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Bi Operation and Rough Sets Generalizations

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Generalization of rough set model is an important aspect of rough set theory research. The problem to be discussed in this paper is to minimize the boundary region and this requires a new approximation approach which increases lower approximation and decreases upper approximation.We generalize both constructive and algebraic method for the theory of rough sets. Instead of one operation used by Jarvinen [1], we use two operations to define, in a lattice theoretical setting, two new mappings which mimic the rough approximations called pairwise lower and pairwise upper approximations. We studied the properties of these approximations by imposing different axioms on the suggested two operations. Also properties of the ordered set of the pairwise lower and upper of an element of a complete atomic Boolean lattice are investigated. Numerical examples are given. Finally an experimental example is given showing that our generalizations can help in expert system and using lower and upper approximations given in this work will minimize the boundary region. This will decrease the uncertainty region that help decision maker to get more accurate results.

Keywords: Complete atomic Boolean lattice, pairwise lower and upper approximations, extensive symmetric and closed mappings.

1 General Introduction

Rough set theory (RST), first proposed by Pawlak [2, 3], is an extension of set theory for the study of the intelligent systems characterized by insufficient and incomplete information. It was introduced as a mathematical tool to deal with uncertainty and based on the premise that lowering the degree of precision in data makes the data pattern more visible. The rough set theory has a wide variety of applications. It can be used for information preserving data reduction, representation of uncertain or imprecise knowledge, knowledge discovery, concept classification, machine learning, data mining [4] economics [5], medical diagnosis [6], and others [7].

A basic notion of rough set theory is the lower and upper approximation, or approximation operators [2, 3, 8]. This theory can be developed in at least two different manners, the constructive and algebraic methods [9]. The constructive methods [10–12] define rough set approximation operators using equivalence relations or their induced partitions and subsystems; the algebraic methods treat approximation operators as abstract operators. There are several definitions of constructive methods, commonly known as the element based, granule based [13, 14], and subsystem based definitions [12]. Each of them offers a unique interpretation of the theory. They can be used to investigate the connections to other theories, and to generalize the basic theory in different directions. The element based definition establishes a connection between approximation operators and the necessity and the possibility operators of modal logic. Under the granule based definition, one may view rough set theory as a concrete example of granular computing [15]. The subsystem based definition relates approximation operators to the interior and closure operators of topological spaces [16], the closure operators of closure systems, and operators in other algebraic systems [1, 17]. Algebraic methods [18] focus on the algebraic system $(2^U, ^c, L, H)$ without directly reference to equivalence where L and H are two abstract unary operators called approximation operator.

The theory of rough sets can be generalized in several directions. Within the set theoretic framework, the generalizations of element based definition can be obtained by using non equivalence relations [19]. For examples, Pawlak [20] and others [17, 21, 22] have studied approximation operators which are defined by tolerance, which is a reflexive and symmetric binary relation. Some authors [23, 24] have studied approximation operators defined by reflexive binary relations. Others [19, 25] have studied approximations determined by arbitrary binary relations. On the other hand, some authors [7, 26] considered approximations based on reflexive and transitive relations. Generalization of the granule based definition can be obtained by using coverings [27–29], and generalization of subsystem based definition can be obtained by using other subsystems [30]. By the fact that, the system $(2^U, ^c, \cap, \cup)$ is Boolean algebra, one can generalize rough set theory using other algebraic systems such as Boolean algebra, lattice [1, 9], and posets.

2 Related Work

The problem to be discussed in this paper is to minimize the boundary region and this requires a new approximation approach which increases lower approximation and decreases upper approximation. Our approach to this problem is to generalize constructive methods for the theory of rough sets by using the atom based definition through two mappings instead in in one used by Jarvinen [1]. We also generalize the algebraic methods by using a complete atomic boolean lattice. By the fact the system $(2^U, ^c, \cap, \cup)$ is an atomic Boolean algebra whose atoms are singleton subsets of U. We replace U with the maximum element 1, with the minimum element 0, set intersection with meet, and set union with join.

In section 3 We recall and develop some notions and notations concerning lattice, ordered set, and properties of maps. Also we discuss the generalization of rough sets by using one operations in a more general setting of complete atomic Boolean lattices which was studied by Jarvinen in 2002. The purpose of section 4 is to introduce a new generalization of the constructive and algebraic definitions of rough sets using another direction by two operations which can be interpreted as the views of two experts. We construct new approximations called pairwise lower and upper approximations and study the properties of these approximations when the operations are extensive, symmetric or closed. Also we study the properties of the ordered set of the pairwise lower and upper of an element of a complete atomic Boolean lattice. Finally experimental examples are given showing that our generalizations can help in expert system and using lower and upper approximation given in this work will minimize the boundary region.

3 Preliminaries

In this section, we restated the basic concepts to make the work self contained.

The emergence of RST and its related notions paved the way for new types of ordering, so some authors [1,9] used these types in lattices [31,32].

Definition 3.1. A semilattice is an ordered set $\mathbf{B} = (B, \leq)$ in which every nonempty finite subset has an infimum(inf.). A sub-semi lattice is an ordered set B in which every nonempty finite subset has a supremum(sup.). An ordered set which is both a semi lattice and a sup-semi lattice is called a Lattice.

Definition 3.2. A complete lattice is an ordered set in which every subset has a (sup.) and an (inf.).

Lemma 3.3. *Let* $\mathbf{B} = (B, \leq)$ *be a complete lattice,* $S, T \subseteq B$ *and* $\{X_i : i \in I\} \subseteq P(B)$

(a) *If* $S \subseteq T$*, then* $\bigvee S \subseteq \bigvee T$
(b) $\bigvee (S \cup T) = (\bigvee S) \bigvee (\bigvee T)$
(c) $\bigvee (\bigcup \{X_i : i \in I\}) = \bigvee \{\bigvee X_i \in I\}$

Definition 3.4. A lattice $\mathbf{B} = (B, \leq)$ is called a Boolean lattice if

(a) B is distributive.
(b) B has a least element 0 and a greatest element 1, and
(c) Each $a \in B$ has a complement $a' \in B$ such that $a \vee a' = 1$ and $a \wedge a' = 0$.

Proposition 3.5. *Let* $\mathbf{B} = (B, \leq)$ *be a Boolean lattice, then for all* $a, b \in B$

(a) $0' = 1$ *and* $1' = 0$
(b) $a'' = a$
(c) $(a \vee b)' = a' \wedge b'$ *and* $(a \wedge b)' = a' \vee b'$.

Lemma 3.6. *Let* $\mathbf{B} = (B, \leq)$ *be a complete Boolean lattice. Then for all* $\{x_i : i \in I\} \subseteq B$ *and* $y \in B$

$$y \wedge (\bigvee_{i \in I} x_i) = \bigvee_{i \in I} (y \wedge x_i)$$

and

$$y \vee (\bigwedge_{i \in I} x_i) = \bigwedge_{i \in I} (y \vee x_i).$$

Definition 3.7. Let $\mathbf{B} = (B, \leq)$ be an ordered set and $x, y \in B$, we say that x is covered by y (or that y covers x), and write, $x \prec y$ if $x < y$ and there is no element z in B with $x < z < y$.

Definition 3.8. Let $\mathbf{B} = (B, \leq)$ be a lattice with a least element 0. Then $a \in B$ is called an atom if $0 \prec a$. The set of atoms of B is denoted by $A(B)$. The lattice B is atomic if every element of B is the supremum of the atoms below it, that is $x = \bigvee \{a \in A(B) : a \leq x\}$.

Remark 1. It is obvious that in a lattice $B = (B, \leq)$ with a least element 0,

$$a \wedge x \neq 0 \Longleftrightarrow a \leq x$$

for all $a \in A(B)$ and $x \in B$. This implies that $a \wedge b = 0$ for all $a, b \in A(B)$ such that $a \neq b$. Furthermore, if B is atomic, then for all $x \neq 0$ there exists an atom $a \in A(B)$ such that $a \leq x$. Namely, if $\{a \in A(B) : a \leq x\} = \phi$, then $x = \bigvee \{a \in A(B) : a \leq x\} = \bigvee \phi = 0$.

Now we recall some definitions concerning properties of maps.

Definition 3.9. Let $\mathbf{B} = (B, \leq)$ be an ordered set. A mapping $f : B \longrightarrow B$ is said to be extensive , if $x \leq f(x)$ for all $x \in B$. the map f is order preserving if $x \leq y$ implies $f(x) \leq f(y)$. Moreover, f is idempotent if $f(f(x)) = f(x)$ for all $x \in B$.

Definition 3.10. A map $c : B \longrightarrow B$ is said to be a closure operator on B, if c is extensive, order-preserving, and idempotent. An element $x \in B$ is c-closed if $c(x) = x$. furthermore, if $i : B \longrightarrow B$ is a closure operator on $B^{\vartheta} = (B, \geq)$ then I is an interior operator on B.

Definition 3.11. Let $\mathbf{B} = (B, \leq)$ be a Boolean lattice. Two maps $f : B \longrightarrow B$ and $g : B \longrightarrow B$ are the duals of each other if $f(x') = g(x)'$ and $g(x') = f(x)'$ for all $x \in B$.

The following obvious lemma shows that the dual of a closure operator is an interior operator.

Lemma 3.12. *Let $\mathbf{B} = (B, \leq)$ be a Boolean lattice and let $f : B \longrightarrow B$ be a closure operator on B. If $g : B \longrightarrow B$ is the dual of , then g is an interior operator on B.*

Definition 3.13. Let $\mathbf{B} = (B, \leq)$ and $\mathbf{Q} = (Q, \leq)$ be ordered sets. $f : B \longrightarrow Q$ is an order embedding, if for any $a, b \in B$, $a \leq b$ in **B** if and only if $f(a) \leq f(b)$ in **Q**; (Note that an order embedding is always an injection). An order-embedding f onto Q is called an order-isomorphism between **B** and **Q**, we say that **B** and **Q** are order-isomorphic and write $\mathbf{B} \cong \mathbf{Q}$. If $(B, \leq)$ and $(Q, \leq)$ are order-isomorphic, then **B** and **Q** are said to be dually order-isomorphic.

Galois connections are found in numerous settings from algebra to computer science and defined in two theoretically equivalent ways. In the one adopted here maps are order-preserving, and in the other maps are order-reversing.

Definition 3.14. Let $\mathbf{B} = (B, \leq)$ be an ordered set. A pair $(^{\nabla}, ^{\triangle})$ of maps $^{\nabla} : B \longrightarrow B$ and $^{\triangle} : B \longrightarrow B$ is called a dual Galois connection on B if $^{\nabla}$ and $^{\triangle}$ are order preserving and $x^{\nabla\triangle} \leq x \leq x^{\triangle\nabla}$ for all $x \in B$.

In 2002 Jouni Jarvinen [1] studied properties of approximations in a more general setting of complete atomic Boolean lattices.

Definition 3.15. Let $\mathbf{B} = (B, \leq)$ be a complete atomic Boolean lattice. We say that $\varphi : A(B) \to B$ is

(a) extensive, if $x \leq \varphi(x)$ for all $x \in A(B)$;
(b) symmetric, if $x \leq \varphi(y)$ implies $y \leq \varphi(x)$ for all $x, y \in A(B)$;
(c) closed, if $y \leq \varphi(x)$ implies $\varphi(y) \leq \varphi(x)$ for all $x, y \in A(B)$.

Let $\approx$ be a binary relation on a set U. The ordered set $(P(U), \subseteq)$ is a complete atomic Boolean lattice. Since the atoms $x(x \in U)$ of $(P(U), \subseteq)$ can be identified with the elements of U, the map

$$\varphi : U \longrightarrow P(U), x \longrightarrow [x]_{\approx}$$

may be considered to be of the form $\varphi : A(B) \longrightarrow B$, where $\mathbf{B} = (B, \leq)$ equals $(P(U), \subseteq)$. The following observations are obvious:

(1) $\approx$ is reflexive $\Longleftrightarrow \varphi$ is extensive;
(2) is symmetric $\Longleftrightarrow \varphi$ is symmetric;
(3) is transitive $\Longleftrightarrow \varphi$ is closed;

Definition 3.16. Let $\mathbf{B} = (B, \leq)$ be a complete atomic Boolean lattice and let $\varphi : A(B) \to B$ be any mapping. For any element $x \in B$, let

$$x^{\nabla} = \bigvee \{a \in A(B) : \varphi(a) \leq x\} \quad \text{and} \quad x^{\triangle} = \bigvee \{a \in A(B) : \varphi(a) \wedge x \neq 0\}$$

The elements x^∇ and $x^\triangle$ are called the *lower* and the *upper* approximation of x with respect to φ respectively. Two elements x and y are equivalent if they have the same upper and lower approximations. The resulting equivalence classes are called rough sets.

Proposition 3.17. *Let* $\mathbf{B} = (B, \leq)$ *be a complete atomic Boolean lattice and let* $\varphi : A(B) \to B$ *be any mapping. Then for all* $a \in A(B)$ *and* $x \in B$

(a) $a \leq x^\nabla \Longleftrightarrow \varphi(a) \leq x$
(b) $a \leq x^\triangle \Longleftrightarrow \varphi(a) \wedge x \neq 0$.

Proposition 3.18. *Let* $\mathbf{B} = (B, \leq)$ *be a complete atomic Boolean lattice and let* $\varphi : A(B) \to B$*be any mapping. Then*

(a) $0^\triangle$*=0 and* $1^\triangledown$*=1;*
(b) $x \leq y$ *implies* $x^\nabla \leq y^\nabla$ *and* $x^\triangle \leq y^\triangle$.

For all $S \subseteq B$*, we denote* $S^\nabla = \{x^\nabla : x \in S\}$ *and* $S^\triangle = \{x^\triangle : x \in S\}$.

Proposition 3.19. *Let* $\mathbf{B} = (B, \leq)$ *be a complete atomic Boolean lattice and let* $\varphi : A(B) \to B$ *be any mapping.*

(a) *The mappings* $^\triangle : B \longrightarrow B$ *and* $^\triangledown : B \longrightarrow B$ *are mutually dual.*
(b) *For all* $S \subseteq B$*,* $\vee S^\triangle = (\vee S)^\triangle$.
(c) *For all* $S \subseteq B$*,*$\wedge S^\nabla = (\wedge S)^\nabla$.

Proposition 3.20. *Let* $\mathbf{B} = (B, \leq)$ *be a complete atomic Boolean lattice and let* $\varphi : A(B) \to B$ *be an extensive mapping. Then*

(a) $0^\nabla = 0$ *and* $1^\triangle = 1$
(b) $x^\nabla \leq x \leq x^\triangle$ *for all* $x \in B$

We end this section by presenting the notion of multi valued information [33].

Definition 3.21. A multi valued information system is a triple $S = (U, A, \{V_a\}_{a \in A})$, where U is a non empty set of objects, A is a non empty set of attributes, and $\{V_a\}_{a \in A}$ an indexed set of values of attributes. Each attribute is a function $a : U \to P(V_a) - \{\phi\}$.

4 Bi Operation Generalization

In practical situations, it is preferred to study the problems from more than one viewpoint. Each one is represented by a relation or a function. Our aim in this article is to construct a new generalization using two operations.

Definition 4.1. Let $\mathbf{B} = (B, \leq)$ be a complete atomic Boolean lattice and let $\varphi_i : A(B) \to B(i = 1.2)$ be any mappings. For any element $x \in B$, let

$$x^{\nabla p} = x^{\nabla 1} \vee x^{\nabla 2} \quad \text{and} \quad x^{\triangle p} = x^{\triangle 1} \wedge x^{\triangle 2},$$

where for $i = 1, 2$

$$x^{\nabla i} = \bigvee \{a \in A(B) : \varphi_i(a) \leq x\} \quad \text{and} \quad x^{\triangle i} = \bigvee \{a \in A(B) : \varphi_i(a) \wedge x \neq 0\}$$

The elements $x^{\nabla p}$ and $x^{\triangle p}$ are called the *pairwise lower* and the *pairwise upper* approximation of x with respect to φ_1 and φ_2 respectively. Two elements x and y are equivalent if they have the same pairwise upper and pairwise lower approximations. The resulting equivalence classes are called pairwise rough sets.

Lemma 4.2. *Let $\mathbf{B} = (B, \leq)$ be a complete atomic Boolean lattice and let $\varphi_i : A(B) \to B(i = 1.2)$ be any mappings. Then for all $x \in B$,*

(1) $x^{\nabla p} = \bigvee \{a \in A(B) : \varphi_1(a) \leq x \ \ or \ \ \varphi_2(a) \leq x\}$
(2) $x^{\triangle p} = \bigvee \{a \in A(B) : \varphi_1(a) \wedge x \neq 0 \ \ and \ \ \varphi_2(a) \wedge x \neq 0\}$

Proof. $x^{\nabla p} = x^{\nabla 1} \vee x^{\nabla 2} = (\bigvee \{a \in A(B) : \varphi_1(a) \leq x\}) \bigvee (\bigvee \{a \in A(B) : \varphi_2(a) \leq x\})$
$=\bigvee (\{a \in A(B) : \varphi_1(a) \leq x\} \bigcup \{a \in A(B) : \varphi_2(a) \leq x\})$ (by lemma 3.3)
$=\bigvee \{a \in A(B) : \varphi_1(a) \leq x \ \ or \ \ \varphi_2(a) \leq x\}$.

$x^{\triangle p} = x^{\triangle 1} \wedge x^{\triangle 2} = (\bigvee \{b \in A(B) : \varphi_1(b) \wedge x \neq 0\}) \bigwedge (\bigvee \{a \in A(B) : \varphi_2(a) \wedge x \neq 0\})$.
Let $y = \bigvee \{b \in A(B) : \varphi_1(b) \wedge x \neq 0\}$, so $x^{\triangle p} = y \wedge \bigvee \{a \in A(B) : \varphi_2(a) \wedge x \neq 0\}=$
$\bigvee \{y \wedge a : a \in A(B) \ \ and \ \ \varphi_2(a) \wedge x \neq 0\}$ (by lemma 3.6)
$=\bigvee \{\bigvee \{b \in A(B) : \varphi_1(b) \wedge x \neq 0\} \wedge a : a \in A(B) \ \ and \ \ \varphi_2(a) \wedge x \neq 0\}\}$
$=\bigvee \{\bigvee \{b \wedge a : a, b \in A(B), \varphi_1(b) \wedge x \neq 0 \ \ and \ \ \varphi_2(a) \wedge x \neq 0\}$
$=\bigvee \{a \wedge b : a, b \in A(B), \varphi_1(b) \wedge x \neq 0 \ \ and \ \ \varphi_2(a) \wedge x \neq 0\}$.
If $a \neq b$, then $a \wedge b = 0$ because $a, b \in a(B)$. Hence $a = b \ i.e. \ a \wedge b = a$. Therefore $x^{\triangle p} = \bigvee \{a \in A(B) : \varphi_1(a) \wedge x \neq 0 \ and \ \varphi_2(a) \wedge x \neq 0\}$. □

Lemma 4.3. *Let $\mathbf{B} = (B, \leq)$ be a complete atomic Boolean lattice and let $\varphi_i : A(B) \to B(i = 1.2)$ be any mappings. Then for all $a \in A(B)$ and $x \in B$*

(a) $a \leq x^{\nabla p} \Longleftrightarrow \varphi_1(a) \leq x \ \ or \ \ \varphi_2(a) \leq x$
(b) $a \leq x^{\triangle p} \Longleftrightarrow \varphi_1(a) \wedge x \neq 0 \ and \ \varphi_2(a) \wedge x \neq 0$.

Proof. **(a)** ($\Longrightarrow$) Suppose that $a \leq x^{\nabla p} = \bigvee \{b \in A(B) : \varphi_1(b) \leq x \ or \ \varphi_2(b) \leq x\}$. If $\varphi_1(a) \nleq x$ and $\varphi_2(a) \nleq x$, then $a \wedge x^{\nabla p} = a \wedge \bigvee \{b \in A(B) : \varphi_1(b) \leq x \ or \ \varphi_2(b) \leq x\}$ $=\bigvee \{a \wedge b : \varphi_1(b) \leq x \ or \ \varphi_2(b) \leq x\}$. Since $\varphi_1(a) \nleq x$and $\varphi_2(a) \nleq x$, then $a \neq b$.So $a \wedge b = 0$ because $a, b \in A(B)$. Hence $a \wedge x^{\nabla p} = 0$. This implies that $a \leq (x^{\nabla p})'$, a

contradiction!
($\Longleftarrow$) Suppose that $\varphi_1(a) \leq x$ or $\varphi_2(a) \leq x$, then $a \leq \bigvee\{a \in A(B) : \varphi_1(a) \leq x \; or \; \varphi_2(a) \leq x\}$=$x^{\nabla p}$
(b)($\Longrightarrow$) suppose that $a \leq x^{\triangle p}$= $x^{\triangle 1} \wedge x^{\triangle 2}$, then $a \leq x^{\triangle 1}$ and $a \leq x^{\triangle 2}$. Hence $\varphi_1(a) \wedge x \neq 0 \; and \; \varphi_2(a) \wedge x \neq 0$ (by Proposition 3.2(b))
($\Longleftarrow$) If $\varphi_1(a) \wedge x \neq 0 \; and \; \varphi_2(a) \wedge x \neq 0$, then $a \leq x^{\triangle 1}$ and $a \leq x^{\triangle 2}$by Proposition 3.2(b)). Therefore $a \leq x^{\triangle 1} \wedge x^{\triangle 2}$=$x^{\triangle p}$. □

Remark 2. The two operations suggested in this work are suitable also for other operators based on binary relations. The following observation illustrate this idea.

If U is any universal set , then $P(U)$ is a complete atomic boolean lattice whose atoms are singleton subset of U. Let R_1 and R_1 be two general relations on U, we define two mapping $\varphi_1 : A(B) \longrightarrow B : U \longrightarrow P(U), x \longrightarrow R_1(x)$ and $\varphi_2 : A(B) \longrightarrow B : U \longrightarrow P(U), x \longrightarrow R_2(x)$ where $R_i(x) = \{y \in U : xR_iy\}$(i=1,2). Let $X^{\nabla i} = \cup\{x \in U : R_i(x) \subseteq X\}$ and $X^{\triangle i} = \cup\{x \in U : R_i(x) \cap X \neq \phi\}$ (i=1,2). Then $X^{\nabla p} = X^{\nabla 1} \cup X^{\nabla 2} = \cup\{x \in U : R_1(x) \subseteq X \; or \; R_2(x) \subseteq X\}$ and $X^{\triangle p} = X^{\triangle 1} \cap X^{\triangle 2} = \cup\{x \in U : R_1(x) \cap X \neq \phi \; and \; R_2(x) \cap X \neq \phi\}$

Next we present some properties of the mappings $^{\nabla p} : B \longrightarrow B$ and $^{\triangle p} : B \longrightarrow B$.

Proposition 4.4. *Let* $\mathbf{B} = (B, \leq)$ *be a complete atomic Boolean lattice and let* $\varphi_i : A(B) \rightarrow B (i = 1.2)$ *be any mappings. Then*

(a) $0^{\triangle p}$*=0 and* 1^{∇^p}*=1;*
(b) $x \leq y$ *implies* $x^{\nabla p} \leq y^{\nabla p}$ *and* $x^{\triangle p} \leq y^{\triangle p}$.

Proof. **(a)** $0^{\triangle p}$= $0^{\triangle 1} \wedge 0^{\triangle 2}$= $0 \wedge 0$=0 (by Proposition 3.3(b)) and $1^{\nabla p}$= $1^{\nabla^1} \vee 1^{\nabla^2}$= $1 \vee 1$ =1 (by Proposition 3.3(a)).
(b) Assume that $x \leq y$, then $x^{\nabla 1} \leq y^{\nabla 1}$ and $x^{\nabla 2} \leq y^{\nabla 2}$ (by Proposition 3.3(b)). Therefore $x^{\nabla p} = x^{\nabla 1} \vee x^{\nabla 2} \leq y^{\nabla 1} \vee y^{\nabla 2} = y^{\nabla p}$.Also $x \leq y$ implies that $x^{\triangle 1} \leq y^{\triangle 1}$and $x^{\triangle 2} \leq y^{\triangle 2}$. Hence $x^{\triangle p} = x^{\triangle 1} \wedge x^{\triangle 2} \leq y^{\triangle 1} \wedge y^{\triangle 2} = y^{\triangle p}$. □

Proposition 4.5. *Let*$\mathbf{B} = (B, \leq)$ *be a complete atomic Boolean lattice and let* $\varphi_i : A(B) \rightarrow B(i = 1.2)$*be any mappings. Then the mappings* $^{\nabla p} : B \longrightarrow B$ *and* $^{\triangle p} : B \longrightarrow B$ *are mutually dual.*

Proof. We must show that $(x^{\triangle p})' = (x')^{\nabla_p}$ and $(x^{\nabla p})' = (x')^{\triangle_p}$ for all $x \in B$. Let $x \in B$, since $x^{\triangle p} = x^{\triangle 1} \wedge x^{\triangle 2}$, then $(x^{\triangle p})' = (x^{\triangle 1} \wedge x^{\triangle 2})'$ =$(x^{\triangle 1})' \vee (x^{\triangle 2})'$ (by Demorgans law).But $(x^{\triangle 1})' = (x')^{\nabla_1}$ and $(x^{\triangle 2})' = (x')^{\nabla_2}$ (by Proposition 3.20(a)). So $(x^{\triangle p})' = (x')^{\nabla_1} \vee (x')^{\nabla_2} = (x')^{\nabla_p}$. Also $(x^{\nabla p})' = (x^{\nabla 1} \vee x^{\nabla 2})' = (x^{\nabla 1})' \wedge (x^{\nabla 2})' = (x')^{\triangle_1} \wedge (x')^{\triangle_2} = (x')^{\triangle_p}$(by Proposition 3.20(a)). □

The following is an example of this idea.

Example 4.1 Let $U = \{0, a, b, c\}$ and $B = P(U)$. Let the mappings $\varphi_i : A(B) \to B$ $i = 1, 2$ be defined as follows:

$$\varphi_1(a) = \{a, b\}, \qquad \varphi_1(b) = \{b\}, \qquad \varphi_1(c) = \{b, c\}$$

and

$$\varphi_2(a) = \{a\}, \qquad \varphi_2(b) = \{b, c\}, \qquad \varphi_2(c) = \{a, c\}.$$

The pairwise lower and upper approximations of elements of B can be described in Table 4.1. The diagrams of ${}^{\nabla p} : B \longrightarrow B$ and ${}^{\triangle p} : B \longrightarrow B$ are in Figure 4.1

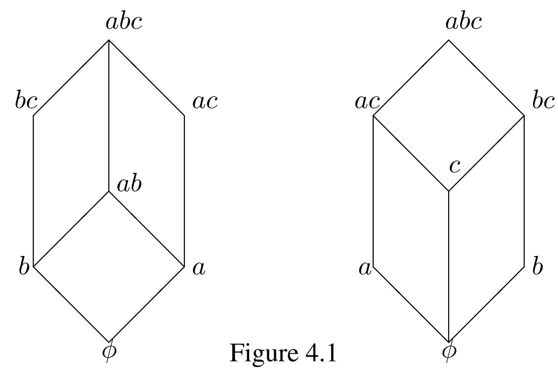

Figure 4.1

x	$x^{\nabla p}$	$x^{\triangle p}$
a	a	a
b	b	b
c	ϕ	c
{a,b}	{a,b}	{a,b,c}
{a,c}	{a,c}	{a,c}
{b,c}	{b,c}	{b,c}
{a,b,c}	{a,b,c}	{a,b,c}

Table 4.1: Pairwise lower and pairwise upper approximations

For all $S \subseteq B$, we denote $S^{\nabla p} = \{x^{\nabla p} : x \in S\}$ and $S^{\triangle p} = \{x^{\triangle p} : x \in S\}$.

Proposition 4.6. *Let* $\mathbf{B} = (B, \leq)$ *be a complete atomic Boolean lattice and let* $\varphi_i : A(B) \to B (i = 1.2)$ *be any mappings.*

(a) *For all* $S \subseteq B$, $\vee S^{\triangle p} = (\vee S)^{\triangle^p}$.

(b) *For all* $S \subseteq B$*,*$\wedge S^{\nabla p} = (\wedge S)^{\nabla^p}$*.*

(c) $(B^{\triangle p}, \leq)$ *is a complete lattice; 0 is the least element and* $1^{\triangle_P}$ *is the greatest element of* $(B^{\triangle p}, \leq)$*.*

(d) $(B^{\nabla p}, \leq)$ *is a complete lattice;* 0^{∇_P} *is the least element and 1 is the greatest element of* $(B^{\nabla p}, \leq)$*.*

(e) *The kernel* $\theta_{\nabla_P} = \{(x, y) : x^{\nabla_p} = y^{\nabla_p}\}$ *of the map* $^{\nabla_P} : B \longrightarrow B$ *is a congruence on the semilattice* $(B, \wedge)$ *such that the* θ_{∇_P}*-class of any x has a least element.*

(f) *The kernel* $\theta_{\triangle_P} = \{(x, y) : x^{\triangle_p} = y^{\triangle_p}\}$ *of the map* $^{\triangle_P} : B \longrightarrow B$ *is a congruence on the semilattice* $(B, \vee)$ *such that the* $\theta_{\triangle_P}$*-class of any x has a greatest element.*

Proof. **(a)** Let $S \subseteq B$, then $(\vee S)^{\triangle_p} = (\vee S)^{\triangle_1} \wedge (\vee S)^{\triangle_2}$ =$\vee S^{\triangle 1} \wedge \vee S^{\triangle 2}$ (by Proposition 3.19(b)). Let $a = \vee S^{\triangle 1}$, so $(\vee S)^{\triangle_p} = a \wedge \vee S^{\triangle 2} = a \wedge \{\vee x^{\triangle 2} : x \in S\} = \vee_{x \in S} (a \wedge x^{\triangle 2})$ (by Lemma 3.6). =$\vee_{x \in S} (\vee \{x^{\triangle 1} : x \in S\} \wedge x^{\triangle 2})$=$\vee_{x \in S} \vee_{x \in S} (x^{\triangle 1} \wedge x^{\triangle 2})$= $\vee_{x \in S} (x^{\triangle 1} \wedge x^{\triangle 2})$ =$\vee \{x^{\triangle 1} \wedge x^{\triangle 2} : x \in S\}$ =$\vee \{x^{\triangle p} : x \in S\}$=$\vee S^{\triangle p}$.
Claim **(b)** can be proved similarly. Assertions **(c)** and **(d)** follow easily from **(a)**, **(b)**, and Lemma 4.2(a).

(e) It can be easily seen that θ_{∇_P} is an equivalence on p. Let $x_1, x_2, y_1, y_2 \in B$ and assume that $(x_1, y_1), (x_2, y_2) \in \theta_{\nabla_P}$ then

$$(x_1 \wedge x_2)^{\nabla_P} = {x_1}^{\nabla_P} \wedge {x_2}^{\nabla_P} = {y_1}^{\nabla_P} \wedge {y_2}^{\nabla_P} = (y_1 \wedge y_2)^{\nabla_P}$$

.

Thus,θ_{∇_P} is a congruence on $(B, \wedge)$.

It is clear that $\wedge [x]_{\theta_{\nabla_P}}$ is the least element in the congruence class of x since

$$(\wedge \{y : y^{\nabla_P} = x^{\nabla_P}\})^{\nabla_P} = \wedge \{y^{\nabla_P} : y^{\nabla_P} = x^{\nabla_P}\} = x^{\nabla_P}$$

Assertion **(f)** can be proved similarly. □

Next we show that $(B^{\triangle p}, \leq)$ and $(B^{\nabla p}, \leq)$ are dually order-isomorphic.

Proposition 4.7. $(B^{\nabla p}, \leq) \cong (B^{\nabla p}, \geq)$

Proof. We show that $x^{\triangle p} \longrightarrow (x')^{\nabla_p}$ is the required dual order isomorphism. It is obvious that $x^{\triangle p} \longrightarrow (x')^{\nabla_p}$ is onto $(B^{\nabla p}, \geq)$. We show that $x^{\triangle p} \longrightarrow (x')^{\nabla_p}$ is order embedding. Suppose that $x^{\triangle p} \leq y^{\triangle p}$ and $(y')^{\nabla_p} \nleq (x')^{\nabla_p}$. So there exists $a \in A(B)$ such that $a \leq (y')^{\nabla_p}$ and $a \nleq (x')^{\nabla_p}$ i.e. $\varphi_1(a) \leq y'$ or $\varphi_2(a) \leq y'$ but $\varphi_1(a) \nleq x'$ and $\varphi_2(a) \nleq x'$. Since $\varphi_1(a) \nleq x'$ and $\varphi_2(a) \nleq x'$ are equivalent to $\varphi_1(a) \wedge x \neq 0$ and $\varphi_2(a) \wedge x \neq 0$, we have $\varphi_1(a) \wedge y \neq 0$ and $\varphi_2(a) \wedge y \neq 0$ because $x^{\triangle p} \leq y^{\triangle p}$. But this means that $\varphi_1(a) \nleq y'$ and $\varphi_2(a) \nleq y'$, a contradiction! Hence $(y')^{\nabla_p} \leq (x')^{\nabla_p}$.

On the other hand, assume that $(y')^{\nabla_p} \leq (x')^{\nabla_p}$ and $x^{\triangle p} \nleq y^{\triangle p}$, so there exists $a \in A(B)$ such that $a \leq x^{\triangle p}$ and $a \nleq y^{\triangle p}$. Hence $\varphi_1(a) \wedge x \neq 0$ and $\varphi_2(a) \wedge x \neq 0$

but either $\varphi_1(a) \wedge y = 0$ or $\varphi_2(a) \wedge y = 0$ (by Lemma 4.3). This implies that either $\varphi_1(a) \leq y'$ or $\varphi_2(a) \leq y'$ and thus $a \leq (y')^{\nabla_p}$. Since $(y')^{\nabla_p} \leq (x')^{\nabla_p}$, then $a \leq (x')^{\nabla_p}$. Therefore $\varphi_1(a) \leq x'$ or $\varphi_2(a) \leq x'$ which equivalent to $\varphi_1(a) \wedge x = 0$ or $\varphi_2(a) \wedge x = 0$ a contradiction! Consequently $x^{\triangle p} \leq y^{\triangle p}$. □

Next we study the properties of approximations more closely in cases when the corresponding mappings $\varphi_i : A(B) \to B(i = 1.2)$ is extensive, symmetric or closed.

Extensiveness

Proposition 4.8. *Let* $\mathbf{B} = (B, \leq)$ *be a complete atomic Boolean lattice and let* $\varphi_i : A(B) \to B(i = 1.2)$ *be any two extensive mappings. Then*

(a) $0^{\nabla p} = 0$ *and* $1^{\triangle p} = 1$
(b) $x^{\nabla p} \leq x \leq x^{\triangle p}$ *for all* $x \in B$

Proof. **(a)** $0^{\nabla p} = 0^{\nabla 1} \vee 0^{\nabla 2} = 0 \vee 0 = 0$, and $1^{\triangle p} = 1^{\triangle 1} \wedge 1^{\triangle 2} = 1 \wedge 1 = 1$ (by Proposition 3.20(a)). **(b)** Let $x \in B$, since $x^{\nabla 1} \leq x \leq x^{\triangle 1}$ and $x^{\nabla 2} \leq x \leq x^{\triangle 2}$ (by Proposition 3.20(b)), then $x^{\nabla 1} \vee x^{\nabla 2} \leq x \leq x^{\triangle 1} \wedge x^{\triangle 2}$ i.e. $x^{\nabla p} \leq x \leq x^{\triangle p}$. □

In the following example we show that if φ_1 and φ_2 are not extensive mappings, then the conditions (a) and (b) are not necessary.

Example 4.2 Let $B = \{0, a, b, c, d, e, f, 1\}$ and let the order $\leq$ be defined as in Figure 4.2.

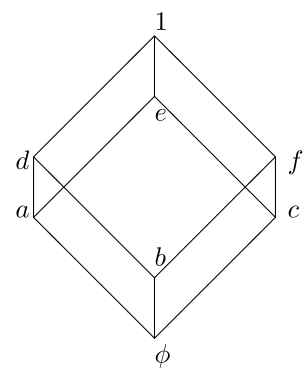

Figure 4.2

The set of atoms of a complete atomic Boolean lattice $\mathbf{B} = (B, \leq)$ is $\{a, b, c\}$. Let the mappings $\varphi_i : A(B) \to B(i = 1.2)$ be defined as follows,

$$\varphi_1(a) = 0, \qquad \varphi_1(b) = e, \qquad \varphi_1(c) = f,$$

and

$$\varphi_2(a) = d, \qquad \varphi_2(b) = 0, \qquad \varphi_2(c) = c.$$

The mappings φ_1 and φ_2 are not extensive because $a \nleq 0 = \varphi_1(a)$ and $b \nleq 0 = \varphi_2(b)$.

We show that the conditions (a) and (b) in the previous proposition are not valid.

$$0^{\nabla p} = \bigvee \{a \in A(B) : \varphi_1(a) \leq 0 \text{ or } \varphi_2(a) \leq 0\} = a \vee b = d \neq 0.$$

$$1^{\triangle p} = \bigvee \{a \in A(B) : \varphi_1(a) \wedge 1 \neq 0 \text{ and } \varphi_2(a) \wedge 1 \neq 0\} = c \neq 1.$$

Also $d^{\nabla p} = a$ and $d^{\triangle p} = 0$, hence $d^{\nabla p} \leq d \nleq d^{\triangle p}$. Also $e^{\nabla p} = b \vee c = f$ and $e^{\triangle p} = c$. Hence $d^{\nabla p} \nleq d \nleq d^{\triangle p}$.

Symmetry

In this article we assume that φ_1 and φ_2 are symmetric mappings. First we show by example that the pair $(^{\nabla^p}, ^{\triangle p})$ is not a dual Galois connection.

Example 4.3 Let $B = \{0, a, b, c, d, e, f, 1\}$ and let the order $\leq$ be defined as in Figure 4.1. Let the mappings $\varphi_i : A(B) \to B (i = 1.2)$ be defined as follows,

$$\varphi_1(a) = d, \qquad \varphi_1(b) = e, \qquad \varphi_1(c) = f,$$

and

$$\varphi_2(a) = b, \qquad \varphi_2(b) = f, \qquad \varphi_2(c) = d,$$

The mappings φ_1 and φ_2 are symmetric.Let x=d then $x^{\nabla p} = a \vee c = e$ and $x^{\nabla^p \triangle p} = b \vee c = f$. Since $f \nleq d$, then $x^{\nabla^p \triangle p} \nleq x$. Hence $(^{\nabla^p}, ^{\triangle p})$ is not a dual Galois connection.

Proposition 4.9. *Let* $\mathbf{B} = (B, \leq)$ *be a chain and* $\varphi_i : A(B) \to B (i = 1.2)$ *be any two symmetric mappings such that* φ_1 *and* φ_2 *are order preserving. Then the pair* $(^{\nabla^p}, ^{\triangle p})$ *is a dual Galois connection on* B.

Proof. The mapping $^{\nabla p} : B \longrightarrow B$ and $^{\triangle p} : B \longrightarrow B$ are order preserving. We show that $x^{\nabla^p \triangle p} \leq x \leq x^{\triangle^p \nabla p}$ for all $x \in B$. Let $a \in A(B)$. If $a \leq x^{\nabla^p \triangle p}$, then $\varphi_1(a) \wedge x^{\nabla^p} \neq 0$ and $\varphi_2(a) \wedge x^{\nabla^p} \neq 0$. This implies that there exist $b_1, b_2 \in A(B)$ such that $b_1 \leq \varphi_1(a) \wedge x^{\nabla^p}$ and $b_2 \leq \varphi_2(a) \wedge x^{\nabla^p}$. So $b_1 \leq \varphi_1(a)$,$b_1 \leq x^{\nabla^p}$, $b_2 \leq \varphi_2(a)$ and $b_2 \leq x^{\nabla^p}$. Since $b_1 \leq x^{\nabla^p}$, $\varphi_1(b_1) \leq x$ or $\varphi_2(b_1) \leq x$. Since $b_2 \leq x^{\nabla^p}$, $\varphi_1(b_2) \leq x$ or $\varphi_2(b_2) \leq x$. Since φ_1 and φ_2 are symmetric, $a \leq \varphi_1(b_1)$ and $a \leq \varphi_2(b_2)$. If $\varphi_1(b_1) \leq x$ or $\varphi_2(b_2) \leq x$, then $a \leq x$. Now suppose that $\varphi_1(b_2) \leq x$ and $\varphi_2(b_1) \leq x$. Since B is chain, then either $b_1 \leq b_2$ or $b_2 \leq b_1$. If $b_1 \leq b_2$, then $\varphi_1(b_1) \leq \varphi_1(b_2)$ because φ_1 is order preserving. Hence $a \leq \varphi_1(b_1) \leq \varphi_1(b_2) \leq x$. If $b_2 \leq b_1$, then $\varphi_2(b_2) \leq \varphi_2(b_1)$ because φ_2 is order preserving and hence $a \leq \varphi_2(b_2) \leq \varphi_2(b_1) \leq x$. Therefore $\{a \in A(B) : a \leq x^{\nabla^p \triangle p}\} \subseteq \{a \in A(B) : a \leq x\}$, which implies $x^{\nabla^p \triangle p} \leq x$.

If we denote $y = x'$ then $y^{\nabla^p \triangle p} \leq y$ implies that $x = y' \leq (y^{\nabla^p \triangle p})' = ((y^{\nabla p})')^{\nabla p} = (y')^{\triangle_p \nabla p} = x^{\triangle_p \nabla p}$. $\square$

Proposition 4.10. *Let* $\mathbf{B} = (B, \leq)$ *be a complete atomic Boolean lattice and let* φ_i : $A(B) \to B(i = 1.2)$ *be two symmetric mappings. Then*

(a) $a^{\triangle_p} = \varphi_1(a) \wedge \varphi_2(a)$ *for all* $a \in A(B)$.
(b) $x^{\triangle_p} = \bigvee \{\varphi_1(a) \wedge \varphi_2(a) : a \in A(B) \;\; and \;\; a \leq x\}$ *for any* $x \in B$

Proof. **(a)** Let $a \in A(B)$, then $a^{\triangle_p} = \bigvee \{b \in A(B) : \varphi_1(b) \wedge a \neq 0$ and $\varphi_2(b) \wedge a \neq 0\}$. But $\varphi_1(b) \wedge a \neq 0$ and $\varphi_2(a) \wedge x \neq 0$ are equivalent to $a \leq \varphi_1(b)$ and $a \leq \varphi_2(b)$. Since φ_1 and φ_2 are symmetric, then $b \leq \varphi_1(a)$ and $b \leq \varphi_2(a)$. So $b \leq \varphi_1(a) \wedge \varphi_2(a)$ and hence $a^{\triangle_p} = \bigvee \{b \in A(B) : b \leq \varphi_1(a) \wedge \varphi_2(a)\}$.

$$
\begin{aligned}
(\mathbf{b}) \;\; x^{\triangle_p} = \left(\bigvee \{a \in a(B) \colon a \leq x\}\right)^{\triangle_p} &= \bigvee \{a^{\triangle_p} : a \in A(B) \;\; and \;\; a \leq x\} \\
&= \bigvee \{\varphi_1(a) \wedge \varphi_2(a) \colon a \in A(B) \;\; \text{and} \;\; a \leq x\}.
\end{aligned}
$$

□

Closeness

Here we studies the case in which φ_1 and φ_2 are closed mappings. First we present the following observation.

Proposition 4.11. *Let* $\mathbf{B} = (B, \leq)$ *be a complete atomic Boolean lattice and let* φ_i : $A(B) \to B(i = 1.2)$ *be any two closed mappings. Then for all* $x \in B$,

(a) $x^{\triangle_p \triangle_p} \leq x^{\triangle_p}$
(b) $x^{\nabla_p} \leq x^{\nabla_p \nabla_p}$

Proof. **(a)** Let $a \in A(B)$, we show that

$$\{a \in A(B) : a \leq x^{\triangle_p \triangle_p}\} \subseteq \{a \in A(B) : a \leq x^{\triangle_p}\}.$$

Assume that $a \leq x^{\triangle_p \triangle_p}$, then $\varphi_1(a) \wedge x^{\triangle_p} \neq 0 \; and \; \varphi_2(a) \wedge x^{\triangle_p} \neq 0$. So there exists $b, c \in A(B)$ such that $b \leq \varphi_1(a)$, $b \leq x^{\triangle_p}$, $c \leq \varphi_2(a)$ and $c \leq x^{\triangle_p}$. Hence $\varphi_1(b) \wedge x \neq 0$, $\varphi_2(b) \wedge x \neq 0$, $\varphi_1(c) \wedge x \neq 0$,and $\varphi_2(c) \wedge x \neq 0$. Since φ_1 is closed and $b \leq \varphi_1(a)$, then $\varphi_1(b) \leq \varphi_1(a)$. Also since φ_2 is closed and $c \leq \varphi_2(a)$, then $\varphi_2(c) \leq \varphi_2(a)$. But $\varphi_1(b) \wedge x \neq 0$ and $\varphi_1(b) \leq \varphi_1(a)$ implies $\varphi_1(a) \wedge x \neq 0$. Also $\varphi_2(c) \wedge x \neq 0$ and $\varphi_2(c) \leq \varphi_2(a)$ implies $\varphi_2(a) \wedge x \neq 0$. But $\varphi_1(a) \wedge x \neq 0$ and $\varphi_2(a) \wedge x \neq 0$ are equivalent to $a \leq x^{\triangle_p}$. Therefore $\{a \in A(B) : a \leq x^{\triangle_p \triangle_p}\} \subseteq \{a \in A(B) : a \leq x^{\triangle_p}\}$. Thus $x^{\triangle_p \triangle_p} = \bigvee \{a \in A(B) : a \leq x^{\triangle_p \triangle_p}\} \leq \bigvee \{a \in A(B) : a \leq x^{\triangle_p}\} = x^{\triangle_p}$.

(b) Let us denote that $y = x'$. Then by (a) $y^{\triangle_p \triangle_p} \leq y^{\triangle_p}$ and $x^{\nabla_p} = (y')^{\nabla_p} = (y^{\triangle_p})' \leq (y^{\triangle_p \triangle_p})' = ((y^{\triangle_p})')^{\nabla_p} = (y')^{\nabla^{p \nabla_p}} = x^{\nabla_p \nabla_p}$. □

In the following example we show that if φ_1 and φ_2 are not closed mappings, then the conditions (a) and (b) are not necessary.

Example 4.4 Let $B = \{0, a, b, c, d, e, f, 1\}$ and let the order $\leq$ be defined as in Figure 4.1. Let the mappings $\varphi_i : A(B) \to B(i = 1.2)$ be defined as follows,

$$\varphi_1(a) = b \qquad \varphi_1(b) = e \qquad \varphi_1(c) = f$$

and

$$\varphi_2(a) = b \qquad \varphi_2(b) = f \qquad \varphi_2(c) = d$$

The mappings φ_1 and φ_2 are not closed because $a \leq e = \varphi_1(b)$ but $\varphi_1(a) = b \nleq e$ and $c \leq f = \varphi_2(b)$ but $\varphi_2(c) = d \nleq f$. We show that conditions (a) and (b) in the previous proposition are not valid. Since $e^{\triangle_p} = b \vee c = f$ and $e^{\triangle_p \triangle_p} = f^{\triangle_p} = a \vee b \vee c = 1 \nleq f = e^{\triangle_p}$, then in general $x^{\triangle_p \triangle_p} \nleq x^{\triangle_p}$. Also since $d^{\nabla p} = a \vee c = e$, $d^{\nabla p \nabla p} = e^{\nabla p} = b$ and $e \nleq b$, then in general $x^{\nabla p} \nleq x^{\nabla p \nabla p}$.

In the following proposition we study the case when the mappings φ_1 and φ_2 are extensive and closed.

Proposition 4.12. *Let* $\mathbf{B} = (B, \leq)$ *be a complete atomic Boolean lattice and let* $\varphi_i : A(B) \to B(i = 1.2)$ *be any two extensive and closed mappings. Then*

(a) *The mapping* $^{\triangle p} : B \longrightarrow B$ *is a closure operator.*
(b) *The mapping* $^{\nabla p} : B \longrightarrow B$ *is an interior operator.*
(c) $(B^{\nabla p}, \leq)$ *and* $(B^{\triangle p}, \leq)$ *are distributive sublattices of* $(B, \leq)$.

Proof. **(a)** The mapping $^{\triangle p} : B \longrightarrow B$ is extensive because φ_1 and φ_2 are extensive (by Proposition 4.8(b)) , and it is order preserving (by Proposition 4.4(b)). Also, $x^{\triangle_p \triangle_p} \leq x^{\triangle_p}$ by Proposition 4.11(a) and $x^{\triangle_p} \leq x^{\triangle_p \triangle_p}$ holds since φ_1 and φ_2 are extensive. Claim (b) follows from Lemma 3.12 and Proposition 3.18(a). **(c)** Suppose that x^{∇^p} , $y^{\nabla^p} \in B^{\nabla p}$. Then $x^{\nabla^p} \wedge y^{\nabla^p} = (x \wedge y)^{\nabla^p}$ (by Proposition 3.19(c)), which implies that $x^{\nabla^p} \wedge y^{\nabla^p} \in B^{\nabla p}$. Next we show that $x^{\nabla^p} \vee y^{\nabla^p} \in B^{\nabla p}$. It is obvious that $x^{\nabla^p} \leq x^{\nabla^p} \vee y^{\nabla^p} \in B^{\nabla p}$ and $x^{\nabla^p} = x^{\nabla^p \nabla^p}(by) \leq (x^{\nabla^p} \vee y^{\nabla^p})^{\nabla p}$. Similarly $y^{\nabla^p} \leq (x^{\nabla^p} \vee y^{\nabla^p})^{\nabla p}$. So $(x^{\nabla^p} \vee y^{\nabla^p})^{\nabla p}$ is an upper bound of x^{∇^p} and y^{∇^p}. We show that it is a greatest upper bound. Let $z \in B$ be an upper bound of x^{∇^p} and y^{∇^p}, then $x^{\nabla^p} \leq z$ and $y^{\nabla^p} \leq z$. Since φ_1 and φ_2 are extensive, $(x^{\nabla^p} \vee y^{\nabla^p})^{\nabla p} \leq x^{\nabla^p} \vee y^{\nabla^p} \leq z$. Thus $x^{\nabla^p} \vee y^{\nabla^p} = (x^{\nabla^p} \vee y^{\nabla^p})^{\nabla p}$ and therefore $x^{\nabla^p} \vee y^{\nabla^p} \in B^{\nabla p}$. The other part can be proved analogously. Since every sublattice of a distributive lattice is distributive, see [31] for example, $(B^{\nabla p}, \leq)$ and $(B^{\triangle p}, \leq)$ are distributive sublattices of $(B, \leq)$. $\square$

Remark 3. An expert system, also known as a knowledge based system, is a computer program that contains some of the subject-specific knowledge, and contains the knowledge and analytical skills of one or more human experts. The most common form of expert systems is a program made up of a set of rules that analyzes information (usually supplied by the user of the system) about a specific class of problems, as well as provides mathematical analysis of the problem(s), and, depending upon its design, recommends a course of user action in order to implement corrections. To construct a set of rules for a given data, it is necessary to pass by a mathematical model. Rough set theory, Lattice theory, fuzzy set theory, differential equations,$\cdots$ are examples for mathematical models. The present work introduced a modification for rough set model, which in turn apply a progress in expert systems.

The following example shows that the use of two operators minimizes the boundary region and increase the accuracy.

Example 4.5 This example is a small form of multi-valued information table of a file containing some persons $U = \{p_1, p_2, \cdots, p_6\}$ applied for a job. It contains the languages they speak and scientific degrees that they have. Some groups of them are chosen and we must assess the accuracy of the decision. This is shown in table 4.2. We have

(1) Measures of the accuracy of the decision with respect to language only.

(2) Measures of the accuracy of the decision with respect to scientific degrees only.

(3) Measures of the accuracy of the decision with respect to language and scientific degrees.

Relationships among the objects (persons) from the set U are determined by their properties. Typically, these relationships have the form of binary relations. These relations are referred to as information relations and they are determined by the problem we have. We prefer persons who speak more languages and have more scientific degrees, so we choose the subset relations.

Table 4.2:

	$Language$	$Degree$
p_1	F, D	Bs, Ms, PhD
p_2	H, R	Bs
p_3	F, D, S	Bs, Ms
p_4	F	Bs, Ms
p_5	R	Bs
p_6	F,S	Bs

We choose the subset relation R_1 between the persons with respect to the attribute languages and R_2 with respect to the attribute degrees. So $p_i R_1 p_j$ iff $Lanp_i \subseteq lanp_j$ and $p_i R_2 p_j$ iff $Degp_i \subseteq Degp_j$. Then

$$\begin{aligned} R_1 = \{&(p_1,p_1),(p_1,p_3),(p_2,p_2),(p_3,p_3),(p_4,p_4)(p_4,p_1),(p_4,p_3),(p_4,p_6),(p_5,p_5),\\ &(p_1,p_3),(p_6,p_6),(p_6,p_3)\},\\ R_2 = \{&(p_1,p_1),(p_2,p_1),(p_2,p_2),(p_2,p_3),(p_2,p_4),(p_2,p_5),(p_2,p_6),(p_3,p_1),(p_3,p_3),\\ &(p_3,p_4),(p_4,p_1),(p_4,p_3),(p_4,p_4),(p_5,p_5),(p_5,p_1),(p_5,p_2),(p_5,p_6),(p_6,p_6),\\ &(p_6,p_1),(p_6,p_2),(p_6,p_5)\}. \end{aligned}$$

Since $P(U)$ is a complete atomic boolean lattice whose atoms are singleton subset of U, we define two mapping $\varphi_1 : A(B) \longrightarrow B : U \longrightarrow P(U), x \longrightarrow R_1(x)$ and $\varphi_2 : A(B) \longrightarrow B : U \longrightarrow P(U), x \longrightarrow R_2(x)$, where $\varphi_1(p_i) = R_1(p_i)$, and $\varphi_2(p_i) = R_1(p_i)$. So $\varphi_1(p_1) = \{p_1, p_3\}$,$\varphi_1(p_2) = \{p_2\}$, $\varphi_1(p_3) = \{p_3\}$, $\varphi_1(p_4) = \{p_1, p_3, p_4, p_6\}$, $\varphi_1(p_5) = \{p_2, p_5\}$ and $\varphi_1(p_6) = \{p_3, p_6\}$. Also $\varphi_2(p_1) = \{p_1\}$, $\varphi_2(p_2) = U$, $\varphi_2(p_3) = \{p_1, p_3, p_4\}$, $\varphi_2(p_4) = \{p_1, p_3, p_4\}$, $\varphi_2(p_5) = \{p_1, p_2, p_5, p_6\}$ and $\varphi_2(p_6) = \{p_1, p_2, p_5.p_6\}$.

We choose the group of persons $X = \{p_2, p_3, p_5\}$ and calculate lower, upper, pairwise lower and pairwise upper approximations of this set. We find that the accuracy with respect to pairwise lower and pairwise upper approximations is higher than the accuracy with respect to lower and upper approximations.

$X^{\nabla p} = \bigcup\{x \in U : \varphi_1(x) \subseteq X \ \ or \ \ \varphi_2(x) \subseteq X\}$

$X^{\triangle_p} = \bigcup\{x \in U : \varphi_1(x) \cap X \neq \phi \ \ and \ \ \varphi_2(x) \cap X \neq \phi\}$.

Hence $X^{\nabla p} = \{p_2, p_3, p_5\}$ and $X^{\triangle_p} = \{p_2, p_3, p_4, p_5, p_6\}$. Therefore $BND(X) = X^{\triangle_p} - X^{\nabla p} = \{p_4, p_6\}$ and accuracy of $X = cardX^{\nabla p}/cardX^{\triangle_p} = 3/5$. On the other hand $X^{\nabla 1} = \{p_2, p_3, p_5\}$ and $X^{\triangle_1} = U$, so boundary of X, $BND(X) = X^{\triangle_1} - X^{\nabla 1} = \{p_2, p_3, p_5\}$ and accuracy of X with respect to $R_1 = cardX^{\nabla 1}/cardX^{\triangle_1} = 3/6 = 1/2$. Also $X^{\nabla 2} = \phi$ and $X^{\triangle_2} = \{p_2, p_3, p_4, p_5, p_6\}$, so $BND(X) = X^{\triangle_2} - X^{\nabla 2} = \{p_2, p_3, p_4, p_5, p_6\}$ and accuracy of X with respect to $R_1 = 0/5 = 0$.

5 Conclusion

In expert systems and artificial intelligence, the input information are transformed to logical statements connected by logical quantifiers. These sentences are connected to obtain any required question. Lattice is the appropriate structure for such treatment. We expect that the suggested concepts and method can help in the process of data bases and expert systems. In our future work we shall indicate the effect of our generalizations on such studies. Our results are generalization of Jarvinen results and both are the same if $\varphi_1 = \varphi_2$.

Using lower and upper approximation given in this work minimizes the boundary region and decreases the uncertainty region so the decision makers may get more accurate results.

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