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Analytical Solutions of Fuzzy Initial Value Problems by HAM

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Abstract: In this paper, series solution of fuzzy initial value problems under strongly generalized differentiability by means of the homotopy analysis method is considered. The new approach provides the solution in the form of a rapidly convergent series with easily computable components using symbolic computation software. Although the homotopy analysis method contains the auxiliary parameter, the convergence region of the series solution can be controlled in a simple way. The proposed technique is applied to a few test examples to illustrate the accuracy and applicability of the method. The results reveal that the method is very effective and straightforward. Meanwhile, analysis results show that the homotopy analysis method is a powerful and easy-to-use analytic tool to solve fuzzy initial value problems.

Keywords: Homotopy analysis method; Fuzzy initial value problems; Characterization theorem

1 Introduction

When a real world problem is transformed into a deterministic initial value problem of ordinary differential equations (ODEs), namely $x'(t) = f(t,x(t))$, $t_0 \leq t \leq t_0 + a$ subject to the initial condition $x(t_0) = x^0$, we cannot usually be sure that the model is perfect. For example, the initial value may not be known exactly and the function f may contain uncertain parameters. If they are estimated through certain measurements, they are necessarily subject to errors. The analysis of the effect of these errors leads to the study of the qualitative behavior of the solutions of aforementioned equation. Thus, it would be natural to employ fuzzy initial value problems (FIVPs) [1].

Fuzzy set theory is a powerful tool for modeling uncertainty and for processing vague or subjective information in mathematical models. Their main directions of development have been diverse and its applications to the very varied real problems, for instance, in the golden mean [2], particle systems [3], quantum optics and gravity [4], synchronize hyperchaotic systems [5], chaotic system [6], medicine [7], and engineering problems [8]. Particularly, FIVP is a topic very important as much of the theoretical point of view (see e.g. [9–11]) as well as of their applications, for example, in population models [12], civil engineering [13], physics [14], and in modeling hydraulic [15].

There are several approaches to studying FIVPs [16–20]. The first approach was the use of the Hukuhara differentiability for fuzzy-valued functions. Under this setting, mainly the existence and uniqueness of the solution of FIVP were studied (see e.g. [9, 16, 20]). This approach has a drawback: the solution becomes fuzzier as time goes by [21, 22]. Hence, the fuzzy solution behaves quite differently from the crisp solution. To alleviate the situation, author in [23] interpreted FIVPs as a family of differential inclusions. The main shortcoming of using differential inclusions is that we do not have a derivative of a fuzzy-valued function.

The strongly generalized differentiability was first introduced in [24] and studied in [21, 25–27]. This concept allows us to resolve the above-mentioned shortcoming. Indeed, the strongly generalized derivative is defined for a larger class of fuzzy-valued functions than the Hukuhara derivative. Hence, we use this differentiability concept in the present paper. Under appropriate conditions, the FIVP considered under this interpretation has locally two solutions [21].

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The purpose of this paper is to extend the application of the homotopy analysis method (HAM) under strongly generalized differentiability to provide symbolic approximate solution for FIVP which is as follows [9]:

$$x'(t) = f(t, x(t)), t_0 \leq t \leq t_0 + a, \tag{1}$$

subject to the fuzzy initial condition

$$x(t_0) = x^0, \tag{2}$$

where $f : [t_0, t_0 + a] \times \mathbb{R}_{\mathscr{F}} \to \mathbb{R}_{\mathscr{F}}$ is a continuous fuzzy-valued function, $x^0 \in \mathbb{R}_{\mathscr{F}}$, and t_0, a are real finite constants with $a > 0$. Here, $\mathbb{R}_{\mathscr{F}}$ denote the set of fuzzy numbers on $\mathbb{R}$.

In general, FIVPs do not always have solutions which we can obtain using analytical methods. In fact, many of real physical phenomena encountered, are almost impossible to solve by this technique. Due to this, some authors have proposed numerical methods to approximate the solutions of FIVPs. To mention a few, the Euler method has been applied to solve Eqs. (1) and (2) as described in [28]. In [29] the authors have developed the Runge-Kutta method (RKM). In [30] also, the author has provided the residual power series method to further investigation to the above equations. Furthermore, the predictor-corrector method (PCM) is carried out in [31]. In [32] the author has discussed the continuous genetic algorithm for solving FIVP (1) and (2). Recently, the artificial neural network approach for solving fuzzy Eqs. (1) and (2) is proposed in [33]. The numerical solvability of other version of differential equations and other related equations can be found in [34–41] and references therein.

Anyway, investigation about FIVPs under strongly generalized differentiability is scarce. However, none of previous studies propose a methodical way to solve FIVPs under this type of differentiability. Moreover, previous studies require more effort to achieve the results, they are not accurate but are usually developed for linear form of FIVP (1) and (2). Meanwhile, the proposed method has an advantage that it is possible to pick any point in the interval of integration and as well the approximate solution and its derivative will be applicable.

The HAM, which proposed by Liao [42–47], is effectively and easily used to solve some classes of linear and nonlinear problems without linearization, perturbation, or discretization. The HAM is based on the homotopy, a basic concept in topology. The auxiliary parameter $\hbar$ is introduced to construct the so-called zero-order deformation equation. Thus, unlike all previous analytic techniques, the HAM provides us with a family of solution expressions in auxiliary parameter $\hbar$. As a result, the convergence region and rate of solution series are dependent upon the auxiliary parameter $\hbar$ and thus can be greatly enlarged by means of choosing a proper value of $\hbar$. This provides us with a convenient way to adjust and control convergence region and rate of solution series given by the HAM.

In the recent years, extensive work has been done using HAM, which provides analytical approximations for linear and nonlinear equations. This method has been implemented in many branches of mathematics and engineering, such as nonlinear water waves [45], unsteady boundary-layer flows [46], solitary waves with discontinuity [47], Klein-Gordon equation [48], boundary value problems for integro-differential equations [49], systems of fractional differential equations [50], fractional differential equations [51, 52], systems of fractional algebraic-differential equations [53], fractional SIR model [54], MHD fluid flow and heat transfer problem [55], and others.

The outline of the paper is as follows: in the next section, we present some necessary definitions and preliminary results that will be used in our work. The theory of solving FIVPs is presented in section 3. In section 4, basic idea of the HAM is introduced. In section 5, we utilize the statement of the HAM for solving FIVPs. In section 6, numerical examples are given to illustrate the capability of HAM. This article ends in section 7 with some concluding remarks.

2 Preliminaries

The material in this section is basic in some sense. For the reader's convenience, we present some necessary definitions from fuzzy calculus theory and preliminary results. For the concept of fuzzy derivative, we will adopt strongly generalized differentiability, which is a modification of the Hukuhara differentiability and has the advantage of dealing properly with FIVPs.

Let X be a nonempty set. A fuzzy set u in X is characterized by its membership function $u : X \to [0,1]$. Thus, $u(s)$ is interpreted as the degree of membership of an element s in the fuzzy set u for each $s \in X$.

A fuzzy set u on $\mathbb{R}$ is called convex if for each $s, t \in \mathbb{R}$ and $\lambda \in [0,1]$, we have

$$u(\lambda s + (1-\lambda)t) \geq \min\{u(s), u(t)\},$$

and is called normal if there is $s \in \mathbb{R}$ such that $u(s) = 1$. The support of a fuzzy set u is defined as $\{s \in \mathbb{R} : u(s) > 0\}$.

Definition 2.1. [9] A fuzzy number u is a fuzzy subset of the real line with a normal, convex, and upper semicontinuous membership function of bounded support.

For each $r \in (0,1]$, set $[u]^r = \{s \in \mathbb{R} : u(s) \geq r\}$ and $[u]^0 = \overline{\{s \in \mathbb{R} : u(s) > 0\}}$. Then, it easily to establish that u is a fuzzy number if and only if $[u]^r$ is compact convex subset of $\mathbb{R}$ for each $r \in [0,1]$ and $[u]^1 \neq \phi$ [56]. Thus, if u is a fuzzy number, then $[u]^r = [\underline{u}(r), \overline{u}(r)]$, where $\underline{u}(r) = \min\{s : s \in [u]^r\}$ and $\overline{u}(r) = \max\{s : s \in [u]^r\}$ for each $r \in [0,1]$. The symbol $[u]^r$ is called the r-cut representation or parametric form of a fuzzy number u.

The previous discussion leads to the following characterization of fuzzy number u in terms of its endpoint functions $\underline{u}(r)$ and $\overline{u}(r)$. This theorem is a fundamental rule in fuzzy numbers theory and their applications.

Theorem 2.1. [56] Suppose that $\underline{u} : [0,1] \to \mathbb{R}$ and $\overline{u} : [0,1] \to \mathbb{R}$ satisfy the following conditions:
(i) $\underline{u}$ is a bounded increasing function,
(ii) $\overline{u}$ is a bounded decreasing function,
(iii) $\underline{u}(1) \leq \overline{u}(1)$,
(iv) for each $k \in (0,1]$, $\lim_{r\to k^-} \underline{u}(r) = \underline{u}(k)$ and $\lim_{r\to k^-} \overline{u}(r) = \overline{u}(k)$,
(v) $\lim_{r\to 0^+} \underline{u}(r) = \underline{u}(0)$ and $\lim_{r\to 0^+} \overline{u}(r) = \overline{u}(0)$.

Then $u : \mathbb{R} \to [0,1]$ defined by

$$u(s) = \sup\{r : \underline{u}(r) \leq s \leq \overline{u}(r)\} \tag{3}$$

is a fuzzy number with parameterization $[\underline{u}(r), \overline{u}(r)]$. Furthermore, if $u : \mathbb{R} \to [0,1]$ is a fuzzy number with parameterization $[\underline{u}(r), \overline{u}(r)]$, then the functions $\underline{u}$ and $\overline{u}$ satisfy conditions (i-v).

Based on Theorem 2.1, we can represent an arbitrary fuzzy number u by an order pair of functions $(\underline{u}, \overline{u})$ which satisfy the requirements (i-v) above. Frequently, we will write simply $\underline{u}_r$ and $\overline{u}_r$ instead of $\underline{u}(r)$ and $\overline{u}(r)$, respectively, for each $r \in [0,1]$.

The metric structure on $\mathbb{R}_{\mathscr{F}}$ is given by the Hausdorff distance $D : \mathbb{R}_{\mathscr{F}} \times \mathbb{R}_{\mathscr{F}} \to \mathbb{R}^+ \cup \{0\}$ such that $D(u,v) = \sup_{0\leq r\leq 1} \max\{|\underline{u}_r - \underline{v}_r|, |\overline{u}_r - \overline{v}_r|\}$ for arbitrary fuzzy numbers u and v. It is shown in [57] that $(\mathbb{R}_{\mathscr{F}}, D)$ is a complete metric space.

Two fuzzy numbers u and v are equal, if $[u]^r = [v]^r$ for each $r \in [0,1]$. For the arithmetic operations on fuzzy numbers we refer to [1]. The following results are well known and follow from the theory of interval analysis.

Theorem 2.2. [1] If u and v are two fuzzy numbers, then for each $r \in [0,1]$, we have
(i) $[u+v]^r = [u]^r + [v]^r = [\underline{u}_r + \underline{v}_r, \overline{u}_r + \overline{v}_r]$,
(ii) $[\lambda u]^r = \lambda [u]^r = \begin{cases} [\lambda \underline{u}_r, \lambda \overline{u}_r], \ \lambda \geq 0, \\ [\lambda \overline{u}_r, \lambda \underline{u}_r], \ \lambda < 0, \end{cases}$
(iii) $[uv]^r = [u]^r [v]^r = [\min S_r, \max S_r]$, where $S_r = \{\underline{u}_r\underline{v}_r, \underline{u}_r\overline{v}_r, \overline{u}_r\underline{v}_r, \overline{u}_r\overline{v}_r\}$.

The collection of all the fuzzy numbers with addition and scalar multiplication defined by part (i) and (ii) of Theorem 2.2 is a convex cone [58]. Also, it can be shown that these parts are equivalent to the addition and scalar multiplication as defined by using r-cut approach [56] and the extension principles [59].

Definition 2.2. [58] Let $u, v \in \mathbb{R}_{\mathscr{F}}$. If there exists a $w \in \mathbb{R}_{\mathscr{F}}$ such that $u = v + w$, then w is called the H-difference of u and v, denoted by $u \ominus v$.

Here, the sign "$\ominus$" stands always for H-difference and let us remark that $u \ominus v \neq u + (-1)v$. Usually we denote $u + (-1)v$ by $u - v$, while $u \ominus v$ stands for the H-difference. It follows that Hukuhara differentiable function has increasing length of support [9]. To avoid this difficulty, we consider the following definition.

Definition 2.3. [21] Let $x : [a,b] \to \mathbb{R}_{\mathscr{F}}$ and $t_0 \in [a,b]$. We say that x is strongly generalized differentiable at t_0, if there exists an element $x'(t_0) \in \mathbb{R}_{\mathscr{F}}$ such that either:
(i) $\forall h > 0$ sufficiently close to 0, the H-differences $x(t_0+h) \ominus x(t_0)$, $x(t_0) \ominus x(t_0-h)$ exist and $x'(t_0) = \lim_{h\to 0^+} \frac{x(t_0+h)\ominus x(t_0)}{h} = \lim_{h\to 0^+} \frac{x(t_0)\ominus x(t_0-h)}{h}$,
or
(ii) $\forall h > 0$ sufficiently close to 0, the H-differences $x(t_0) \ominus x(t_0+h)$, $x(t_0-h) \ominus x(t_0)$ exist and $x'(t_0) = \lim_{h\to 0^+} \frac{x(t_0)\ominus x(t_0+h)}{-h} = \lim_{h\to 0^+} \frac{x(t_0-h)\ominus x(t_0)}{-h}$.

Here the limit is taken in the metric space $(\mathbb{R}_{\mathscr{F}}, D)$ and at the endpoints of $[a,b]$, we consider only one-sided derivatives. In the previous definition, case (i) corresponds to the H-derivative introduced in [58], so this differentiability concept is a generalization of the Hukuhara derivative. Moreover, in [21], the authors consider four cases for derivatives. Here, we only consider the two first cases of Definition 5 in [21]. In the other cases, the derivative is trivial because it is reduced to a crisp element in $\mathbb{R}$. The reader is asked to refer to [21, 24–26] in order to know more details about strongly generalized differentiability and its applications.

Definition 2.4. [26] Let $x : [a,b] \to \mathbb{R}_{\mathscr{F}}$. We say x is (1)-differentiable on $[a,b]$ if x is differentiable in the sense (i) of Definition 2.3 and its derivative is denoted D_1x, and similarly for (2)-differentiability we have D_2x.

The principal properties of the defined derivatives are well known and can be found in [21, 26]. In this paper, we make use of the following theorem.

Theorem 2.3. [26] Let $x : [a,b] \to \mathbb{R}_{\mathscr{F}}$ and put $[x(t)]^r = [\underline{x}_r(t), \overline{x}_r(t)]$ for each $r \in [0,1]$.
(i) if x is (1)-differentiable, then $\underline{x}_r$ and $\overline{x}_r$ are differentiable functions and $[D_1x(t)]^r = [\underline{x}'_r(t), \overline{x}'_r(t)]$,
(ii) if x is (2)-differentiable, then $\underline{x}_r$ and $\overline{x}_r$ are differentiable functions and $[D_2x(t)]^r = [\overline{x}'_r(t), \underline{x}'_r(t)]$.

Theorem 2.3 shows us a way to translate FIVP into a system of ODEs. As a conclusion one does not need to rewrite the numerical methods for ODEs in fuzzy setting, but instead, we can use the numerical methods directly on the obtained ordinary differential system.

3 Theory of solving FIVPs

In this section, we define a first-order FIVP under strongly generalized differentiability, then we replace it by its parametric form. Furthermore, we present an algorithm to solve the new system which consists of two classical ODEs for each type of differentiability.

Consider the following crisp ODE:

$$x'(t) = f(t, x(t)), \, t_0 \leq t \leq t_0 + a, \tag{4}$$

subject to the initial condition

$$x(t_0) = x^0, \tag{5}$$

where $f : [t_0, t_0 + a] \times \mathbb{R} \to \mathbb{R}$ is a continuous real-valued function, $x^0 \in \mathbb{R}$, and t_0, a are real finite constants with $a > 0$.

Suppose that the initial condition x^0 in Eq. (5) is uncertain and modeled by a fuzzy number. Also, assume that the function f in Eq. (4) contain uncertain parameters modeled by a fuzzy number. Then we obtain the FIVP (1) and (2).

In order to solve this problem, we write the fuzzy function $x(t)$ in its r-cut representation form to get $[x(t)]^r = [\underline{x}_r(t), \overline{x}_r(t)]$ and $[x(0)]^r = [\underline{x}_r(0), \overline{x}_r(0)] = [\underline{x}_r^0, \overline{x}_r^0]$.

On the other hand, the extension principle of Zadeh leads to the following definition of $f(t, x(t))$ when $x(t)$ is a fuzzy number [1]:

$$f(t, x(t))(s) = \sup\{x(t)(\tau) : s = f(t, \tau), s \in \mathbb{R}\}.$$

From this according to Nguyen theorem [59] it follows that

$$\begin{aligned}
&[f(t, x(t))]^r \\
&\quad = [\underline{f}_r(t, x(t)), \overline{f}_r(t, x(t))] = f(t, [x(t)]^r) \\
&\quad = \{f(t, y) : y \in [x(t)]^r\} \\
&\quad = [f_{1,r}(t, \underline{x}_r(t), \overline{x}_r(t)), f_{2,r}(t, \underline{x}_r(t), \overline{x}_r(t))],
\end{aligned}$$

where the two term endpoint functions are given as

$$\begin{aligned}
f_{1,r}(t, \underline{x}_r(t), \overline{x}_r(t)) &:= \min\{f(t, y) : y \in [x(t)]^r\}, \\
f_{2,r}(t, \underline{x}_r(t), \overline{x}_r(t)) &:= \max\{f(t, y) : y \in [x(t)]^r\}.
\end{aligned}$$

The reader is asked to refer to [59,60] in order to know more details about Zadeh's extension principle, including its justification and conditions for use, properties, and its applications.

It is worth stating that in many cases, since FIVPs are often derived from problems in physical world, existence and uniqueness are often "obvious" for physical reasons. Notwithstanding this, a mathematical statement about existence and uniqueness is worthwhile. Uniqueness would be of importance if, for instance, we wished to approximate the solution. If two solutions passed through a point, then successive approximations could very well jump from one solution to the other-with misleading consequences.

The following definition is needed by the succeeding theorem regarding the existence and unicity of two solutions (one solution for each lateral derivative) to first order FIVPs under strongly generalized differentiability.

Definition 3.1. [26] Let $x : [t_0, t_0 + a] \to \mathbb{R}_{\mathcal{F}}$ such that $D_1 x$ or $D_2 x$ exists. If x and $D_1 x$ satisfy FIVP (1) and (2), we say x is a (1)-solution of FIVP (1) and (2). Similarly, if x and $D_2 x$ satisfy FIVP (1) and (2), we say x is a (2)-solution of FIVP (1) and (2).

Theorem 3.1. [26] Let $f : [t_0, t_0 + a] \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_{\mathcal{F}}$ is a continuous fuzzy-valued function. If there exists $k > 0$ such that $D(f(t, x), f(t, y)) \leq kD(x, y)$ for each $t \in [t_0, t_0 + a]$ and $x, y \in \mathbb{R}_{\mathcal{F}}$. Then, the FIVP (1) and (2) has two unique solutions on $[t_0, t_0 + a]$. One is (1)-differentiable solution and the other one is (2)-differentiable solution.

The object of the next algorithm is to implement a procedure to solve FIVP (1) and (2) in parametric form in term of its r-cut representation. Following [26], we observe that the relations of $[D_1 x]^r$ and $[D_2 x]^r$ in Theorem 2.3 give us a useful procedure to solve FIVPs.

Algorithm 3.1. To find the solutions of FIVP (1) and (2), we discuss the following cases:

Case I. If $x(t)$ is (1)-differentiable, then $[D_1 x(t)]^r = [\underline{x}'_r(t), \overline{x}'_r(t)]$ and solving FIVP (1) and (2) translates into the following subroutine:

Step (i): solve the following system of ODEs for $\underline{x}_r(t)$ and $\overline{x}_r(t)$:

$$\begin{aligned}
\underline{x}'_r(t) &= f_{1,r}(t, \underline{x}_r(t), \overline{x}_r(t)), \\
\overline{x}'_r(t) &= f_{2,r}(t, \underline{x}_r(t), \overline{x}_r(t)),
\end{aligned} \tag{6}$$

subject to the initial conditions

$$\begin{aligned}
\underline{x}_r(t_0) &= \underline{x}_r^0, \\
\overline{x}_r(t_0) &= \overline{x}_r^0,
\end{aligned} \tag{7}$$

Step (ii): ensure that the solution $[\underline{x}_r(t), \overline{x}_r(t)]$ and its derivative $[\underline{x}'_r(t), \overline{x}'_r(t)]$ are valid level sets for each $r \in [0, 1]$,

Step (iii): use Eq. (3) to construct a (1)-solution $x(t)$ such that $[x(t)]^r = [\underline{x}_r(t), \overline{x}_r(t)]$ for each $r \in [0, 1]$.

Case II. If $x(t)$ is (2)-differentiable, then $[D_2 x(t)]^r = [\overline{x}'_r(t), \underline{x}'_r(t)]$ and solving FIVP (1) and (2) translates into the following subroutine:

Step (i): solve the following system of ODEs for $\underline{x}_r(t)$ and $\overline{x}_r(t)$:

$$\begin{aligned}
\underline{x}'_r(t) &= f_{2,r}(t, \underline{x}_r(t), \overline{x}_r(t)), \\
\overline{x}'_r(t) &= f_{1,r}(t, \underline{x}_r(t), \overline{x}_r(t)),
\end{aligned} \tag{8}$$

subject to the initial conditions

$$\begin{aligned}
\underline{x}_r(t_0) &= \underline{x}_r^0, \\
\overline{x}_r(t_0) &= \overline{x}_r^0,
\end{aligned} \tag{9}$$

Step (ii): ensure that the solution $[\underline{x}_r(t), \overline{x}_r(t)]$ and its derivative $[\overline{x}'_r(t), \underline{x}'_r(t)]$ are valid level sets for each $r \in [0, 1]$,

Step (ii): use Eq. (3) to construct a (2)-solution $x(t)$ such that $[x(t)]^r = [\underline{x}_r(t), \overline{x}_r(t)]$ for each $r \in [0, 1]$.

Sometimes, we can't decompose the membership function of the solution $[x(t)]^r$ as a function defined on $\mathbb{R}$

for each $t \in [t_0, t_0 + a]$. Then, using identity (3) we can leave the solution in term of its r-cut representation. We mention here that, Case II above is an extension of the procedure used in [61] for solving FIVPs, where the derivative is considered in the first form of Definition 2.3 only; which is coincident with our Case I. Thus, from Case II we obtain new solution for FIVP (1) and (2).

The characterization theorem states that under certain conditions a FIVP is equivalent to a system of ODEs. The next result utilizes the characterization theorem of FIVPs under strongly generalized differentiability.

Theorem 3.2. [27] Consider the FIVP (1) and (2) where $f : [t_0, t_0 + a] \times \mathbb{R}_{\mathscr{F}} \to \mathbb{R}_{\mathscr{F}}$ is such that

(i) $[f(t, x(t))]^r = [f_{1,r}(t, \underline{x}_r(t), \overline{x}_r(t)), f_{2,r}(t, \underline{x}_r(t), \overline{x}_r(t))]$,

(ii) $f_{1,r}$ and $f_{2,r}$ are equicontinuous functions,

(iii) there exists $L > 0$ such that for each $r \in [0,1]$

$$|f_{1,r}(t, x_1, y_1) - f_{1,r}(t, x_2, y_2)| \le L \max\{|x_1 - x_2|, |y_1 - y_2|\},$$

$$|f_{2,r}(t, x_1, y_1) - f_{2,r}(t, x_2, y_2)| \le L \max\{|x_1 - x_2|, |y_1 - y_2|\}.$$

Then, for (1)-differentiability, the FIVP (1) and (2) and the system of ODEs (6) and (7) are equivalent and in (2)-differentiability, the FIVP (1) and (2) and the system of ODEs (8) and (9) are equivalent.

We mention here that the requirement $[f(t, x(t))]^r = [f_{1,r}(t, \underline{x}_r(t), \overline{x}_r(t)), f_{2,r}(t, \underline{x}_r(t), \overline{x}_r(t))]$ is achieved by any fuzzy-valued function obtained from a continuous function by Zadeh's extension principle. So this condition is not too restrictive.

4 Basic idea of the HAM

The principles of the HAM and its applicability for various kinds of differential equations are given in [42–55]. For convenience of the reader, we will present a review of the HAM [42–47] then we will implement the HAM to construct a symbolic approximate solution to FIVPs.

To achieve our goal, we consider the nonlinear differential equation

$$N[x(t)] = 0, t \ge t_0, \tag{10}$$

where N is a nonlinear differential operator and $x(t)$ is an unknown function of the independent variable t.

Liao [42–47] constructs the so-called zeroth-order deformation equation

$$(1-q)\mathscr{L}[\Phi(t;q) - x_0(t)] = q\hbar H(t) N[\Phi(t;q)], \tag{11}$$

where $q \in [0,1]$ is an embedding parameter, $\hbar \neq 0$ is an auxiliary parameter, $H(t) \neq 0$ is an auxiliary function, N is a nonlinear differential operator, $\Phi(t;q)$ is an unknown function, $x_0(t)$ is an initial guess of $x(t)$, which satisfies the initial condition, and $\mathscr{L}$ is an auxiliary linear operator with the property

$$\mathscr{L}[f(t)] = 0 \text{ when } f(t) = 0. \tag{12}$$

It should be emphasized that one has great freedom to choose the initial guess $x_0(t)$, the auxiliary linear operator $\mathscr{L}$, the auxiliary parameter $\hbar$, and the auxiliary function $H(t)$. According to the property (12) and the suitable initial condition, when $q = 0$, we have

$$\Phi(t;0) = x_0(t), \tag{13}$$

and when $q = 1$, since $\hbar \neq 0$ and $H(t) \neq 0$, the zeroth-order deformation equation (11) is equivalent to Eq. (10), hence

$$\Phi(t;1) = x(t). \tag{14}$$

Thus, according to Eqs. (13) and (14), as q increasing from 0 to 1, the solution $\Phi(t;q)$ various continuously from the initial approximation $x_0(t)$ to the exact solution $x(t)$.

Define the so-called mth-order deformation derivatives

$$x_m(t) = \frac{1}{m!} \frac{\partial^m \Phi(t;q)}{\partial q^m}\bigg|_{q=0}, \tag{15}$$

expanding $\Phi(t;q)$ in a Taylor series with respect to the embedding parameter q, by using Eqs. (13) and (15), we have

$$\Phi(t;q) = x_0(t) + \sum_{m=1}^{\infty} x_m(t) q^m. \tag{16}$$

Assume that the auxiliary parameter $\hbar$, the auxiliary function $H(t)$, the initial approximation $x_0(t)$, and the auxiliary linear operator $\mathscr{L}$ are properly chosen so that the series (16) of $\Phi(t;q)$ converges at $q = 1$. Then, we have under these assumptions the series solution $x(t) = x_0(t) + \sum_{m=1}^{\infty} x_m(t)$.

Define the vector

$$\vec{x}_n(t) = \{x_0(t), x_1(t), \ldots, x_n(t)\}.$$

Differentiating Eq. (11) m-times with respect to embedding parameter q, and then setting $q = 0$ and finally dividing them by $m!$, we have, using Eq. (15), the so-called mth-order deformation equation

$$\mathscr{L}[x_m(t) - \chi_m x_{m-1}(t)] = \hbar H(t) \Re x_m(\vec{x}_{m-1}(t)), \tag{17}$$

where $m = 1, 2, \ldots, n$, $\chi_m = \begin{cases} 0, & m \le 1, \\ 1, & m > 1, \end{cases}$ and

$$\Re x_m(\vec{x}_{m-1}(t)) = \frac{1}{(m-1)!} \frac{\partial^{m-1} N[\Phi(t;q)]}{\partial q^{m-1}}\bigg|_{q=0}. \tag{18}$$

For any given nonlinear operator N, the term $\Re x_m(\vec{x}_{m-1}(t))$ can be easily expressed by Eq. (18). Thus, we can gain $x_0(t), x_1(t), \ldots, x_n(t)$ by means of solving the linear high-order deformation Eq. (17) one after the other in order. The mth-order approximation of $x(t)$ is given by $x(t) = \sum_{k=0}^{m-1} x_k(t)$.

It should be emphasized that the so-called mth-order deformation equation (17) is linear, which can be easily solved by symbolic computation software's such as Maple or Mathematica.

5 Solution of FIVPs by HAM

In this section, we employ our technique to construction the HAM solution for FIVPs with respect to (1)-differentiability only in order not to increase the length of the paper without the loss of generality for the remaining type. However, similar construct can be implemented for the (2)-differentiability.

Let $q \in [0,1]$ be the so-called embedding parameter. The HAM is based on a kind of continuos mappings $\underline{x}_r(t) \to \underline{\Phi}_r(t;q)$ and $\overline{x}_r(t) \to \overline{\Phi}_r(t;q)$ such that, as the embedding parameter q increases from 0 to 1, $\underline{\Phi}_r(t;q)$ and $\overline{\Phi}_r(t;q)$ varies from the initial approximation to the exact solution.

Define the nonlinear operators

$$\begin{aligned}N_1\left[\underline{\Phi}_r(t;q)\right] &= \tfrac{d}{dt}\left[\underline{\Phi}_r(t;q)\right] - f_{1,r}\left(t,\underline{\Phi}_r(t;q),\overline{\Phi}_r(t;q)\right),\\ N_2\left[\overline{\Phi}_r(t;q)\right] &= \tfrac{d}{dt}\left[\overline{\Phi}_r(t;q)\right] - f_{2,r}\left(t,\underline{\Phi}_r(t;q),\overline{\Phi}_r(t;q)\right).\end{aligned}$$

Let $\hbar_i \neq 0$ and $H_i(t) \neq 0$, $i=1,2$, denote the so-called auxiliary parameter and auxiliary function, respectively. Using the embedding parameter q, we construct a family of zeroth-order deformation equations

$$\begin{aligned}(1-q)\mathscr{L}_1\left[\underline{\Phi}_r(t;q) - \underline{x}_{r,0}(t)\right] &= q\hbar_1 H_1(t) N_1\left[\underline{\Phi}_r(t;q)\right],\\ (1-q)\mathscr{L}_2\left[\overline{\Phi}_r(t;q) - \overline{x}_{r,0}(t)\right] &= q\hbar_2 H_2(t) N_2\left[\overline{\Phi}_r(t;q)\right],\end{aligned} \tag{19}$$

subject to the initial conditions $\underline{\Phi}_r(t_0;q) = \underline{x}_{r,0}(t)$ and $\overline{\Phi}_r(t_0;q) = \overline{x}_{r,0}(t)$, where $\underline{x}_{r,0}(t)$ and $\overline{x}_{r,0}(t)$ are the initial guesses approximations of $\underline{x}_r(t)$ and $\overline{x}_r(t)$, respectively.

Obviously, when $q=0$, since $\underline{x}_{r,0}(t)$ and $\overline{x}_{r,0}(t)$ satisfy the initial conditions (7) and according to the property (12), we have $\underline{\Phi}_r(t;0) = \underline{x}_{r,0}(t)$ and $\overline{\Phi}_r(t;0) = \overline{x}_{r,0}(t)$. Also, when $q=1$, since $\hbar_i \neq 0$ and $H_i(t) \neq 0$, $i=1,2$, the zeroth-order deformation equation (19) gives $\underline{\Phi}_r(t;1) = \underline{x}_r(t)$ and $\overline{\Phi}_r(t;1) = \overline{x}_r(t)$.

By Taylor's theorem, we expand $\underline{\Phi}_r(t;q)$ and $\overline{\Phi}_r(t;q)$ by a power series of the embedding parameter q as $\underline{\Phi}_r(t;q) = \underline{x}_{r,0}(t) + \sum_{m=1}^{\infty}\underline{x}_{r,m}(t)q^m$ and $\overline{\Phi}_r(t;q) = \overline{x}_{r,0}(t) + \sum_{m=1}^{\infty}\overline{x}_{r,m}(t)q^m$, where $\underline{x}_{r,m}(t)$ and $\overline{x}_{r,m}(t)$ are given, respectively, as $\frac{1}{m!}\frac{\partial^m\underline{\Phi}_r(t;q)}{\partial q^m}\Big|_{q=0}$ and $\frac{1}{m!}\frac{\partial^m\overline{\Phi}_r(t;q)}{\partial q^m}\Big|_{q=0}$. Thus, at $q=1$, the series becomes

$$\begin{aligned}\underline{x}_r(t) &= \underline{x}_{r,0}(t) + \sum_{m=1}^{\infty}\underline{x}_{r,m}(t),\\ \overline{x}_r(t) &= \overline{x}_{r,0}(t) + \sum_{m=1}^{\infty}\overline{x}_{r,m}(t).\end{aligned} \tag{20}$$

From the so-called mth-order deformation equations (17) and (18), we have

$$\begin{aligned}&\mathscr{L}_1\left[\underline{x}_{r,m}(t) - \chi_m\underline{x}_{r,m-1}(t)\right]\\ &\quad= \hbar_1 H_1(t)\Re\underline{x}_{r,m}\left(\overrightarrow{\underline{x}}_{r,m-1}(t),\overrightarrow{\overline{x}}_{r,m-1}(t)\right),\\ &\mathscr{L}_2\left[\overline{x}_{r,m}(t) - \chi_m\overline{x}_{r,m-1}(t)\right]\\ &\quad= \hbar_2 H_2(t)\Re\overline{x}_{r,m}\left(\overrightarrow{\underline{x}}_{r,m-1}(t),\overrightarrow{\overline{x}}_{r,m-1}(t)\right),\end{aligned} \tag{21}$$

where

$$\begin{aligned}&\Re\underline{x}_{r,m}\left(\overrightarrow{\underline{x}}_{r,m-1}(t),\overrightarrow{\overline{x}}_{r,m-1}(t)\right)\\ &= \tfrac{d}{dt}\underline{x}_{r,m}(t) - \frac{1}{(m-1)!}\frac{\partial^{m-1}f_{1,r}\left[t,\underline{\Phi}_r(t;q),\overline{\Phi}_r(t;q)\right]}{\partial q^{m-1}}\Big|_{q\to 0},\\ &\Re\overline{x}_{r,m}\left(\overrightarrow{\underline{x}}_{r,m-1}(t),\overrightarrow{\overline{x}}_{r,m-1}(t)\right)\\ &= \tfrac{d}{dt}\overline{x}_{r,m}(t) - \frac{1}{(m-1)!}\frac{\partial^{m-1}f_{2,r}\left[t,\underline{\Phi}_r(t;q),\overline{\Phi}_r(t;q)\right]}{\partial q^{m-1}}\Big|_{q\to 0}.\end{aligned} \tag{22}$$

For simplicity, we can choose $H_i(t) = 1$, $\hbar_i = \hbar$, and $\mathscr{L}_i = \frac{d}{dt}$, $i=1,2$. Then, the right inverse of $\frac{d}{dt}$ will be $\int_{t_0}^{t}(\cdot)\,d\tau$. Hence, the mth-order deformation equation (21) for $m \geq 1$ becomes

$$\begin{aligned}\underline{x}_{r,m}(t) &= \chi_m\underline{x}_{r,m-1}(t)\\ &\quad+\hbar\int_{t_0}^{t}\Re\underline{x}_{r,m}\left(\overrightarrow{\underline{x}}_{r,m-1}(\tau),\overrightarrow{\overline{x}}_{r,m-1}(\tau)\right)d\tau,\\ \overline{x}_{r,m}(t) &= \chi_m\overline{x}_{r,m-1}(t)\\ &\quad+\hbar\int_{t_0}^{t}\Re\overline{x}_{r,m}\left(\overrightarrow{\underline{x}}_{r,m-1}(\tau),\overrightarrow{\overline{x}}_{r,m-1}(\tau)\right)d\tau.\end{aligned} \tag{23}$$

If we choose $\underline{x}_{r,0}(t) = \underline{x}_r(t_0) = \underline{x}_r^0$ and $\overline{x}_{r,0}(t) = \overline{x}_r(t_0) = \overline{x}_r^0$ as initial guesses approximations of $\underline{x}_r(t)$ and $\overline{x}_r(t)$, respectively, then we can calculate $\underline{x}_{r,i}(t)$ and $\overline{x}_{r,i}(t)$, $i=1,2,\ldots,n$ by using the iteration formula (23). Finally, we approximate the solution $\underline{x}_r(t)$ and $\overline{x}_r(t)$ of system (6) and (7) by the kth-truncated series $\psi_{\underline{x}_{r,k}}(t) = \sum_{m=0}^{k-1}\underline{x}_{r,m}(t)$ and $\psi_{\overline{x}_{r,k}}(t) = \sum_{m=0}^{k-1}\overline{x}_{r,m}(t)$.

6 Numerical results and discussion

The HAM provides an analytical approximate solution in terms of an infinite power series. However, there is a practical need to evaluate this solution, and to obtain numerical values from the infinite power series. The consequent series truncation and the practical procedure are conducted to accomplish this task. In this section, we consider three examples to demonstrate the performance and efficiency of the present technique.

Throughout this paper, we will try to give the results of the all examples; however, in some cases we will switch between the results obtained for the examples in order not to increase the length of the paper without the loss of generality for the remaining examples and results.

In the process of computation, all the symbolic and numerical computations performed by using Mathematica 7.0 software package.

Example 4.1. Consider the following linear FIVP:

$$x'(t) = 2tx(t) + tu,\ 0 \le t \le \sqrt{\ln(3)}, \tag{24}$$

subject to the fuzzy initial condition

$$x(0) = u, \tag{25}$$

where $u(s) = \max(0, 1 - |s|)$, $s \in \mathbb{R}$.

The fuzzy (1)-differentiable exact solution is $x(t) = \frac{1}{2}\left(3e^{t^2} - 1\right)u$, while the fuzzy (2)-differentiable exact solution is $x(t) = \frac{1}{2}\left(3e^{-t^2} - 1\right)u$.

For finding the fuzzy (1)- and (2)-solutions of FIVP (24) and (25) which are corresponding to their parametric form, we have the following two cases of ODEs system:

Case I. The system of the ODEs corresponding to (1)-differentiability is:

$$\begin{aligned} \underline{x}_r'(t) &= 2t\underline{x}_r(t) + t(r-1), \\ \overline{x}_r'(t) &= 2t\overline{x}_r(t) + t(1-r), \end{aligned} \tag{26}$$

subject to the initial conditions

$$\begin{aligned} \underline{x}_r(0) &= r-1, \\ \overline{x}_r(0) &= 1-r. \end{aligned} \tag{27}$$

According to Eqs. (15) and (22), we have

$$\begin{aligned} &\Re\underline{x}_{r,m}\left(\overrightarrow{\underline{x}}_{r,m-1}(t), \overrightarrow{\overline{x}}_{r,m-1}(t)\right) \\ &\quad = \tfrac{d}{dt}\underline{x}_{r,m-1}(t) - 2t\underline{x}_{r,m-1}(t) - t(r-1)(1-\chi_m), \\ &\Re\overline{x}_{r,m}\left(\overrightarrow{\underline{x}}_{r,m-1}(t), \overrightarrow{\overline{x}}_{r,m-1}(t)\right) \\ &\quad = \tfrac{d}{dt}\overline{x}_{r,m-1}(t) - 2t\overline{x}_{r,m-1}(t) - t(1-r)(1-\chi_m). \end{aligned}$$

As we mentioned earlier, if we select the initial guesses approximations as $\underline{x}_{r,0}(t) = r-1$ and $\overline{x}_{r,0}(t) = 1-r$, then according to the iteration formula (23), the first few terms of the HAM series solution for Eqs. (26) and (27), are as follows:

$$\begin{aligned} &\underline{x}_{r,1}(t) = \tfrac{3}{2}(1-r)\hbar t^2, \\ &\overline{x}_{r,1}(t) = \tfrac{3}{2}(r-1)\hbar t^2, \\ &\underline{x}_{r,2}(t) \\ &\quad = \tfrac{3}{2}(1-r)\hbar t^2 + \tfrac{3}{2}(1-r)\hbar^2 t^2 + \tfrac{3}{4}(r-1)\hbar^2 t^4, \\ &\overline{x}_{r,2}(t) \\ &\quad = \tfrac{3}{2}(r-1)\hbar t^2 + \tfrac{3}{2}(r-1)\hbar^2 t^2 + \tfrac{3}{4}(1-r)\hbar^2 t^4. \end{aligned}$$

It is to be noted that the series solution contains the auxiliary parameter $\hbar$ and the truth parameter r. So, similarly to the so-called $\hbar$-curve, we can consider the so called $\hbar r$-plane which provide us with a way to determine the valid region of the auxiliary parameter $\hbar$ at various values of r which corresponds to the horizontal line segment resulting by the intersection of the plane $r = r_0$ with the flat surface. Figure 1 shows the $\hbar r$-plane corresponding to the 6th-order approximation HAM solution of $\underline{x}_r(t)$ and $\overline{x}_r(t)$ for each $r \in [0,1]$. It is evident that the valid region of the auxiliary parameter $\hbar$ for both components $\underline{x}_r(t)$ and $\overline{x}_r(t)$ is about $-1.8 < \hbar < -0.4$ for each $r \in [0,1]$. Thus, -1 is available value for $\hbar$.

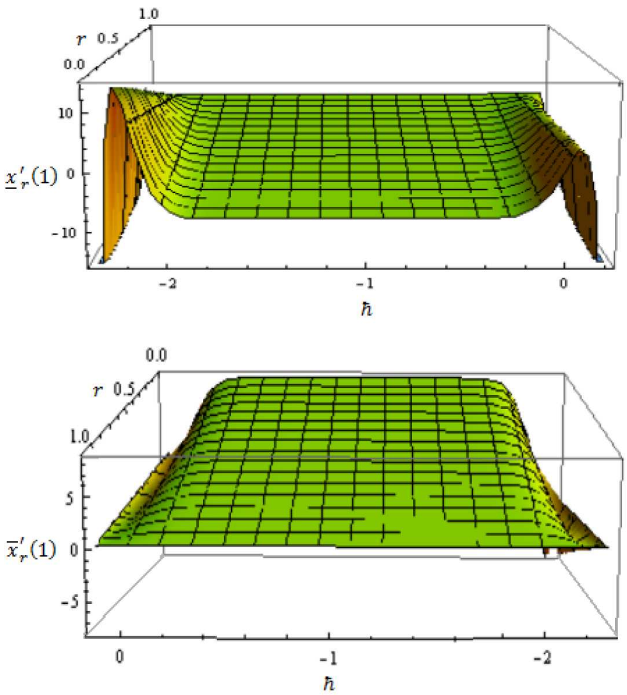

Fig. 1: The $\hbar r$-plane of $\underline{x}_r'(1)$ and $\overline{x}_r'(1)$ which are corresponding to the 6th-order approximation HAM solution of $\underline{x}_r(t)$ and $\overline{x}_r(t)$ for Eqs. (26) and (27).

Now, substitute $\hbar = -1$ into the 6th-order approximation HAM solution of $\underline{x}_r(t)$ and $\overline{x}_r(t)$ to get the following series:

$$\begin{aligned} &\psi_{\underline{x}_{r,6}}(t) \\ &= \tfrac{1}{2}\left(3 + 3t^2 + \tfrac{3}{2}t^4 + \tfrac{3}{6}t^6 + \tfrac{3}{24}t^8 + \tfrac{3}{120}t^{10} - 1\right)(r-1) \\ &= \tfrac{1}{2}\left(3\sum_{j=0}^{5}\frac{(t^2)^j}{j!} - 1\right)(r-1), \end{aligned}$$

$$\begin{aligned} &\psi_{\overline{x}_{r,6}}(t) \\ &= \tfrac{1}{2}\left(3 + 3t^2 + \tfrac{3}{2}t^4 + \tfrac{3}{6}t^6 + \tfrac{3}{24}t^8 + \tfrac{3}{120}t^{10} - 1\right)(1-r) \\ &= \tfrac{1}{2}\left(3\sum_{j=0}^{5}\frac{(t^2)^j}{j!} - 1\right)(1-r). \end{aligned}$$

Thus, the exact solution of Eqs. (26) and (27) has the general form which are coinciding with the exact solutions

$$\begin{aligned}\underline{x}_r(t) &= \tfrac{1}{2}\left(3\sum_{j=0}^{\infty}\frac{(t^2)^j}{j!}-1\right)(r-1)\\ &= \tfrac{1}{2}\left(3e^{t^2}-1\right)(r-1),\\ \overline{x}_r(t) &= \tfrac{1}{2}\left(3\sum_{j=0}^{\infty}\frac{(t^2)^j}{j!}-1\right)(1-r)\\ &= \tfrac{1}{2}\left(3e^{t^2}-1\right)(1-r).\end{aligned}$$

So, the exact solution of FIVP (24) and (25) in the sense of (1)-differentiability is $x(t) = \frac{1}{2}\left(3e^{t^2}-1\right)u$, where $u(s) = \max(0, 1-|s|)$, $s \in \mathbb{R}$.

Case II. The system of the ODEs corresponding to (**2**)-differentiability is:

$$\begin{aligned}\underline{x}_r'(t) &= 2t\overline{x}_r(t) + t(r-1),\\ \overline{x}_r'(t) &= 2t\underline{x}_r(t) + t(1-r),\end{aligned} \tag{28}$$

subject to the initial conditions

$$\begin{aligned}\underline{x}_r(0) &= r-1,\\ \overline{x}_r(0) &= 1-r.\end{aligned} \tag{29}$$

Choose the initial guesses approximations as $\underline{x}_{r,0}(t) = r-1$ and $\overline{x}_{r,0}(t) = 1-r$, then according to the iteration formula (23), the first few terms of the HAM series solution for Eqs. (28) and (29) are as follows:

$$\begin{aligned}&\underline{x}_{r,1}(t) = \tfrac{3}{2}(1-r)\hbar t^2,\\ &\overline{x}_{r,1}(t) = \tfrac{3}{2}(r-1)\hbar t^2,\\ &\underline{x}_{r,2}(t)\\ &\quad = \tfrac{3}{2}(1-r)\hbar t^2 + \tfrac{3}{2}(1-r)\hbar^2 t^2 + \tfrac{3}{4}(r-1)\hbar^2 t^4,\\ &\overline{x}_{r,2}(t)\\ &\quad = \tfrac{3}{2}(r-1)\hbar t^2 + \tfrac{3}{2}(r-1)\hbar^2 t^2 + \tfrac{3}{4}(1-r)\hbar^2 t^4.\end{aligned}$$

If we set the auxiliary parameter $\hbar = -1$, then the 6th-order approximation HAM solution of $\underline{x}_r(t)$ and $\overline{x}_r(t)$ are

$$\begin{aligned}&\psi_{\underline{x}_{r,6}}(t)\\ &= \tfrac{1}{2}\left(3 - 3t^2 + \tfrac{3}{2}t^4 - \tfrac{3}{6}t^6 + \tfrac{3}{24}t^8 - \tfrac{3}{120}t^{10} - 1\right)(r-1)\\ &= \tfrac{1}{2}\left(3\sum_{j=0}^{5}\frac{(-t^2)^j}{j!}-1\right)(r-1),\end{aligned}$$

$$\begin{aligned}&\psi_{\overline{x}_{r,6}}(t)\\ &= \tfrac{1}{2}\left(3 - 3t^2 + \tfrac{3}{2}t^4 - \tfrac{3}{6}t^6 + \tfrac{3}{24}t^8 - \tfrac{3}{120}t^{10} - 1\right)(1-r)\\ &= \tfrac{1}{2}\left(3\sum_{j=0}^{5}\frac{(-t^2)^j}{j!}-1\right)(1-r).\end{aligned}$$

Thus, the exact solution of Eqs. (28) and (29) has the general form which are coinciding with the exact solutions

$$\begin{aligned}\underline{x}_r(t) &= \tfrac{1}{2}\left(3\sum_{j=0}^{\infty}\frac{(-t^2)^j}{j!}-1\right)(r-1)\\ &= \tfrac{1}{2}\left(3e^{-t^2}-1\right)(r-1),\\ \overline{x}_r(t) &= \tfrac{1}{2}\left(3\sum_{j=0}^{\infty}\frac{(-t^2)^j}{j!}-1\right)(1-r)\\ &= \tfrac{1}{2}\left(3e^{-t^2}-1\right)(1-r).\end{aligned}$$

So, the exact solution of FIVP (24) and (25) in the sense of (2)-differentiability is $x(t) = \frac{1}{2}\left(3e^{-t^2}-1\right)u$, where $u(s) = \max(0, 1-|s|)$, $s \in \mathbb{R}$.

Next, we show by example that the crisp IVPs can be modeled in a natural way as FIVPs. Consider a simple *RL* circuit (The ”*RL*-circuit” is an abbreviation of Resistance-Inductance circuit). The ODE corresponding to this electrical circuit is $i'(t) = -\frac{R}{L}i(t) + v(t)$, $t_0 \le t \le t_0 + a$, subject to the initial condition $i(t_0) = i^0$, where R is the circuit resistance, L is the coefficient corresponding to the solenoid, and v is the voltage function. Environmental conditions, inaccuracy in element modeling, electrical noise, leakage, and other parameters cause uncertainty in the aforementioned equation. Considering it instead as a FIVP yields more realistic results. This innovation helps to detect unknown conditions in circuit analysis [33].

Example 4.2. Consider an electrical *RL* circuit with an *AC* source:

$$i'(t) = -\tfrac{R}{L}i(t) + v(t),\ 0 \le t \le 1, \tag{30}$$

subject to the uncertain initial condition

$$i(0) = u. \tag{31}$$

Suppose that $R = 1$ Ohm, $L = 1$ Henry, $v(t) = \sin(t)$, and $u(s) = \begin{cases} 25s - 24, & 0.96 \le s \le 1,\\ -100s + 101, & 1 \le s \le 1.01,\\ 0, & \text{otherwise.}\end{cases}$ Then the fuzzy (1)-differentiable exact solution is $i(t) = \frac{1}{2}(\sin(t) - \cos(t)) + \frac{1}{2}e^{-t} + \cosh(t)u - \sinh(t)u$, while the fuzzy (2)-differentiable exact solution is $i(t) = \frac{1}{2}(\sin(t) - \cos(t)) + \frac{1}{2}e^{-t} + e^{-t}u$.

For finding the fuzzy (1)- and (2)-solutions of FIVP (30) and (31) which are corresponding to their parametric form, we have the following two cases of ODEs system:

Case I. The system of the ODEs corresponding to (1)-differentiability is:

$$\begin{aligned}\underline{i}_r'(t) &= -\overline{i}_r(t) + \sin(t),\\ \overline{i}_r'(t) &= -\underline{i}_r(t) + \sin(t),\end{aligned} \tag{32}$$

subject to the initial conditions

$$\begin{aligned}\underline{i}_r(0) &= \tfrac{24}{25}+\tfrac{1}{25}r,\\ \bar{i}_r(0) &= \tfrac{101}{100}-\tfrac{1}{100}r.\end{aligned} \tag{33}$$

As we mentioned earlier, if we select the initial guesses approximations as $\underline{i}_{r,0}(t)=\frac{24}{25}+\frac{1}{25}r$ and $\bar{i}_{r,0}(t)=\frac{101}{100}-\frac{1}{100}r$, then according to the iteration formula (23), the first few terms of the HAM series solution for Eqs. (32) and (33) are

$$\begin{aligned}
\underline{i}_{r,1}(t) &= \hbar(\cos(t)-1)+\left(\tfrac{101}{100}-\tfrac{1}{100}r\right)\hbar t,\\
\bar{i}_{r,1}(t) &= \hbar(\cos(t)-1)+\left(\tfrac{24}{25}+\tfrac{1}{25}r\right)\hbar t,\\
\underline{i}_{r,2}(t) &= \hbar(\cos(t)-1)+\left(\tfrac{101}{100}-\tfrac{1}{100}r\right)\hbar t\\
&\quad+\hbar^2(\cos(t)+\sin(t)-1)+\left(\tfrac{1}{100}-\tfrac{1}{100}r\right)\hbar^2 t\\
&\quad+\left(\tfrac{12}{25}+\tfrac{1}{50}r\right)\hbar^2 t^2,\\
\bar{i}_{r,2}(t) &= \hbar(\cos(t)-1)+\left(\tfrac{24}{25}+\tfrac{1}{25}r\right)\hbar t,\\
&\quad+\hbar^2(\cos(t)+\sin(t)-1)+\left(\tfrac{1}{25}r-\tfrac{1}{25}\right)\hbar^2 t\\
&\quad+\left(\tfrac{101}{200}-\tfrac{1}{200}r\right)\hbar^2 t^2.
\end{aligned}$$

Choose the auxiliary parameter as $\hbar=-1$. Then the $4k$th, $(4k+1)$th, $(4k+2)$th, and $(4k+3)$th-truncated series of the HAM solution of $\underline{i}_r(t)$ and $\bar{i}_r(t)$ are as follows:

$$\begin{aligned}
\psi_{\underline{i}_{r,4k}}(t) &= \tfrac{1}{2}\sum_{j=0}^{4k}\frac{(-1)^j t^{2j+1}}{(2j+1)!}-\tfrac{1}{2}\sum_{j=0}^{4k}\frac{(-1)^j t^{2j}}{(2j)!}\\
&\quad+\tfrac{1}{2}\sum_{j=0}^{4k}\frac{(-1)^j t^{j}}{j!}+\left(\tfrac{24}{25}+\tfrac{1}{25}r\right)\sum_{j=0}^{4k}\frac{t^{2j}}{(2j)!}\\
&\quad-\left(\tfrac{101}{100}-\tfrac{1}{100}r\right)\sum_{j=0}^{4k}\frac{t^{2j+1}}{(2j+1)!},
\end{aligned}$$

$$\begin{aligned}
\psi_{\bar{i}_{r,4k}}(t) &= \tfrac{1}{2}\sum_{j=0}^{4k}\frac{(-1)^j t^{2j+1}}{(2j+1)!}-\tfrac{1}{2}\sum_{j=0}^{4k}\frac{(-1)^j t^{2j}}{(2j)!}\\
&\quad+\tfrac{1}{2}\sum_{j=0}^{4k}\frac{(-1)^j t^{j}}{j!}+\left(\tfrac{101}{100}-\tfrac{1}{100}r\right)\sum_{j=0}^{4k}\frac{t^{2j}}{(2j)!}\\
&\quad-\left(\tfrac{24}{25}+\tfrac{1}{25}r\right)\sum_{j=0}^{4k}\frac{t^{2j+1}}{(2j+1)!},
\end{aligned}$$

$$\begin{aligned}
\psi_{\underline{i}_{r,4k+1}}(t) &= -\cos(t)+\tfrac{1}{2}\sum_{j=0}^{4k+1}\frac{(-1)^j t^{2j+1}}{(2j+1)!}\\
&\quad+\tfrac{1}{2}\sum_{j=0}^{4k+1}\frac{(-1)^j t^{2j}}{(2j)!}+\tfrac{1}{2}\sum_{j=0}^{4k+1}\frac{(-1)^j t^{j}}{j!}\\
&\quad+\left(\tfrac{24}{25}+\tfrac{1}{25}r\right)\sum_{j=0}^{4k+1}\frac{t^{2j}}{(2j)!}\\
&\quad-\left(\tfrac{101}{100}-\tfrac{1}{100}r\right)\sum_{j=0}^{4k+1}\frac{t^{2j+1}}{(2j+1)!},
\end{aligned}$$

$$\begin{aligned}
\psi_{\bar{i}_{r,4k+1}}(t) &= -\cos(t)+\tfrac{1}{2}\sum_{j=0}^{4k+1}\frac{(-1)^j t^{2j+1}}{(2j+1)!}\\
&\quad+\tfrac{1}{2}\sum_{j=0}^{4k+1}\frac{(-1)^j t^{2j}}{(2j)!}+\tfrac{1}{2}\sum_{j=0}^{4k+1}\frac{(-1)^j t^{j}}{j!}\\
&\quad+\left(\tfrac{101}{100}-\tfrac{1}{100}r\right)\sum_{j=0}^{4k+1}\frac{t^{2j}}{(2j)!}\\
&\quad-\left(\tfrac{24}{25}+\tfrac{1}{25}r\right)\sum_{j=0}^{4k+1}\frac{t^{2j+1}}{(2j+1)!},
\end{aligned}$$

$$\begin{aligned}
\psi_{\underline{i}_{r,4k+2}}(t) &= \sin(t)-\cos(t)\\
&\quad-\tfrac{1}{2}\sum_{j=0}^{4k+2}\frac{(-1)^j t^{2j+1}}{(2j+1)!}+\tfrac{1}{2}\sum_{j=0}^{4k+2}\frac{(-1)^j t^{2j}}{(2j)!}\\
&\quad+\tfrac{1}{2}\sum_{j=0}^{4k+2}\frac{(-1)^j t^{j}}{j!}+\left(\tfrac{24}{25}+\tfrac{1}{25}r\right)\sum_{j=0}^{4k+2}\frac{t^{2j}}{(2j)!}\\
&\quad-\left(\tfrac{101}{100}-\tfrac{1}{100}r\right)\sum_{j=0}^{4k+2}\frac{t^{2j+1}}{(2j+1)!},
\end{aligned}$$

$$\begin{aligned}
\psi_{\bar{i}_{r,4k+2}}(t) &= \sin(t)-\cos(t)\\
&\quad-\tfrac{1}{2}\sum_{j=0}^{4k+2}\frac{(-1)^j t^{2j+1}}{(2j+1)!}+\tfrac{1}{2}\sum_{j=0}^{4k+2}\frac{(-1)^j t^{2j}}{(2j)!}\\
&\quad+\tfrac{1}{2}\sum_{j=0}^{4k+2}\frac{(-1)^j t^{j}}{j!}+\left(\tfrac{101}{100}-\tfrac{1}{100}r\right)\sum_{j=0}^{4k+2}\frac{t^{2j}}{(2j)!}\\
&\quad-\left(\tfrac{24}{25}+\tfrac{1}{25}r\right)\sum_{j=0}^{4k+2}\frac{t^{2j+1}}{(2j+1)!},
\end{aligned}$$

$$\begin{aligned}
\psi_{\underline{i}_{r,4k+3}}(t) &= \sin(t)-\tfrac{1}{2}\sum_{j=0}^{4k+3}\frac{(-1)^j t^{2j+1}}{(2j+1)!}\\
&\quad-\tfrac{1}{2}\sum_{j=0}^{4k+3}\frac{(-1)^j t^{2j}}{(2j)!}+\tfrac{1}{2}\sum_{j=0}^{4k+3}\frac{(-1)^j t^{j}}{j!}\\
&\quad+\left(\tfrac{24}{25}+\tfrac{1}{25}r\right)\sum_{j=0}^{4k+3}\frac{t^{2j}}{(2j)!}\\
&\quad-\left(\tfrac{101}{100}-\tfrac{1}{100}r\right)\sum_{j=0}^{4k+3}\frac{t^{2j+1}}{(2j+1)!},
\end{aligned}$$

$$\begin{aligned}
\psi_{\bar{i}_{r,4k+3}}(t) &= \sin(t)-\tfrac{1}{2}\sum_{j=0}^{4k+3}\frac{(-1)^j t^{2j+1}}{(2j+1)!}\\
&\quad-\tfrac{1}{2}\sum_{j=0}^{4k+3}\frac{(-1)^j t^{2j}}{(2j)!}+\tfrac{1}{2}\sum_{j=0}^{4k+3}\frac{(-1)^j t^{j}}{j!}\\
&\quad+\left(\tfrac{101}{100}-\tfrac{1}{100}r\right)\sum_{j=0}^{4k+3}\frac{t^{2j}}{(2j)!}\\
&\quad-\left(\tfrac{24}{25}+\tfrac{1}{25}r\right)\sum_{j=0}^{4k+3}\frac{t^{2j+1}}{(2j+1)!}.
\end{aligned}$$

Thus, in all cases the analytic approximate solutions of Eqs. (32) and (33) agree well with the exact solutions

$$\begin{aligned}
\underline{i}_r(t) &= \tfrac{1}{2}(\sin(t)-\cos(t))+\tfrac{1}{2}e^{-t}\\
&\quad+\left(\tfrac{24}{25}+\tfrac{1}{25}r\right)\cosh(t)-\left(\tfrac{101}{100}-\tfrac{1}{100}r\right)\sinh(t),
\end{aligned}$$

$$\begin{aligned}
\bar{i}_r(t) &= \tfrac{1}{2}(\sin(t)-\cos(t))+\tfrac{1}{2}e^{-t}\\
&\quad+\left(\tfrac{101}{100}-\tfrac{1}{100}r\right)\cosh(t)-\left(\tfrac{24}{25}+\tfrac{1}{25}r\right)\sinh(t).
\end{aligned}$$

So, the exact solution in the sense of (1)-differentiability is

$$i(t)=\frac{1}{2}(\sin(t)-\cos(t))+\frac{1}{2}e^{-t}+\cosh(t)u-\sinh(t)u,$$

where $[u]^r=\left[\frac{24}{25}+\frac{1}{25}r,\frac{101}{100}-\frac{1}{100}r\right]$.

The solution set $[i(t)]^r$ and its derivative $[i'(t)]^r$ are plotted in Figure 2. It can be shown that theses sets satisfies the FIVP (30) and (31) with respect to (1)-differentiability. On the other hand, it is clear from the figure that for each $t\in[0,1]$ and $r\in[0,1]$ the r-cut

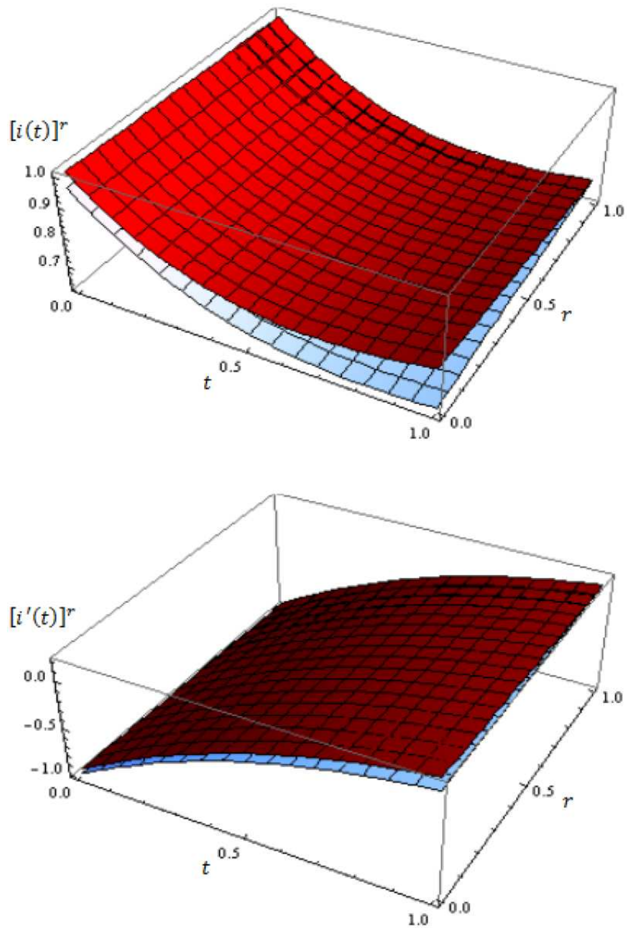

Fig. 2: The solution $[i(t)]^r$ and its derivative $[i'(t)]^r$ of Eqs. (32) and (33): lower surfaces denote the $\underline{i}_r(t)$ and $\underline{i}'_r(t)$, while upper surfaces denote the $\bar{i}_r(t)$ and $\bar{i}'_r(t)$.

representation of the solution and its derivative are valid level sets. In fact these results are in agreement with Case I of Algorithm 3.1.

Case II. The system of the ODEs corresponding to $(\mathbf{2})$-differentiability is:

$$\begin{aligned} \underline{i}'_r(t) &= -\underline{i}_r(t) + \sin(t), \\ \bar{i}'_r(t) &= -\bar{i}_r(t) + \sin(t), \end{aligned} \tag{34}$$

subject to the initial conditions

$$\begin{aligned} \underline{i}_r(0) &= \tfrac{24}{25} + \tfrac{1}{25}r, \\ \bar{i}_r(0) &= \tfrac{101}{100} - \tfrac{1}{100}r. \end{aligned} \tag{35}$$

As in the previous, if we select the initial guesses approximations as $\underline{i}_{r,0}(t) = \frac{24}{25} + \frac{1}{25}r$ and $\bar{i}_{r,0}(t) = \frac{101}{100} - \frac{1}{100}r$, then according to the iteration formula (23), the first few terms of the HAM series solution for Eqs. (34) and (35) are

$$\begin{aligned}
\underline{i}_{r,1}(t) &= \hbar(\cos(t) - 1) + \left(\tfrac{24}{25} + \tfrac{1}{25}r\right)\hbar t, \\
\bar{i}_{r,1}(t) &= \hbar(\cos(t) - 1) + \left(\tfrac{101}{100} - \tfrac{1}{100}r\right)\hbar t, \\
\underline{i}_{r,2}(t) &= \hbar(\cos(t) - 1) + \left(\tfrac{101}{100} - \tfrac{1}{100}r\right)\hbar t \\
&\quad + \hbar^2(\cos(t) + \sin(t) - 1) + \left(\tfrac{1}{100} - \tfrac{1}{100}r\right)\hbar^2 t \\
&\quad + \left(\tfrac{12}{25} + \tfrac{1}{50}r\right)\hbar^2 t^2, \\
\bar{i}_{r,2}(t) &= \hbar(\cos(t) - 1) + \left(\tfrac{24}{25} + \tfrac{1}{25}r\right)\hbar t \\
&\quad + \hbar^2(\cos(t) + \sin(t) - 1) + \left(\tfrac{r}{25} - \tfrac{1}{25}\right)\hbar^2 t \\
&\quad + \left(\tfrac{101}{200} - \tfrac{1}{200}r\right)\hbar^2 t^2.
\end{aligned}$$

Similarly to the previous discussion, if we set the auxiliary parameter $\hbar = -1$, then the $4k$th, $(4k+1)$th, $(4k+2)$th, and $(4k+3)$th-truncated series of the HAM solution of $\underline{i}_r(t)$ and $\bar{i}_r(t)$ are as follows:

$$\begin{aligned}
\psi_{\underline{i}_{r,4k}}(t) &= \frac{1}{2}\sum_{j=0}^{4k}\frac{(-1)^j t^{2j+1}}{(2j+1)!} - \frac{1}{2}\sum_{j=0}^{4k}\frac{(-1)^j t^{2j}}{(2j)!} \\
&\quad + \frac{1}{2}\sum_{j=0}^{4k}\frac{(-1)^j t^j}{j!} + \left(\frac{24}{25} + \frac{1}{25}r\right)\sum_{j=0}^{4k}\frac{(-1)^j t^j}{j!},
\end{aligned}$$

$$\begin{aligned}
\psi_{\bar{i}_{r,4k}}(t) &= \frac{1}{2}\sum_{j=0}^{4k}\frac{(-1)^j t^{2j+1}}{(2j+1)!} - \frac{1}{2}\sum_{j=0}^{4k}\frac{(-1)^j t^{2j}}{(2j)!} \\
&\quad + \frac{1}{2}\sum_{j=0}^{4k}\frac{(-1)^j t^j}{j!} + \left(\frac{101}{100} - \frac{1}{100}r\right)\sum_{j=0}^{4k}\frac{(-1)^j t^j}{j!},
\end{aligned}$$

$$\begin{aligned}
\psi_{\underline{i}_{r,4k+1}}(t) &= -\cos(t) \\
&\quad + \frac{1}{2}\sum_{j=0}^{4k+1}\frac{(-1)^j t^{2j+1}}{(2j+1)!} + \frac{1}{2}\sum_{j=0}^{4k+1}\frac{(-1)^j t^{2j}}{(2j)!} \\
&\quad + \frac{1}{2}\sum_{j=0}^{4k+1}\frac{(-1)^j t^j}{j!} + \left(\frac{24}{25} + \frac{1}{25}r\right)\sum_{j=0}^{4k+1}\frac{(-1)^j t^j}{j!},
\end{aligned}$$

$$\begin{aligned}
\psi_{\bar{i}_{r,4k+1}}(t) &= -\cos(t) \\
&\quad + \frac{1}{2}\sum_{j=0}^{4k+1}\frac{(-1)^j t^{2j+1}}{(2j+1)!} + \frac{1}{2}\sum_{j=0}^{4k+1}\frac{(-1)^j t^{2j}}{(2j)!} \\
&\quad + \frac{1}{2}\sum_{j=0}^{4k+1}\frac{(-1)^j t^j}{j!} + \left(\frac{101}{100} - \frac{1}{100}r\right)\sum_{j=0}^{4k+1}\frac{(-1)^j t^j}{j!},
\end{aligned}$$

$$\begin{aligned}
\psi_{\underline{i}_{r,4k+2}}(t) &= \sin(t) - \cos(t) \\
&\quad - \frac{1}{2}\sum_{j=0}^{4k+2}\frac{(-1)^j t^{2j+1}}{(2j+1)!} + \frac{1}{2}\sum_{j=0}^{4k+2}\frac{(-1)^j t^{2j}}{(2j)!} \\
&\quad + \frac{1}{2}\sum_{j=0}^{4k+2}\frac{(-1)^j t^j}{j!} + \left(\frac{24}{25} + \frac{1}{25}r\right)\sum_{j=0}^{4k+2}\frac{(-1)^j t^j}{j!},
\end{aligned}$$

$$\begin{aligned}
\psi_{\bar{i}_{r,4k+2}}(t) &= \sin(t) - \cos(t) \\
&\quad - \frac{1}{2}\sum_{j=0}^{4k+2}\frac{(-1)^j t^{2j+1}}{(2j+1)!} + \frac{1}{2}\sum_{j=0}^{4k+2}\frac{(-1)^j t^{2j}}{(2j)!} \\
&\quad + \frac{1}{2}\sum_{j=0}^{4k+2}\frac{(-1)^j t^j}{j!} + \left(\frac{101}{100} - \frac{1}{100}r\right)\sum_{j=0}^{4k+2}\frac{(-1)^j t^j}{j!},
\end{aligned}$$

$$\psi_{\underline{i}_{r,4k+3}}(t) = \sin(t) - \frac{1}{2}\sum_{j=0}^{4k+3}\frac{(-1)^j t^{2j+1}}{(2j+1)!} - \frac{1}{2}\sum_{j=0}^{4k+3}\frac{(-1)^j t^{2j}}{(2j)!} + \frac{1}{2}\sum_{j=0}^{4k+3}\frac{(-1)^j t^j}{j!} + \left(\frac{24}{25}+\frac{1}{25}r\right)\sum_{j=0}^{4k+3}\frac{(-1)^j t^j}{j!},$$

$$\psi_{\bar{i}_{r,4k+3}}(t) = \sin(t) - \frac{1}{2}\sum_{j=0}^{4k+3}\frac{(-1)^j t^{2j+1}}{(2j+1)!} - \frac{1}{2}\sum_{j=0}^{4k+3}\frac{(-1)^j t^{2j}}{(2j)!} + \frac{1}{2}\sum_{j=0}^{4k+3}\frac{(-1)^j t^j}{j!} + \left(\frac{101}{100}-\frac{1}{100}r\right)\sum_{j=0}^{4k+3}\frac{(-1)^j t^j}{j!}.$$

It is clear that in all cases the analytic approximate solutions of Eqs. (34) and (35) agree well with the exact solutions

$$\underline{i}_r(t) = \frac{1}{2}(\sin(t)-\cos(t)) + \frac{1}{2}e^{-t} + \left(\frac{24}{25}+\frac{1}{25}r\right)e^{-t},$$
$$\bar{i}_r(t) = \frac{1}{2}(\sin(t)-\cos(t)) + \frac{1}{2}e^{-t} + \left(\frac{101}{100}-\frac{1}{100}r\right)e^{-t}.$$

So, the exact solution in the sense of (2)-differentiability is

$$i(t) = \frac{1}{2}(\sin(t)-\cos(t)) + \frac{1}{2}e^{-t} + e^{-t}u,$$

where $[u]^r = \left[\frac{24}{25}+\frac{1}{25}r, \frac{101}{100}-\frac{1}{100}r\right]$.

In most real-life situations, the differential equation that models the uncertainty problem is too complicated to solve exactly, and there is a practical need to approximate the solution. In the next example, the fuzzy (1)- and (2)-differentiable exact solutions cannot be found analytically.

Example 3.3. Consider the following nonlinear FIVP:

$$x'(t) = \sinh(tx(t)),\ 0 \le t \le 1.5, \tag{36}$$

subject to the fuzzy initial condition

$$x(0) = u, \tag{37}$$

where $u(s) = \max\left(0, 1-4s^2\right)$, $s \in \mathbb{R}$.

According to Zadeh's extension principal and Nguyen's theorem, it is clear that

$$[\sinh(tx(t))]^r = [\sinh(t\underline{x}_r(t)), \sinh(t\bar{x}_r(t))].$$

This is due to the fact that $\sinh(s)$ is strictly increasing function on $(-\infty,\infty)$ and $t \ge 0$.

For finding the fuzzy (1)- and (2)-approximate solutions of FIVP (36) and (37) which are corresponding to their parametric form, we have the following two cases of ODEs system:

Case I. The system of the ODEs corresponding to (1)-differentiability is:

$$\begin{aligned}\underline{x}_r'(t) &= \sinh(t\underline{x}_r(t)),\\ \bar{x}_r'(t) &= \sinh(t\bar{x}_r(t)),\end{aligned} \tag{38}$$

subject to the initial conditions

$$\begin{aligned}\underline{x}_r(0) &= -\tfrac{1}{2}\sqrt{1-r},\\ \bar{x}_r(0) &= \tfrac{1}{2}\sqrt{1-r}.\end{aligned} \tag{39}$$

According to Eqs. (15) and (22), we have

$$\Re\underline{x}_{r,m}\left(\overrightarrow{\underline{x}}_{r,m-1}(t), \overrightarrow{\bar{x}}_{r,m-1}(t)\right) = \frac{d}{dt}\underline{x}_{r,m-1}(t) - \frac{1}{(m-1)!}\frac{\partial^{m-1}\sinh(t\underline{\Phi}_r(t;q))}{\partial q^{m-1}}\bigg|_{q\to 0},$$

$$\Re\bar{x}_{r,m}\left(\overrightarrow{\underline{x}}_{r,m-1}(t), \overrightarrow{\bar{x}}_{r,m-1}(t)\right) = \frac{d}{dt}\bar{x}_{r,m-1}(t) - \frac{1}{(m-1)!}\frac{\partial^{m-1}\sinh\left(t\overline{\Phi}_r(t;q)\right)}{\partial q^{m-1}}\bigg|_{q\to 0},$$

where $\underline{\Phi}_r(t;q)$ and $\overline{\Phi}_r(t;q)$ are unknown functions to be determined. Assuming that the initial guesses approximations have the form $\underline{x}_{r,0}(t) = -\frac{1}{2}\sqrt{1-r}$ and $\bar{x}_{r,0}(t) = \frac{1}{2}\sqrt{1-r}$. Then according to the iteration formula (23), we have

$$\underline{x}_{1,m}(t) = \bar{x}_{1,m}(t) = 0,\ m = 0,1,2,\ldots,n,$$

and for $r \in [0,1)$ the 6th-truncated series of the HAM solution of $\underline{x}_r(t)$ and $\bar{x}_r(t)$ for Eqs. (38) and (39) are

$$\begin{aligned}\psi_{\underline{x}_{r,6}}(t) = &-\tfrac{1}{2}\sqrt{1-r} + \tfrac{20\hbar}{\sqrt{1-r}}\sinh^2\left(\tfrac{1}{4}\sqrt{1-r}t\right)\\ &- \tfrac{2\hbar^2}{\sqrt{(1-r)^3}}[-10+t^2-r\left(4+t^2\right)\\ &+4(3+r)\cosh\left(\tfrac{1}{2}\sqrt{1-r}t\right) - 2\cosh\left(\sqrt{1-r}t\right)\\ &+2\sqrt{1-r}t(-4\sinh\left(\tfrac{1}{2}\sqrt{1-r}t\right)+\sinh\left(\sqrt{1-r}t\right))]\\ &- \tfrac{3\hbar^2}{2\sqrt{(1-r)^3}}[t^2-2\hbar t^2-rt^2-2\hbar rt^2\\ &+(8r-8h-8)\sinh^2\left(\tfrac{1}{4}\sqrt{1-r}t\right)+\ldots],\end{aligned}$$

$$\begin{aligned}\psi_{\bar{x}_{r,6}}(t) = &\tfrac{1}{2}\sqrt{1-r} - \tfrac{20\hbar}{\sqrt{1-r}}\sinh^2\left(\tfrac{1}{4}\sqrt{1-r}t\right)\\ &+ \tfrac{2\hbar^2}{\sqrt{(1-r)^3}}[-10+t^2-r\left(4+t^2\right)\\ &+4(3+r)\cosh\left(\tfrac{1}{2}\sqrt{1-r}t\right) - 2\cosh\left(\sqrt{1-r}t\right)\\ &+2\sqrt{1-r}t(-4\sinh\left(\tfrac{1}{2}\sqrt{1-r}t\right)+\sinh\left(\sqrt{1-r}t\right))]\\ &+ \tfrac{3\hbar^2}{2\sqrt{(1-r)^3}}[t^2-2\hbar t^2-rt^2-2\hbar rt^2\\ &+(8r-8h-8)\sinh^2\left(\tfrac{1}{4}\sqrt{1-r}t\right)+\ldots].\end{aligned}$$

The HAM yields rapidly convergent series solution by using a few iterations. For the convergence of the HAM, the reader is referred to [42]. According to [62], it is to be noted that the series solution contains the auxiliary parameter $\hbar$ for each $r \in [0,1]$ which provides a simple way to adjust and control the convergence of the series solution. In fact, it is very important to ensure that the

Table 1: The valid region of the auxiliary parameter $\hbar$ derived from Figure 3.

Component	$r=0$	$r=0.25$	$r=0.5$	$r=0.75$	$r=1$
$\underline{x}_r(t)$	$(-1.3,-0.7)$	$(-1.35,-0.7)$	$(-1.4,-0.6)$	$(-1.45,-0.6)$	$(-\infty,\infty)$
$\overline{x}_r(t)$	$(-1.3,-0.7)$	$(-1.35,-0.7)$	$(-1.4,-0.6)$	$(-1.45,-0.6)$	$(-\infty,\infty)$

Table 2: The optimal values of the auxiliary parameter $\hbar$ for $\underline{x}_r(t)$ and $\overline{x}_r(t)$ when $r=0$, $r=0.5$, and $r=0.99$.

x	$\underline{x}_0(t)$	$\overline{x}_0(t)$	$\underline{x}_{0.5}(t)$	$\overline{x}_{0.5}(t)$	$\underline{x}_{0.99}(t)$	$\overline{x}_{0.99}(t)$
0.1	−1.0000000013	−1.0000000013	−1.0000000013	−1.0000000013	−0.5410000001	−0.5410000001
0.3	−1.0130452602	−1.0130452602	−1.0129118647	−1.0129118647	−0.5498010110	−0.5498010110
0.5	−1.0354529642	−1.0354529642	−1.0351219648	−1.0351219648	−0.5587990098	−0.5587990098
0.7	−1.0766553248	−1.0766553248	−1.0719138996	−1.0719138996	−0.5988990098	−0.5988990098
0.9	−1.1508810120	−1.1508810120	−1.1305725598	−1.1305725598	−0.6320000000	−0.6320000000
1.1	−1.2958134301	−1.2958134301	−1.2284778981	−1.2284778981	−0.6756323170	−0.6756323170
1.3	−1.6162720895	−1.6162720895	−1.4116937600	−1.4116937600	−0.7357323170	−0.7357323170
1.5	−2.2546652710	−2.2546652710	−1.8260061609	−1.8260061609	−0.7931021973	−0.7931021973

Table 3: The value of $|\mathrm{E}\underline{x}_r(t)|$ and $\underline{x}_r(t)$ at $\hbar=-1$ and $\hbar=\hbar^*$ when $r=0.5$.

x	$\mathrm{E}\underline{x}_{0.5}(t,\hbar=-1)$	$\mathrm{E}\underline{x}_{0.5}(t,\hbar=\hbar^*)$	$\underline{x}_{0.5}(t,\hbar=\hbar^*)$
0.1	3.67898×10^{-12}	8.72864×10^{-13}	−0.35532577
0.3	2.47288×10^{-10}	1.99886×10^{-12}	−0.36984328
0.5	1.32890×10^{-7}	2.65782×10^{-12}	−0.40078310
0.7	9.68787×10^{-6}	1.16784×10^{-11}	−0.45250017
0.9	2.74747×10^{-4}	4.95293×10^{-12}	−0.53329974
1.1	4.54620×10^{-3}	4.98122×10^{-11}	−0.65948125
1.3	5.55665×10^{-2}	4.06397×10^{-11}	−0.86927464
1.5	6.13456×10^{-1}	3.34371×10^{-10}	−1.29064532

series formula (20) are convergent. To this end, we have plotted $\hbar$-curves of $\underline{x}_r'(0)$ and $\overline{x}_r'(0)$ which are corresponding to the 6th-order approximation of the HAM in Figure 3 when $r=0$, $r=0.25$, $r=0.5$, $r=0.75$, and $r=1$.

Again, according to these $\hbar$-curves, it is easy to discover the valid region of $\hbar$ which corresponds to the line segment nearly parallel to the horizontal axis. These valid regions have been listed in Table 1 for the various r in $[0,1]$. Furthermore, these valid regions ensure us the convergence of the obtained series.

To determine the optimal values of $\hbar$ in a neighborhood of t_0, an error analysis is performed. We substitute the approximations $\psi_{\underline{x}_{r,6}}(t)$ and $\psi_{\overline{x}_{r,6}}(t)$ into Eq. (38) and obtain the residual functions $\mathrm{E}\underline{x}_r$ and $\mathrm{E}\overline{x}_r$ as follows:

$$\mathrm{E}\underline{x}_r(t,\hbar)=\tfrac{d}{dt}\psi_{\underline{x}_{r,6}}(t)-\sinh\left(t\psi_{\underline{x}_{r,6}}(t)\right),$$

$$\mathrm{E}\overline{x}_r(t,\hbar)=\tfrac{d}{dt}\psi_{\overline{x}_{r,6}}(t)-\sinh\left(t\psi_{\overline{x}_{r,6}}(t)\right).$$

Following [63], we define the square residual error for approximation solutions on the interval $[t_0-\varepsilon,t_0+\varepsilon]$ as

$$\mathrm{SE}\underline{x}_r(\hbar)=\int_{t_0-\varepsilon}^{t_0+\varepsilon}\left(\mathrm{E}\underline{x}_r(t,\hbar)\right)^2dt,$$

$$\mathrm{SE}\overline{x}_r(\hbar)=\int_{t_0-\varepsilon}^{t_0+\varepsilon}\left(\mathrm{E}\overline{x}_r(t,\hbar)\right)^2dt.$$

By using the first derivative test, we can easily determine the values of $\hbar$ for which the $\mathrm{SE}\underline{x}_r$ and $\mathrm{SE}\overline{x}_r$ are minimum.

In Table 2, the optimal values of $\hbar$, denoted by $\hbar^*$, when $r=0$, $r=0.5$, and $r=0.99$ for the two components $\underline{x}_r(t)$ and $\overline{x}_r(t)$ are tabulated.

In Tables 3 and 4, the absolute residual errors $|\mathrm{E}\underline{x}_r(t)|$ and $|\mathrm{E}\overline{x}_r(t)|$ have been calculated for the various t in the dependent interval $[0,1.5]$ at $\hbar=-1$ and $\hbar=\hbar^*$ when $r=0.5$. From the tables, it can be seen that the HAM provides us with the accurate approximate solution for Eqs. (38) and (39). Also, we can note that the approximate solutions more accurate at the optimal value of $\hbar$.

In Table 5, the absolute errors $|\mathrm{E}\underline{x}_r(t)|$ and $|\mathrm{E}\overline{x}_r(t)|$ have been calculated for the various r in $[0,1]$ at $\hbar=-1$ and $\hbar=\hbar^*$ when $t=1.5$. From the table, it can be seen that the HAM at the optimal value $\hbar^*$ provides us with the less absolute residual error for Eqs. (38) and (39) in comparison with $\hbar=-1$.

Our next goal is to show how the optimal value $\hbar^*$ of the auxiliary parameter $\hbar$ affects the approximate solutions. Tables 6 and 7, show a comparison between the approximate solutions of the HAM at $\hbar=-1$ and $\hbar=\hbar^*$ together with RKM of order four and PCM of the same order for $\underline{x}_r(t)$ and $\overline{x}_r(t)$, respectively. Throughout these tables, the step size for the RKM and PCM is fixed at 0.1. The starting values of the PCM obtained from the classical fourth-order RKM.

Table 4: The value of $|E\bar{x}_r(t)|$ and $\bar{x}_r(t)$ at $\hbar=-1$ and $\hbar=\hbar^*$ when $r=0.5$.

x	$E\bar{x}_{0.5}(t,\hbar=-1)$	$E\bar{x}_{0.5}(t,\hbar=\hbar^*)$	$\bar{x}_{0.5}(t,\hbar=\hbar^*)$
0.1	3.67898×10^{-12}	8.72864×10^{-13}	0.35532577
0.3	2.47288×10^{-10}	1.99886×10^{-12}	0.36984328
0.5	1.32890×10^{-7}	2.65782×10^{-12}	0.40078310
0.7	9.68787×10^{-6}	1.16784×10^{-11}	0.45250017
0.9	2.74747×10^{-4}	4.95293×10^{-12}	0.53329974
1.1	4.54620×10^{-3}	4.98122×10^{-11}	0.65948125
1.3	5.55665×10^{-2}	4.06397×10^{-11}	0.86927464
1.5	6.13456×10^{-1}	3.34371×10^{-10}	1.29064532

Table 5: The value of $|E\underline{x}_r(t)|$ and $|E\bar{x}_r(t)|$ at $\hbar=-1$ and $\hbar=\hbar^*$ when $t=1.5$ for different values of r.

r	$\hbar^*$	$\lvert E\underline{x}_r(1.5,\hbar=-1)\rvert$	$\lvert E\bar{x}_r(1.5,\hbar=-1)\rvert$	$\lvert E\underline{x}_r(1.5,\hbar=\hbar^*)\rvert$	$\lvert E\bar{x}_r(1.5,\hbar=\hbar^*)\rvert$
0	-2.2546652710	4.613980	4.613980	1.97757×10^{-9}	1.97757×10^{-9}
0.25	-2.0578582473	1.829180	1.829180	7.65811×10^{-10}	7.65811×10^{-10}
0.5	-1.8260061609	0.613456	0.613456	3.34371×10^{-10}	3.34371×10^{-10}
0.75	-1.5709161475	0.135063	0.135063	1.24671×10^{-9}	1.24671×10^{-9}
0.99	-0.7931021973	0.026892	0.026892	1.15143×10^{-8}	1.15143×10^{-8}
1	arbitrary	0	0	0	0

Table 6: The approximate solution of $\underline{x}_r(t)$ at various r in $[0,1]$ when $t=1.5$.

r	PCM	RKM	HAM ($\hbar=-1$)	HAM ($\hbar=\hbar^*$)
0	-2.77313316	-2.87698498	-1.97719903	-2.09095862
0.25	-1.83692573	-1.83614143	-1.60002889	-1.70174542
0.5	-1.29653388	-1.29547421	-1.22481988	-1.29064532
0.75	-0.83032636	-0.83009093	-0.81477887	-0.83483167
0.99	-0.15443177	-0.15442550	-0.13553931	-0.14478679
1	0	0	0	0

However, it is evident from Tables 6 and 7 that the both approximate nodal values corresponding to the 6th-order approximation HAM solution of $\underline{x}_r(t)$ and $\bar{x}_r(t)$ are differ by the sign only. That is $\underline{x}_r(t)=-\bar{x}_r(t)$ for each $t\in[0,1.5]$ and $r\in[0,1]$. On the other hand, if we set the auxiliary parameters $\hbar=-1$, then the HAM solution is the same as the Adomian decomposition solution [64, 65] and the homotopy perturbation solution [66].

Case II. The system of the ODEs corresponding to (2)-differentiability is:

$$\begin{aligned}\underline{x}'_r(t) &= \sinh\left(t\bar{x}_r(t)\right),\\ \bar{x}'_r(t) &= \sinh\left(t\underline{x}_r(t)\right),\end{aligned}\tag{40}$$

subject to the initial conditions

$$\begin{aligned}\underline{x}_r(0) &= -\tfrac{1}{2}\sqrt{1-r},\\ \bar{x}_r(0) &= \tfrac{1}{2}\sqrt{1-r}.\end{aligned}\tag{41}$$

As in the previous case, if we select the initial guesses approximations as $\underline{x}_{r,0}(t)=-\frac{1}{2}\sqrt{1-r}$ and $\bar{x}_{r,0}(t)=\frac{1}{2}\sqrt{1-r}$, then according to the iteration formula (23), we have

$$\underline{x}_{1,m}(t)=\bar{x}_{1,m}(t)=0,\; m=0,1,2,\ldots,n,$$

and for $r\in[0,1)$ the 6th-truncated series of the HAM solution of $\underline{x}_r(t)$ and $\bar{x}_r(t)$ for Eqs. (40) and (41) are

$$\begin{aligned}\psi_{\underline{x}_{r,6}}(t) &= -\tfrac{1}{2}\sqrt{1-r}-\tfrac{20\hbar}{\sqrt{1-r}}\sinh^2\left(\tfrac{1}{4}\sqrt{1-r}t\right)\\ &-\tfrac{2\hbar^2}{\sqrt{(1-r)^3}}[-18+4r+(1-r)t^2\\ &-4(r-5)\cosh\left(\tfrac{1}{2}\sqrt{1-r}t\right)-2\cosh\left(\sqrt{1-r}t\right)\\ &+\sqrt{1-r}\left(2t\sinh\left(\sqrt{1-r}t\right)-8\sinh\left(\tfrac{1}{2}\sqrt{1-r}t\right)\right)\\ &+\tfrac{3\hbar^2}{2\sqrt{(1-r)^3}}[(2\hbar r+r-6\hbar-1)t^2\\ &+(8r-8\hbar-8)\sinh^2\left(\tfrac{1}{4}\sqrt{1-r}t\right)+\ldots],\end{aligned}$$

$$\begin{aligned}\psi_{\bar{x}_{r,6}}(t) &= \tfrac{1}{2}\sqrt{1-r}+\tfrac{20\hbar}{\sqrt{1-r}}\sinh^2\left(\tfrac{1}{4}\sqrt{1-r}t\right)\\ &+\tfrac{2\hbar^2}{\sqrt{(1-r)^3}}[-18+4r+(1-r)t^2\\ &-4(r-5)\cosh\left(\tfrac{1}{2}\sqrt{1-r}t\right)-2\cosh\left(\sqrt{1-r}t\right)\\ &+\sqrt{1-r}\left(2t\sinh\left(\sqrt{1-r}t\right)-8\sinh\left(\tfrac{1}{2}\sqrt{1-r}t\right)\right)\\ &-\tfrac{3\hbar^2}{2\sqrt{(1-r)^3}}[(2\hbar r+r-6\hbar-1)t^2\\ &+(8r-8\hbar-8)\sinh^2\left(\tfrac{1}{4}\sqrt{1-r}t\right)+\ldots].\end{aligned}$$

These results are plotted in Figure 4 at $\hbar=-1$ for the two components solutions $\underline{x}_r(t)$ and $\bar{x}_r(t)$ together with their derivatives $\underline{x}'_r(t)$ and $\bar{x}'_r(t)$. As the plots show, while

Table 7: The approximate solution of $\overline{x}_r(t)$ at various r in $[0,1]$ when $t=1.5$.

r	PCM	RKM	HAM ($\hbar=-1$)	HAM ($\hbar=\hbar^*$)
0	2.77313316	2.87698498	1.97719903	2.09095862
0.25	1.83692573	1.83614143	1.60002889	1.70174542
0.5	1.29653388	1.29547421	1.22481988	1.29064532
0.75	0.83032636	0.83009093	0.81477887	0.83483167
0.99	0.15443177	0.15442550	0.13553931	0.14478679
1	0	0	0	0

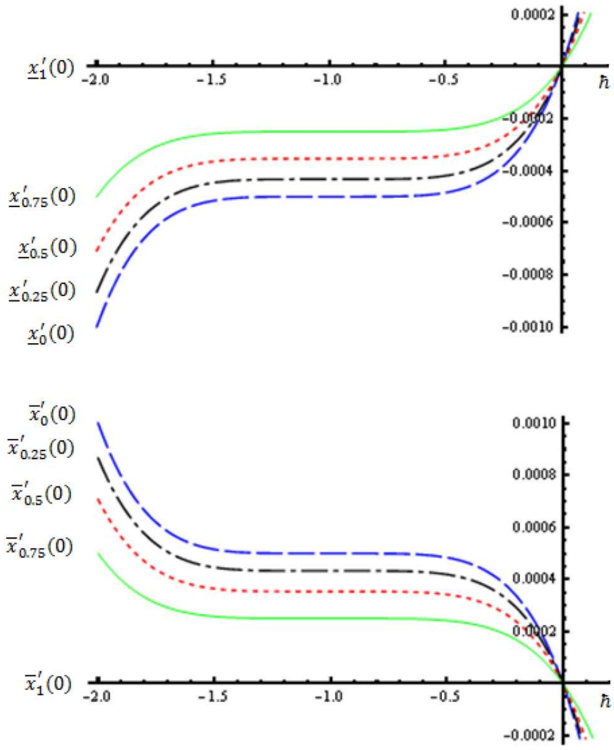

Fig. 3: The $\hbar$-curves of $\underline{x}'_r(0)$ and $\overline{x}'_r(0)$ which are corresponding to the 6th-order approximation HAM solution of $\underline{x}_r(t)$ and $\overline{x}_r(t)$ for Eqs. (38) and (39) when $r=0$, $r=0.25$, $r=0.5$, $r=0.75$, and $r=1$.

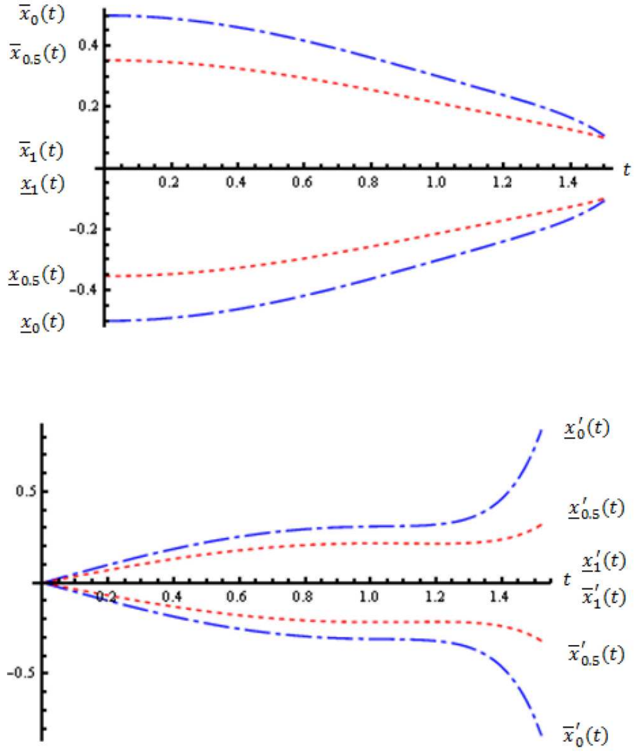

Fig. 4: The 6th-truncated series of the HAM solution set $[\underline{x}_r(t), \overline{x}_r(t)]$ and its derivative $[\overline{x}'_r(t), \underline{x}'_r(t)]$ for Eqs. (40) and (41) when $r=0$, $r=0.5$, and $r=1$.

the truth value r decreases, the component $\underline{x}_r(t)$ expanding to down for each $t\in[0,1.5]$. Meanwhile, the component $\overline{x}_r(t)$ expanding to up, but the solution $[\underline{x}_r(t), \overline{x}_r(t)]$ over $[0,1.5]$ for each $r\in[0,1]$ is a valid level set. Similar conclusion can be achieved for $\underline{x}'_r(t)$ and $\overline{x}'_r(t)$ with full agreement with Case II of Algorithm 3.1.

7 Concluding remarks

The main concern of this work has been to propose an efficient algorithm for the solution of FIVPs. The goal has been achieved by extending the HAM to solve this class of fuzzy differential equations. We can conclude that the HAM is powerful and efficient technique in finding approximate solutions for linear and nonlinear FIVPs. The proposed algorithm produced a rapidly convergent series by choosing suitable values of the auxiliary parameter $\hbar$.

There are two important points to make here. First, the HAM provides us with a simple way to adjust and control the convergence region of the series solution by introducing the auxiliary parameter $\hbar$. Second, the results obtained by the HAM are very effective and convenient in linear and nonlinear cases with less computational work. This confirms our belief that the efficiency of our technique gives it much wider applicability for general classes of linear and nonlinear problems.

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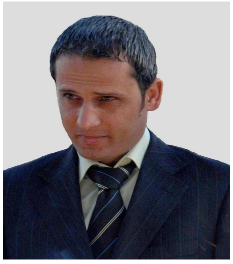

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