Applied Mathematics & Information Sciences **3**(2) (2009), 151–158
– An International Journal

Bosonic Channel Capacity of a Nonlinear Dispersive Optical Fiber

Navneet K Sharma[1] and Anirban Pathak[2]

[1]Jaypee Institute of Information Technology, A-10, Sector-62, Noida, UP-201307, India

Email Address: navneetk.sharma@jiit.ac.in

[2]Jaypee Institute of Information Technology, A-10, Sector-62, Noida, UP-201307, India

Email Address: anirban.pathak@jiit.ac.in

Received May 24, 2008; Revised October 13, 2008

Bosonic channel capacity of some important nonlinear optical channels is studied. In particular, the bosonic channel capacity of a nonlinear dispersive (silica) optical fiber interacting with an intense laser beam has been studied in detail and it has been shown that the channel capacity can be enhanced/tuned by tuning coupling constant, temperature, frequency of the radiations, etc. Further it is shown that channel capacity is not proportional to nonclassicality.

Keywords: Channel capacity, partition function, bosonic channels.

1 Introduction

Channel capacity of physical systems has been an object of extensive study since the pioneering work of Shannon [16]. The interest in this field has been increased in recent past with the advent of quantum communication. Actually it is now realized that all the communication channels are essentially quantum mechanical. This realization and the fact that the communication technologies are approaching the quantum level, have worked as motivation for the community to investigate the physical limits of quantum communication channels in general [2, 3, 5, 8, 11, 12, 14, 15, 20, and references therein]. All the initial studies in this field were limited to the memoryless channels [2, 20]. Only recently people have studied capacities of channels having memories [6] and noise [10]. Among the recent studies on quantum channels the major attention has been drawn by bosonic quantum channels and consequently a measure of classical capacity of quantum channels called "bosonic channel capacity" [5, 8, 14, 15] is introduced. Bosonic channel capacity of several physical systems have been studied so far [2, 3, 5, 8, 11, 12, 14, 15, 20]. But apart from [5] all the

The 2nd author thanks DST, India for partial support through project number SR/FTP/PS-13/2004.

studies known to us have used a strong assumption that the channel is linear. The capacity of nonlinear bosonic systems have been first studied by Giovannetti *et al.* [5] in the limit when the dispersion effects are negligible. They had chosen the bosonic channels in such a way that the Hamiltonian representing the channel can be reduced to that of a harmonic oscillator under suitable canonical transformation. Although nonlinear optical channels are very common in practical use, the effect of nonlinearity on bosonic channel capacity is not studied properly till now. This fact has motivated us to go beyond the consideration of Giovannetti *et al.* and to study the bosonic channel capacity of most commonly used nonlinear channel consisting of a single mode electromagnetic field and a nonlinear dispersive optical fiber. Apart from the effect of nonlinearity the effect of nonclassicality on the bosonic channel capacity is also studied in recent past. For example, Lloyd [11, 12] has shown that the entangled quantum channels can communicate at a potentially higher rate than an unentangled quantum channel given the same power. This has motivated us to check the relation between the depth of nonclassicality and the capacity of the channel.

From a mathematical point of view the bosonic channel capacity is just a specific case of entropy maximization problems [5]. To be precise, the maximization of von-Neumann entropy with some constrains condition. The problem of finding out bosonic channel capacity can essentially be reduced to finding out the partition function [5]. Finding out exact analytic expression of partition function is possible only in a number of simple physical systems. But there exist some methods [4, 7, 17–19] for finding out approximate expression for partition function. It is an outstanding curiosity to note that the introduction of bosonic channel capacity has linked quantum information theory to an apparently unrelated field of mathematical physics, namely, partition function problems, which has a rich literature [4, 7, 17–19]. We have used one existing trick by Witschel to study the variation of bosonic channel capacity of some important nonlinear optical channels.

In the present work we aim to go beyond the consideration of Giovannetti *et al.* and wish to consider more physical systems in which the nonlinear system is modelled as an anharmonic oscillators. It is correctly pointed out in [5] that the effect of dispersion in nonlinear optical channel can not be studied precise as the Hamiltonian is not exactly solvable. To circumvent this, we have used rotating wave approximation and adiabatic assumption. Under this assumptions the Hamiltonian that represents the propagation of a single mode electromagnetic field through a nonlinear optical fiber becomes exactly solvable. We have used this fact to study the bosonic channel capacity of a nonlinear dispersive (silica) optical fiber interacting with an intense laser beam. Further, it is shown that the bosonic channel capacity does not follow any linear relation with nonclassicality (amount of squeezing). This is in sharp contrast with the earlier results. In the next section we have briefly introduced the idea of bosonic channel capacity and have shown that the problem of finding the bosonic channel capacity can essentially be reduced to the finding of partition function. In section 3 we have studied the bosonic channel capacity of a third order nonlinear medium with inversion symmetry (with and without the application of rotating wave approximation). The physical system studied here essentially represents a nonlinear dispersive optical fiber. The analytic and numerical results obtained in the present study are shown graphically and it is established that the bosonic channel capacity can be tuned with the help of tunable laser or heat bath. Finally, section 4 is dedicated to conclusions.

2 Information Capacity

The information capacity of a noiseless channel is defined as the maximum number of bits that can be reliably sent per channel use [5]. In case of a classical channel it is given by the maximum of Shanon's entropy and analogously in case of a quantum channel it is given by the maximum of the von Neumann entropy $S(\rho) = -Tr[\rho \log_2 \rho]$ over all the possible input states ρ of the channel. In the present study we are interested in the interaction of an intense laser beam with a nonlinear medium. Therefore physically, the nonlinear medium constitutes the channel, and the photons of the laser beam carries the information. So the quantum states (ρ) of our interest are essentially the states of the photons which are bosons with zero rest mass. Now for a mass less bosonic field, the associated Hilbert space is infinite dimensional and as a natural consequence of the infinite dimensional Hilbert space, the maximum entropy is infinite. In a real situation a quantum channel having infinite capacity does not make sense and for all realistic scenarios a cutoff is introduced by constraining the energy required in the storage or in the transmission, e.g., requiring the entropy $S(\rho)$ to be maximized only over those states having average energy E, i.e.,

$$E = Tr[\rho H] \tag{2.1}$$

where H is the system Hamiltonian [5]. The constrained maximization of $S(\rho)$ can be solved by standard variational methods (see references in [5]), which entail the solution of

$$\delta \left\{ S(\rho) - \frac{\lambda_1}{\ln 2} Tr[H\rho] - \frac{\lambda_2}{\ln 2} Tr[\rho] \right\} = 0 \tag{2.2}$$

where λ_1 and λ_2 are Lagrange multipliers that take into account the energy constraint (2.1) and the normalization constraint $Tr[\rho] = 1$, and the $\ln 2$ factor is introduced so that all subsequent calculations can be performed using natural logarithms. Equation (2.2) can be solved by the density matrix $\rho = \exp(-\beta H)/Z(\beta)$, where

$$Z(\beta) = Tr(\exp(-\beta H)) \tag{2.3}$$

is the partition function of the system and is determined from the constraint (2.1) by solving the equation

$$E = -\frac{\partial}{\partial \beta} \ln (Z(\beta)) . \tag{2.4}$$

The corresponding capacity is thus given by

$$C = S\left[\frac{\exp(-\beta H)}{Z(\beta)}\right] = \frac{\beta E + \ln Z(\beta)}{\ln 2} \tag{2.5}$$

which means that we can evaluate the system capacity only from its partition function $Z(\beta)$. In [5] the same procedure is followed and bosonic channel capacity has been found for some simple optical systems for which the Hamiltonian can be reduced to that of a simple harmonic oscillator (under a suitable canonical transformation). But for more realistic and physical systems one can't obtain exact analytic expression for C since exact analytic expression for partition function can not be obtained. We have chosen two such systems and in the next section we studied those modelled systems.

3 Bosonic Channel Capacity for a Third Order Nonlinear Medium

It is clear from the above that if we know the partition function then we can find out the bosonic channel capacity. In [5] they have derived bosonic channel capacity for some simple cases where the bosonic Hamiltonian can be reduced to harmonic oscillator Hamiltonian by some simple canonical transformation. Other people have done some calculation for noisy channel [10], or for channel with memory [6]. In the present work we have tried to follow [5] and to find out partition functions for the physical systems in which an intense laser beam interacts with a third order nonlinear medium of inversion symmetry, the system can be modelled by the following Hamiltonian [13]

$$H = \left(a^{\dagger}a + \frac{1}{2}\right)\hbar\omega + \lambda\left(a + a^{\dagger}\right)^{4}, \tag{3.1}$$

where λ is the anharmonic constant which is a function of third order susceptibility χ^3 and thus a functional of the frequency ω. This system is not exactly solvable. But there exist methods for obtaining approximate partition function [4, 7, 17–19]. Among the existing technique we find suitable to use [17, 18]. In [17] Witschel has given an excellent scheme for finding out the partition function if a bosonic Hamiltonian $H(a, a^{\dagger})$ is given, in [19] they have done it for the Hamiltonian (3.1) and obtained the partition function (see equation (24)-(27) of [19]). Therefore, in principle we can now find out bosonic channel capacity for the physical system represented by quartic oscillator. We have done the same with the help of Mathematica 5.2 and the results are presented in the Fig 3.1 below.

There are simpler models which can help us to understand the effect of nonlinearity on the bosonic channel capacity. To be precise, let us consider the propagation of a single mode electromagnetic field of frequency ω through a nonlinear optical fiber. The medium fiber can be modelled as anharmonic oscillator of frequency ω_0. Let a (b) and $a^{\dagger}$ ($b^{\dagger}$) be the annihilation and creation operators for the field (medium). Essentially, we are assuming

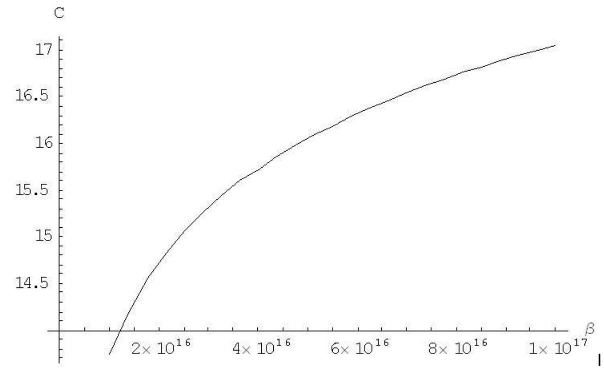

Figure 3.1: Variation of bosonic channel capacity of the third order nonlinear medium described by Hamiltonian (3.1) when anharmonic constant λ=0.01 and wavelength of the incident photon beam is 6000Å.

that the medium is contained in a very good quality single mode cavity. Now by using the rotating wave approximation we can write the total Hamiltonian for the system as

$$H = \hbar\omega_0 b^\dagger b + \hbar\omega a^\dagger a + \hbar q b^{\dagger 2} b^2 + \hbar g(b^\dagger a + a^\dagger b), \tag{3.2}$$

where q is the anahromicity parameter and g is the strength of coupling of medium with the field mode [1]. A further simplified Hamiltonian can be obtained in adiabatic limit. In the adiabatic limit, it is assumed that the two oscillator frequencies (i.e. ω and ω_0) are far from each other. In this limit the effective Hamiltonian of the system reduces to

$$H = a^\dagger a \hbar\bar{\omega} + \lambda a^{\dagger 2} a^2, \tag{3.3}$$

where $\lambda = \hbar\chi = qg^4/\Delta^4$, $\Delta = \omega_0 - \omega$ and the modified frequency $\bar{\omega} = \omega - g^2/\Delta$. Here χ is the dispersive part of the third order nonlinear medium. Thus the the effective Hamiltonian (3.3) describes the propagation of single mode light through a high quality nonlinear dispersive fiber [9]. This system is extensively studied in nonlinear optics [1, 9] and references therein, but the information capacity of this channel is not reported till date. In order to obtain the bosonic channel capacity of this system we need to find out the partition function of the system. To obtain the partition function we have followed the Witschel's algorithm prescribed in [17, 18]. Absence of the offdiagonal terms in the effective Hamiltonian makes the system exactly solvable and the generalized solution can be written as an arbitrary superposition of Fock states as

$$|\psi(t)\rangle = \sum_{n=0}^{\infty} C_n \exp[-it\{\bar{\omega}n - \chi(n^2 - n)\}]|n\rangle. \tag{3.4}$$

Now we can use (2.3) and (3.4) to obtain the partition function corresponding to (3.3) as

$$Z(\beta) = Tr\left(\exp(-\beta H)\right)$$

$$= \sum_{n=0}^{\infty} \exp\left[-\beta\left(\bar{\omega}\hbar n + \lambda(n^2 - n)\right)\right]$$
$$= \sum_{n=0}^{\infty} \exp\left[-\beta\lambda\left(n^2 + \frac{\bar{\omega}\hbar - \lambda}{\lambda}n\right)\right]$$
$$= \sum_{n=0}^{\infty} \exp\left[-\beta\lambda\left(\left(n + \frac{\bar{\omega}\hbar - \lambda}{2\lambda}\right)^2 - \left(\frac{\bar{\omega}\hbar - \lambda}{2\lambda}\right)^2\right)\right]$$
$$= \exp\left(\beta\lambda k^2\right) \sum_{n=0}^{\infty} \exp\left(-\beta\lambda(n+k)^2\right), \tag{3.5}$$

where

$$k = \frac{\hbar\bar{\omega} - \lambda}{2\lambda} = \frac{1 - \chi/\bar{\omega}}{2\chi/\bar{\omega}}.$$

Here we would like to note that under rotating wave approximation and adiabatic assumption we have obtained a Hamiltonian free of off-diagonal terms and thus the corresponding partition function is exact. Once the partition function is obtained we can use (2.4) and (2.5) to obtain the bosonic channel capacity corresponding to the Hamiltonian (3.3). This task is done with the help of Mathematica 5.2 and the results are graphically presented in Fig 3.2a and 3.2b.

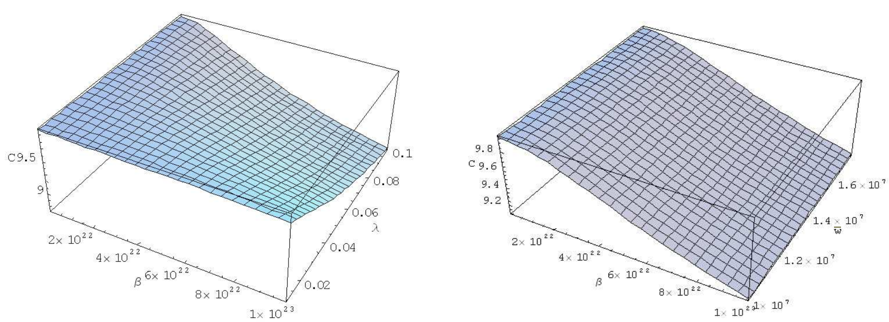

Figure 3.2: A channel is constituted by an incident laser beam of wavelength 6000Å and a non-dispersive optical fiber having Kerr type nonlinearity. Variation of bosonic channel capacity C is shown with respect to (a) anharmonic constant λ and temperature T ($\beta = 1/(kT)$), (b) temperature T ($\beta = 1/(kT)$) and modified frequency $\bar{\omega}$.

From the figures it is clear that with the help of a tunable laser one can control the channel capacity of a dispersive fiber. Even a heat bath can be used to tune the bosonic channel capacity of a nonlinear dispersive optical fiber.

4 Conclusions

In [13] we have reported the possibility of observing squeezing in a third order nonlinear medium described by the Hamiltonian (3.1) and it was found that depth of nonclassicality (see equation 34-39 in [13]) oscillates with the increase of anharmonic constant λ but in the present work we have seen that bosonic channel capacity decreases monotonically with increase in λ (Fig. 3.2a). Thus we observe that the bosonic channel capacity does not follow any linear relation with nonclassicality (amount of squeezing). This is in sharp contrast with the earlier results. It is also shown that in a nonlinear dispersive optical fiber, bosonic channel capacity can be tuned by tuning the temperature, frequency of the incident beam etc. Further, the present work introduces a bridge between two apparently unrelated subfield of physical science, namely, evaluation of partition function and quantum information. We expect that this link will be further exploited to provide bosonic channel capacity of several other physical systems of specific interests.

References

[1] G. S. Agarwal and R. R. Puri, Collapse and revival phenomenon in the evolution of a resonant field in a Kerr-like medium, *Phys. Rev. A* **39** (1989), 2969–2977.

[2] C. M. Caves and P. D. Drummond, Quantum limits on bosonic communication rates, *Rev. Mod. Phys.* **66** (1994), 481–537.

[3] A. Fahmi and M. Golshani, Transition of d-level quantum systems through quantum channels with correlated noise, *Phys. Rev. A* **75** (2007), 042301.

[4] I. D. Feranchuk and A. A. Ivanov, Operator method for nonperturbative calculation of the thermodynamic values in quantum statistics: diatomic molecular gas, *J. Phys. A: Math. Gen.* **37** (2004), 9841–9860.

[5] V. Giovannetti, S. Lloyd, and L. Maccone, Capacity of nonlinear bosonic systems, *Phys. Rev. A* **70** (2004), 012307.

[6] V. Giovannetti and S Mancini, Bosonic memory channels, *Phys. Rev A* **71** (2005), 062304.

[7] B. J. Harnigton, Nonlinear contributions to the partition function of a double-well oscillator, *Phys Rev. D* **18** (1978), 2982–2989.

[8] A. S. Holevo and R. F. Wernera, Evaluating capacities of bosonic Gaussian channels, *Phys Rev A* **63** (2001), 032312.

[9] A. Joshi and A. K. Pati, Geometric phase with photon statistics and squeezed light for the dispersive fiber, *Phys. Rev. A* **49** (1994), 5131–5134.

[10] S. Lloyd, Capacity of the noisy quantum channel, *Phys. Rev. A* **55** (1997), 1613–1622.

[11] S. Lloyd, Power of nntanglement in quantum communication, *Phys. Rev. Lett.* **90** (2003), 167902.

[12] S. Lloyd, The power of entangled quantum channels, *quant-ph/0112034V1*.

[13] A. Pathak and S. Mandal, Classical and quantum oscillators of quartic anharmonicities: second-order solution, *Phys. Lett. A* **286** (2001), 261–276.
[14] A Sen, U. Sen, B. Gromek, D. Bruss, and M. Lewenstein, Capacities of Quantum Channels for Massive Bosons and Fermions, *Phys. Rev. Lett.* **95** (2005), 260503.
[15] A Sen, U. Sen, B. Gromek, D. Bruss, and M. Lewenstein, Capacities of noiseless quantum channels for massive indistinguishable particles: Bosons versus fermions, *Phys. Rev. A* **75** (2007), 022331.
[16] C. E. Shannon, A mathematical theory of communication, *The Bell System Technical Journal*, **27** (1948), 379–423 and 623–656.
[17] W. Witschel, A statistical variation-perturbation method for the partition function of anharmonic oscillators, *Phys. Lett.* **77** (1980), 107–111.
[18] W. Witschel, An improved variation-perturbation technique for the partition function of anharmonic oscillators, *Phys. Lett. A* **97** (1983), 315–319.
[19] W. Witschel, Density matrix of anharmonic oscillators by a thermodynamic variation-perturbation technique, a closed form approach, *Physica A* **300** (2001), 116–138.
[20] H. P. Yuen and M. Ozawa, Ultimate information carrying limit of quantum systems, *Phys. Rev. Lett.* **70** (1992), 363–366.

Navneet K. Sharma was born on June 27, 1975 in Najibabad (District-Bijnor, U.P.), India. Mr. Sharma received his M.Sc. degree in Physics (1997) from University of Roorkee (India) and Ph.D. in Physics (2004) from IIT, Delhi (India). Since June 15, 2004, Mr. Sharma has been working as Senior Lecturer in Physics in Jaypee Institute of Information Technology University, A-10, Sector-62, Noida, U.P.-201307, India. His major research interest is fiber optics and optical sensors.

Anirban Pathak is an Associate Professor in the Department of Physics. He has done his M.Sc and PhD from Department of Physics Visva Bharati. He has joined JIITU in August 2002. During his PhD work and afterwards he has published several papers in reputed journals. He was a category A speaker in the Theoretical Physics Seminar Circuit (TPSC). At present he has active research collaborations with different research groups in India, Argentina, Canada and Germany. His major research interests are: Quantum Optics, Nonlinear Optics, Mathematical Physics and Quantum Computing.