Mathematical Sciences Letters
An International Journal

http://dx.doi.org/10.12785/msl/040208

Variational Iteration Technique for Constructing Methods for Solving Nonlinear Equations

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Received: 2 Feb. 2014, Revised: 6 Apr. 2014, Accepted: 8 Apr. 2014
Published online: 1 May 2015

Abstract: In this paper, we use the variational iteration technique to suggest some iterative methods for solving the nonlinear equations involving an auxiliary function. For appropriate and suitable choice of the auxiliary function,one can obtain a wide class of iterative methods for solving the nonlinear equations, which is a novel aspect of this technique. Convergence analysis of the proposed method is investigated. Several examples are given to illustrate the efficiency and implementation of the proposed new methods. Comparison with other methods is also given. These new methods can be considered as alternative to the developed methods. This technique can be used to suggest a wide class of new iterative methods for solving nonlinear equations.

Keywords: Variational iteration technique, Iterative method, Convergence, Newtons method, Taylor series, Examples

1 Introduction

Finding the solution of the nonlinear equations $f(x) = 0$, is one of the most important and challenging problems in science and engineering applications. Various iterative methods are being developed for finding the simple roots of the nonlinear equation $f(x) = 0$, by using several different techniques such as Taylor series, quadrature formulas, homotopy perturbation method, variational iteration technique and decomposition methods, see [1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18].

It is well known one usually use Newton method for finding the approximate solution of nonlinear equation, which can be written as

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \qquad n = 0, 1, 2, \cdots.$$

which has quadratic convergence, see [17]. To improve the local order of convergence, many modified methods have been proposed. See [1, 2, 3] and [11]. We use the variational iteration technique to suggest and analyze some new iterative methods for solving the nonlinear equations, the origin of which can be traced back to Inokuti et al [6]. However, it was He [4] who realized the potential of this technique for solving a wide class of both linear and nonlinear problems which arise in various branches of pure and applied sciences. See also Noor and Mohyud-Din [10] and the references therein. Essentially using the idea and technique of He [4], Noor [10] and Noor and Shah [12, 16, 17, 18, **?**, **?**] has suggested and analyzed some iterative methods for solving the nonlinear equations using this technique. Now again we have used this technique to suggest third order convergent iterative methods free from higher-order derivatives. Several examples are given to illustrate the efficiency and performance of these new methods. Comparison with other methods show that the proposed methods are robust and perform better. These new methods can be considered as alternative to the existing methods. The ideas and technique of this paper may stimulate further research in this area.

2 Iterative methods

In this section, we construct some new iterative methods for solving nonlinear equations using the variational iteration technique. We develop the main iteration scheme involving the auxiliary function for finding the approximate solution of nonlinear equation. Finite

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difference scheme is used to approximate $f'(y_n)$ and diversify the relation with better efficiency index.
Consider the nonlinear equation of the type

$$f(x) = 0, \tag{1}$$

which can be written in the following equivalent form as:

$$x = H(x), \tag{2}$$

where

$$H(x) = \phi(x) + [f(\phi(x))]g(x), \tag{3}$$

where $g(x)$, is the arbitrary auxiliary function and λ is the unknown Lagrange multiplier. The unknown Lagrange multiplier is determined by using the optimality condition. The function $\phi(x)$ be an iteration function. We observe that if $\phi(x) = x$, then scheme (3) reduces to the relation suggested by Noor [7].

Using the optimality criteria, we obtain the value of λ as:

$$\lambda = -\frac{\phi'(x)}{f'(\phi(x))g(x)\phi'(x) + f(\phi(x))g'(x)}. \tag{4}$$

From (3) and (4), we obtain

$$H(x) = \phi(x) - \frac{\phi'(x) f(\phi(x))g(x)}{f'(\phi(x))g(x)\phi'(x) + f(\phi(x))g'(x)}. \tag{5}$$

Now combining (2) and (5), we obtain

$$x = H(x) = \phi(x) - \frac{\phi'(x) f(\phi(x))g(x)}{f'(\phi(x))g(x)\phi'(x) + f(\phi(x))g'(x)}. \tag{6}$$

This fixed point formulation enables us to suggest the following iterative scheme as:

Algorithm 2.1. For a given x_0, find the approximation solution x_{n+1} by the following iterative scheme:

$$x_{n+1} = \phi(x_n) - \frac{\phi'(x_n) f(\phi(x_n))g(x_n)}{f'(\phi(x_n))g(x_n)\phi'(x_n) + f(\phi(x_n))g'(x_n)}.$$

This is the main recurrence relation involving the iteration function $\phi(x_n)$ and the auxiliary function $g(x.)$ With appropriate and suitable choice of the iteration function and the auxiliary function, one can find a wide class of iterative methods for solving the nonlinear equations and related problems.

Let

$$\phi(x_n) = y_n = x_n - \frac{f(x_n)}{f'(x_n)}$$

Then Algorithm 2.1 reduces to the following form as:

Algorithm 2.2. For a given x_0, find the approximation solution x_{n+1} by the following iterative scheme:

$$x_{n+1} = y_n - \frac{y_n' f(y_n)g(x_n)}{f'(y_n)g(x_n)y_n' + f(y_n)g'(x_n)}.$$

One can easily obtain

$$y_n' = \frac{f(x_n) f''(x_n)}{f(x_n)^2}. \tag{7}$$

Noor suggested the relation

$$f(y_n) = \frac{f(x_n)^2 f''(x_n)}{2f(x_n)^2}. \tag{8}$$

Using (7) and (8) and replacing

$$f'(y_n) = \frac{f(y_n) - f(x_n)}{y_n - x_n}. \tag{9}$$

in Algorithm 2.2, we obtain the following method free from the higher-order derivatives.

Algorithm 2.3. For a given x_0, find the approximation solution x_{n+1} by the following iterative scheme:

$$x_{n+1} = y_n - \frac{2f(y_n) f(x_n)g(x_n)}{2f'(x_n)[f(x_n) - f(y_n)]g(x_n) + f(x_n)^2 g'(x_n)}.$$

Algorithm 2.3 is the main iterative method, which is the main motivation of this paper.

We now discuss the following some special cases for some values of $g(x_n)$.

Case I. Let $g(x) = e^{-\alpha x_n}$. Then from Algorithm 2.2, we obtain the following iterative method for solving the nonlinear equations.

Algorithm 2.4. For a given x_0, find the approximation solution x_{n+1} by the following iterative scheme:

$$y_n = x_n - \frac{f(x_n)}{\acute{f}(x_n)},$$

$$x_{n+1} = y_n - \frac{2f(y_n) f(x_n)}{2f'(x_n)[f(x_n) - f(y_n)] - \alpha f(x_n)^2}, n = 0, 1, 2 \cdots.$$

Case II. Let $g(x) = e^{-\alpha f(x_n)}$. Then, from Algorithm 2.3, we obtain the following iterative method for solving the nonlinear equations.

Algorithm 2.5. For a given x_0, find the approximation solution x_{n+1} by the following iterative scheme:

$$y_n = x_n - \frac{f(x_n)}{\acute{f}(x_n)},$$

$$x_{n+1} = y_n - \frac{2f(y_n) f(x_n)}{f'(x_n)(2[f(x_n) - f(y_n)] - \alpha f(x_n)^2)}, n = 0, 1, 2, \cdots.$$

Case III. Let $g(x) = e^{\frac{\alpha}{f'(x_n)}}$. Then, from Algorithm 2.3, we obtain the following iterative method for solving the nonlinear equations having unknown zeros of multiplicity.

Algorithm 2.6. For a given x_0, find the approximation solution x_{n+1} by the following iterative scheme:

$$y_n = x_n - \frac{f(x_n)}{\acute{f}(x_n)},$$

$$x_{n+1} = y_n - \frac{f(y_n)f(x_n)}{f'(x_n)[f(x_n) - f(y_n)] + \alpha f(y_n)}, n = 0, 1, 2, \cdots.$$

Case IV. Let $g(x) = e^{-\alpha\frac{f(x_n)}{f'(x_n)}}$. Then, from Algorithm 2.3, we obtain the following iterative method for solving the nonlinear equations having unknown zeros of multiplicity.

Algorithm 2.7. For a given x_0, find the approximation solution x_{n+1} by the following iterative scheme:

$$y_n = x_n - \frac{f(x_n)}{\acute{f}(x_n)},$$

$$x_{n+1} = y_n - \frac{2f(y_n)f(x_n)}{2f'(x_n)[f(x_n) - f(y_n)] - \alpha f(x_n)[f(x_n) - 2f(y_n)]}$$

.

Sign of α, should be selected to make the denominator largest in magnitude in above methods to obtain the good results.

3 Convergence analysis

In this section, we consider the convergence criteria of the main iterative scheme Algorithm 2.3 developed in section 2.

Theorem 1. *Assume that the function $f : \mathscr{D} \subset \mathbb{R} \to \mathbb{R}$ has a simple root $r \in \mathscr{D}$ in an open interval in $\mathscr{D}$. Let $f(x)$ be a smooth sufficiently in some neighborhood of root, then Algorithm 2.3 has third order convergence.*

Proof. Let r ba a simple root of the nonlinear equation $f(x)$. Since f is sufficiently differentiable. Expanding $f(x)$ and $\acute{f}(x)$ in Taylor's series at r, we obtain

$$f(x_n) = \acute{f}(r)[e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + c_5e_n^5 + c_6e_n^6 + O(e_n^7)]. \quad (10)$$

and

$$\acute{f}(x_n) = \acute{f}(r)[1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + 5c_5e_n^4 + 6c_6e_n^5 + O(e_n^7)]. \quad (11)$$

where

$$e_n = x_n - r, c_k = \frac{f^k(r)}{k!\acute{f}(r)} \text{ and } k = 2, 3, \cdots$$

Using (10) and (11), we get

$$y_n = c_2e_n^2 + 2c_3 3 - 2c_2^2e_n^3 + (3c_4 - 7c_2c_3 + 4c_2^3)e_n^4 + O(e_n^5). \quad (12)$$

From (12), we obtain

$$f(y_n) = f'(r)(c_2e_n^2 + 2(c_3 - c_2^2)e_n^3 + (-3c_4 + 7c_2c_3 - 5c_2^3)e_n^4 + O(e_n^5)). \quad (13)$$

From(10) and (13), we get

$$f(y_n)f(x_n)g(x_n) = \acute{f}(r)[g(r)c_2e_n^3$$

$$+(2g(r)c_3 - g(r)c_2^2 + c_2g'(r))e_n^4 + O(e_n^5)], \quad (14)$$

and

$$f(x_n) - f(y_n) = \acute{f}(r)[e_n + (-c_3 + 2c_2^2)e_n^3$$

$$-(2c_4 - 7c_2c_3 + 5c_2^3)e_n^4 + O(e_n^5) \quad (15)$$

Using (11) and (14), we obtain

$$\acute{f}(x_n)[f(x_n) - f(y_n)] = f'(r)^2[g(r)e_n + (g'(r) + 2c_2g(r))e_n^2$$

$$+(1/2g''(r) + 2g(r)c_3 + 2g(r)c_2^2 + 2c_2g'(r))e_n^3 + O(e_n^4)], \quad (16)$$

and

$$\acute{f}(x_n)[f(x_n) - f(y_n)]g(x_n) + f(x_n)g'(x_n) = f'(r)^2[(g(r)$$

$$+f'(r)g'(r)))e_n + (g'(r) + 2c_2f'(r)g(r)$$

$$+g''(r) + c_2f'(r)g'(r))e_n^2 + O(e_n^3)]. \quad (17)$$

Now using (10) and (17), we get

$$2\acute{f}(x_n)[f(x_n) - f(y_n)]g(x_n) + f(x_n)^2g'(x_n) =$$

$$f'(r)^{[}2(g(r))e_n + (3g'(r) + 4c_2g(r))e_n^2 + (2g''(r)$$

$$+4c_3g(r) + 4c_2^2g(r) + 6c_2g'(r))e_n^3 + O(e_n^4)] \quad (18)$$

Using (14) and (18), we obtain

$$\frac{2f(y_n)f(x_n)g(x_n)}{2\acute{f}(x_n)[f(x_n) - f(y_n)]g(x_n) + f(x_n)^2g'(x_n)} =$$

$$c_2e_n^2 + \left(2c_3 - 3c_2^2 - c_2\frac{2g'(r)}{g(r)}\right)e_n^3 + O(e_n^4) \quad (19)$$

From (12) and (19), we get

$$x_{n+1} = r + \left(c_2^2 + \frac{g\acute{(}r)}{2g(r)}\right)e_n^3 + +O(e_n^4) \qquad (20)$$

Finally, we get the error equation as

$$e_{n+1} = \left(c_2^2 + \frac{g\acute{(}r)}{2g(r)}\right)e_n^3 + +O(e_n^4) \qquad (21)$$

This shows that Algorithm 2.3 has at least third order convergence. It is worth mentioning that all the methods derived from this scheme are also of third order convergence .

4 Numerical results

We now present some examples to illustrate the efficiency of these new iterative methods (see Tables 4.1-4.5). We compare the Newton method (NM) [18], Noor's method (NR) [9], Algorithm 2.4, Algorithm 2.5, Algorithm 2.6 and Algorithm 2.7, which are introduced here in this article. We also note that these methods do not require the computation of second derivative to carry out the successive iterations. All computations are done using the MAPLE using 60 digits floating point arithmetics (Digits: =60). We will use $\varepsilon = 10^{-32}$. The following stopping criteria are used for computer programs.

$$(i)\ |x_{n+1} - x_n| \le \varepsilon,\ (ii)\ |f(x_n)| \le \varepsilon.$$

The computational order of convergence p approximated for all the examples in Tables 4.1-4.6, (see [**?**]) by means of

$$\rho = \frac{ln(|x_{n+1} - x_n|/|x_n - x_{n-1}|)}{ln(|x_n - x_{n-1}|/|x_{n-1} - x_{n-2}|)}$$

We consider the following examples

$$f_1(x) = sin^2 x - x^2 + 1,$$

$$f_2(x) = x^2 - e^{-x} - 3x + 2.$$

$$f_3(x) = (x-1)^2 - 1,$$

$$f_4(x) = xe^{x^2} - sin^2 x + 3cosx + 5,$$

$$f_5(x) = e^{x^2+7x-30} - 1.$$

Table 4.1 depicts the numerical results of $f_1(x)$. We use $\alpha = 1$, $\alpha = 0.5$, $\alpha = 0.25$ and $\alpha = 0.$ for all methods by using the initial guess $x_0 = 1.$ for the computer program. Table 4.2 shows the numerical results of $f_2(x)$. We use the initial guess $x_0 = 2$, for different values of α. Table 4.3 shows the efficiency of the methods for $f_3(x)$. We use the initial guess $x_0 = 3.5$, for the computer program for different values of α. Number of iterations and

Table 4.1 (Numerical Comparison for $f_1(x)$)

Method	IT	x_n	δ	ρ
For$\alpha = 1$				
NM	7	1.404491648	1.04e-50	2.00003
NR	7	1.404491648	0.00e-01	2.85765
Alg 2.4	4	1.404491648	0.00e-01	2.85988
Alg 2.5	4	1.404491648	0.00e-01	3.05759
Alg 2.6	5	1.404491648	0.00e-01	2.98064
Alg 2.7	5	1.404491648	0.00e-01	2.98355
For$\alpha = 0.5$				
NM	7	1.404491648	1.04e-50	2.00003
NR	7	1.404491648	0.00e-01	2.85765
Alg 2.4	4	1.404491648	4.07e-26	3.09088
Alg 2.5	4	1.404491648	0.00e-01	2.99759
Alg 2.6	5	1.404491648	0.00e-01	2.98064
Alg 2.7	5	1.404491648	1.04e-50	3.00055
For $\alpha = 0.25$				
NM	7	1.404491648	1.04e-50	2.00003
NR	7	1.404491648	0.00e-01	2.85765
Alg 2.4	4	1.404491648	0.00e-01	2.95988
Alg 2.5	4	1.404491648	0.00e-01	3.00959
Alg 2.6	5	1.404491648	0.00e-01	2.98064
Alg 2.7	5	1.404491648	0.00e-01	2.98955
For $\alpha = 0$				
NM	7	1.404491648	1.04e-50	2.00003
NR	7	1.404491648	0.00e01	2.85765
Alg 2.4	5	1.404491648	0.00e-01	2.99988
Alg 2.5	4	1.404491648	1.26e-26	3.00859
Alg 2.6	5	1.404491648	0.00e-01	2.98064
Alg 2.7	5	1.404491648	1.06e-44	2.99965

Table 4.2 (Numerical Comparison for $f_2(x)$)

Method	IT	x_n	δ	ρ
For $\alpha = 1$				
NM	6	0.257530285	9.10e-28	2.00050
NR	5	0.257530285	1.10e-25	2.90051
Alg 2.2	4	0.257530285	1.06e-11	2.876988
Alg 2.3	5	0.257530285	6.13e-14	2.99989
Alg 2.4	4	0.257530285	9.55e-18	2.99964
Alg 2.5	4	0.257530285	3.04e-22	3.00000
For $\alpha = 0.5$				
NM	6	0.257530285	9.10e-28	2.00050
NR	5	0.257530285	1.10e-25	2.90051
Alg 2.2	4	0.257530285	4.00e-13	2.85988
Alg 2.3	5	0.257530285	3.03e-14	2.85759
Alg 2.4	4	0.257530285	5.11e-18	.98064
Alg 2.5	4	0.257530285	4.44e-18	2.98355
For $\alpha = 0.25$				
NM	6	0.257530285	9.10e-28	2.00050
NR	5	0.257530285	1.10e-25	2.90051
Alg 2.2	4	0.257530285	1.03e-33	2.85988
Alg 2.3	4	0.257530285	3.00e-24	2.85759
Alg 2.4	4	0.257530285	2.50e-22	.98064
Alg 2.5	4	0.257530285	6.07e-23	2.98355
For $\alpha = 0$				
NM	6	0.257530285	9.10e-28	2.00050
NR	5	0.257530285	1.10e-25	2.90051
Alg 2.2	4	0.257530285	2.00e-21	2.85988
Alg 2.3	4	0.257530285	1.08e-22	2.85759
Alg 2.4	4	0.257530285	1.03e-22	.98064
Alg 2.5	4	0.257530285	2.07e-22	2.98355

Table 4.3 (Numerical Comparison for $f_3(x)$)

Method	IT	x_n	δ	ρ
For $\alpha = 1$				
NM	7	2.000000000	1.04e-50	2.00003
NR	7	2.000000000	0.00e-01	2.85765
Alg 2.2	4	2.000000000	1.44e-13	3.00000
Alg 2.3	4	2.000000000	1.00e-14	2.99999
Alg 2.4	6	2.000000000	4.22e-25	2.9889
Alg 2.5	6	2.000000000	6.23e-33	3.00355
For $\alpha = 0.5$				
NM	7	2.000000000	1.04e-50	2.00003
NR	7	2.000000000	0.00e-01	2.85765
Alg 2.2	5	2.000000000	0.00e-01	3.99988
Alg 2.3	5	2.000000000	4.06e-21	2.99999
Alg 2.4	6	2.000000000	9.22e-21	2.99994
Alg 2.5	5	2.000000000	5.03e-11	3.00355
For $\alpha = 0.25$				
NM	7	2.000000000	1.04e-50	2.00003
NR	7	2.000000000	0.00e-01	2.85765
Alg 2.2	5	2.000000000	0.00e-01	3.99988
Alg 2.3	5	2.000000000	4.16e-23	2.99999
Alg 2.4	5	2.000000000	7.22e-19	2.99994
Alg 2.5	6	2.000000000	5.33e-17	3.00355
For $\alpha = 0$				
NM	7	2.000000000	1.04e-50	2.00003
NR	7	2.000000000	0.00e-01	3.00065
Alg 2.2	5	2.000000000	8.00e-21	3.00000
Alg 2.3	5	2.000000000	3.33e-31	2.99999
Alg 2.4	5	2.000000000	2.23e-41	3.00064
Alg 2.5	5	2.000000000	1.07e-31	3.00955

Table 4.4 (Numerical Comparison for $f_4(x)$)

Method	IT	x_n	δ	ρ
For $\alpha = 1$				
NM	9	-1.20764782	1.04e-50	2.00003
NR	7	-1.20764782	0.00e-01	3.00022
Alg 2.2	6	-1.20764782	0.00e-01	2.99988
Alg 2.3	6	-1.20764782	0.00e-01	2.99959
Alg 2.4	7	-1.20764782	0.00e-01	3.00064
Alg 2.5	6	-1.20764782	0.00e-01	3.00355
For $\alpha = 0.5$				
NM	9	-1.20764782	1.04e-50	2.00003
NR	7	-1.20764782	0.00e-01	3.00022
Alg 2.2	6	-1.20764782	0.00e-01	2.85988
Alg 2.3	6	-1.20764782	0.00e-01	3.00759
Alg 2.4	7	-1.20764782	0.00e-01	2.98064
Alg 2.5	6	-1.20764782	0.00e-01	2.98355
For $\alpha = 0.25$				
NM	9	-1.20764782	1.04e-50	2.00003
NR	7	-1.20764782	0.00e-01	3.00022
Alg 2.2	6	-1.20764782	0.00e-01	2.99988
Alg 2.3	6	-1.20764782	0.00e-01	3.00009
Alg 2.4	7	-1.20764782	0.00e-01	2.98994
Alg 2.5	6	-1.20764782	0.00e-01	2.99955
For $\alpha = 0$				
NM	9	-1.20764782	1.04e-50	2.00003
NR	7	-1.20764782	0.00e-01	3.00022
Alg 2.2	6	-1.20764782	0.00e-01	2.85988
Alg 2.3	6	-1.20764782	0.00e-01	2.99999
Alg 2.4	6	-1.20764782	0.00e-01	2.98984
Alg 2.5	5	-1.20764782	0.00e-01	3.00000

Table 4.5 (Numerical Comparison for $f_5(x)$)

Method	IT	x_n	δ	ρ
For $\alpha = 1$				
NM	13	3.000000000	1.04e-50	2.00003
NR	11	3.000000000	0.00e-01	3.00005
Alg 2.2	8	3.000000000	1.33e-12	3.11988
Alg 2.3	8	3.000000000	6.03e-14	3.00759
Alg 2.4	11	3.000000000	4.22e-24	2.99064
Alg 2.5	10	3.000000000	6.14e-12	2.98355
For $\alpha = 0.5$				
NM	13	3.000000000	1.04e-50	2.00003
NR	11	3.000000000	0.00e-01	3.00005
Alg 2.2	8	3.000000000	3.33e-13	2.85988
Alg 2.3	8	3.000000000	4.65e-15	2.85759
Alg 2.4	10	3.000000000	6.44e-11	.98064
Alg 2.5	10	3.000000000	9.01e-14	2.98355
For$\alpha = 0.25$				
NM	13	3.000000000	1.04e-50	2.00003
NR	11	3.000000000	0.00e-01	3.00005
Alg2.2	8	3.000000000	4.99e-14	2.85988
Alg 2.3	8	3.000000000	1.03e-14	2.85759
Alg 2.4	10	3.000000000	3.05e-14	2.99994
Alg 2.5	10	3.000000000	4.29e-16	2.98355
For$\alpha = 0$				
NM	13	3.000000000	1.04e-50	2.00003
NR	11	3.000000000	0.00e-01	3.00005
Alg 2.2	8	3.000000000	3.99e-15	3.00000
Alg 2.3	8	3.000000000	3.67e-15	3.00059
Alg 2.4	8	3.000000000	3.77e-15	3.00064
Alg 2.5	8	3.000000000	3.98e-22	3.00055

computational order of convergence gives us an idea about the better performance of the newly developed methods. Table 4.4 shows the efficiency of the methods for example $f_4(x)$. We use the initial guess $x_0 = -2$, $\alpha = 1$, $\alpha = 0.5$, $\alpha = 0.25$ and $\alpha = 0$. for all methods. Number of iterations and computational order of convergence give us an idea about the better performance of the new methods. In Table 4.5, the numerical results for example $f_5(x)$. are described. We use the initial guess $x_0 = 3.5$ for the computer program for different values of α. We observe that all the newly derived methods approach to the approximate solution after equal or less number of iterations and the computational order of convergence can also be observed from the Table.

5 Conclusion

In this work, we have presented some third order convergent methods for solving nonlinear equations, which are free from higher-order derivatives. These methods are compared with Newton method and the proposed methods have been observed to have at least better performance. If we consider the definition of efficiency index [17] as $p^{\frac{1}{m}}$ where p is the order of convergence of the method and m is the number of functional evaluations per iteration required by the method, we have that all of the methods obtained have the efficiency index equal to $3^{\frac{1}{3}} \approx 1.442$, which is better than the one of Newton method $2^{\frac{1}{2}} \approx 1.414$. The presented approach can also be applied further to obtain higher order convergent methods for solving nonlinear equations.

Acknowledgement

The authors are grateful to Dr. S. M. Junaid Zaidi, Rector COMSATS Institute of information Technology, Islamabad, Pakistan for providing excellent research environment and facilities. The authors are also grateful to the anonymous referees for theri constructive comments and suggestions.

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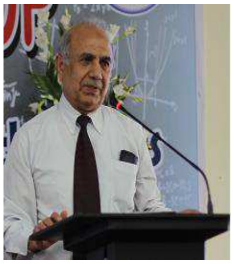

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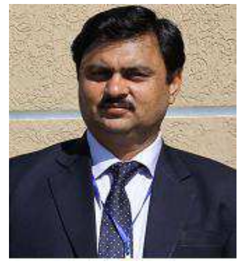

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