies that,

$$|d(fa,ga)| \leq 0,$$

that is, $fa = ga$.

Now, we show that fa is the common fixed point of f and g. Let, if possible $fa \neq ffa$.

Since f and g are weakly compatible, $gfa = fga$, implies that, $ffa = fga = gfa = gga$.

From (2.2), we have

$$\begin{aligned} d(ffa, fa) \precsim\ & Ad(gfa,ga) + B\frac{d(gfa, ffa)d(fa,ga)}{1+d(gfa,ga)} \\ &+ C\frac{d(gfa, fa)d(gfa,ga)}{1+d(gfa,ga)} \\ &+ D\frac{d(gfa, ffa)d(gfa,ga)}{1+d(gfa,ga)} \\ &+ E\frac{d(gfa, fa)d(fa,ga)}{1+d(gfa,ga)} \\ =\ & Ad(ffa, fa) + C\frac{d(ffa, fa)d(ffa,ga)}{1+d(ffa,ga)}, \end{aligned}$$

that is,

$$\begin{aligned} |d(ffa, fa)| \leq\ & A|d(ffa, fa)| \\ &+ C\frac{|d(ffa, fa)|\,|d(ffa, fa)|}{|1+d(ffa, fa)|}. \end{aligned}$$

Since

$$|1+d(ffa, fa)| > |d(ffa, fa)|,$$

we have

$$|d(ffa, fa)| \leq (A+C)|d(ffa, fa)|,$$

implies that, $A+C \geq 1$, a contradiction.

Hence $ffa = fa = gfa$.

Thus, fa is the common fixed point of f and g.

Finally, we show that the common fixed point is unique.

For this, let u and v be two common fixed points of f and g such that $u \neq v$.

$$\begin{aligned} d(v,u) =\ & d(fv, fu) \\ \precsim\ & Ad(gv,gu) + B\frac{d(gv, fv)d(fu,gu)}{1+d(gv,gu)} \\ &+ C\frac{d(gv, fu)d(gv, fu)}{1+d(gv,gu)} \\ &+ D\frac{d(gv, fv)d(gv,gu)}{1+d(gv,gu)} \\ &+ E\frac{d(gv, fu)d(fu,gu)}{1+d(gv,gu)} \\ =\ & Ad(v,u) + C\frac{d(v,u)d(v,u)}{1+d(v,u)}, \end{aligned}$$

that is,

$$|d(v,u)| \leq A|d(v,u)| + C\frac{|d(v,u)|\,|d(v,u)|}{|1+d(v,u)|}.$$

Since

$$|1+d(v,u)| > |d(v,u)|,$$

we have

$$|d(v,u)| \leq (A+C)|d(v,u)|,$$

implies that, $A+C \geq 1$, a contradiction.

Hence f and g have a unique common fixed point.

Corollary 6.*Let f and g be self mappings of a complex valued metric space (X,d) satisfying (2.1), (3.2), (4.1) and the following:*

(4.4)$d(fx, fy) \precsim Ad(gx,gy) + C\dfrac{d(gx, fy)d(gx,gy)}{1+d(gx,gy)}$, *for all x, y in X,*

where A and C are non-negative constants with $A+C < 1$.

Then f and g have a unique common fixed point.

Proof. By putting $B = D = E = 0$ in Theorem 4.1, we get the Corollary 6.

Corollary 7. *Let f and g be self mappings of a complex valued metric space (X, d) satisfying (2.1), (3.2), (4.1) and the following:*

(4.5) $d(fx, fy) \precsim C\dfrac{d(gx, fy)d(gx, gy)}{1 + d(gx, gy)}$, *for all x, y in X,*
where C is a non-negative constant with $C < 1$.

Then f and g have a unique common fixed point.

Proof. By putting $A = 0$ in Corollary 6, we get the Corollary 7.

Corollary 8. *Let f and g be self mappings of a complex valued metric space (X, d) satisfying (2.1), (3.2), (4.1) and the following:*

(4.6) $d(fx, fy) \precsim Ad(gx, gy)$, *for all x, y in X,*
where A is a non-negative constant with $A < 1$.
Then f and g have a unique common fixed point.

Proof. By putting $C = 0$ in Corollary 6, we get the Corollary 8.

Example 5. Let $X = [0, 1]$ and define $d : X \times X \to \mathbb{C}$ by $d(x, y) = i|x - y|$, for all $x, y \in X$.

Then (X, d) is a complex valued metric space.

Define the functions $f, g : X \to X$ by $fx = \dfrac{x}{6}$ and $gx = \dfrac{x}{2}$.

Clearly $fX = \left[0, \dfrac{1}{6}\right] \subseteq \left[0, \dfrac{1}{2}\right] = gX$.

Also, f and g are weakly compatible.

Consider the sequence $\{x_n\} = \left\{\dfrac{1}{n}\right\}$, $n \in \mathbb{N}$.

Since $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = 0$, where $0 \in X$, so f and g satisfy E.A. property.

Also, for $A = \dfrac{1}{3} < 1$, we have

$$d(fx, fy) \precsim Ad(gx, gy), \quad \text{for all } x, y \in X.$$

Here 0 is the unique common fixed point of f and g.

Hence all the conditions of Corollary 8 are satisfied.

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