On Soft Topological Spaces

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Abstract: The classical mathematical theories have their difficulties which are pointed out in [18] for the solution complicated problems in engineering and environment. To overcome these difficulties, Molodtsov [18] introduced the concept of soft set as a new mathematical tool. Furthermore, D. Pei and D. Miao [22] showed that soft sets are a class of special information systems. In [9] for the soft set theory: new definitions, examples, new classes of soft sets, and properties for mappings between different classes of soft sets are introduced and studied. Moreover, the theory of soft topological spaces is investigated. This paper continues the study of the theory of soft topological spaces and presents for this theory new definitions, characterizations, and results concerning separation axioms, convergence, Cartesian product, soft θ-topology, and soft θ-continuity.

1 Preliminaries

In 1999 D. Molodtsov (see [18]) introduced the notion of soft set. Later, he applied this theory to several directions (see [19], [20], and [21]).

The soft set theory has been applied to many different fields (see, for example, [1], [2], [4], [5], [7], [8], [11], [12], [13], [14], [15], [16], [22], [25], [27]).

In 2011 and 2012 few researches (see, for example, [3], [6], [10], [17], [23], [24], [26]) introduced and studied the notion of soft topological spaces which are defined over an initial universe with a fixed set of parameters.

It is known that topology is an important area of mathematics with many applications in the domains of computer science and physical sciences. Soft topology is a relatively new and promising domain which can lead to the development of new mathematical models and innovative approaches that will significantly contribute to the solution of complex problems in natural sciences.

The paper is organized as follows. In section 2 we give known basic notions and results concerning the theory of soft sets and soft topological spaces. In section 3 we present separation axioms and we investigate the notion of convergence. In section 4 the Cartesian product of two soft topological spaces is studied. In section 5 we introduce and study notions concerning the soft θ-open sets, the soft θ-closed sets, and the soft θ-continuity. In

section 6 we give some questions on soft topological spaces. The last section of this paper is the conclusion.

2 Basic known notions for the theory of soft topological spaces

Definition 2.1 (see [18]) Let X be an initial universe set, $\mathscr{P}(X)$ the power set of X, that is the set of all subsets of X, and A a set of parameters. A pair (F,A), where F is a map from A to $\mathscr{P}(X)$, is called a *soft set* over X.

In what follows by $\mathrm{SS}(X,A)$ we denote the family of all soft sets (F,A) over X.

Definition 2.2 (see [18]) Let $(F,A),(G,A) \in \mathrm{SS}(X,A)$. We say that the pair (F,A) is a *soft subset* of (G,A) if $F(p) \subseteq G(p)$, for every $p \in A$. Symbolically, we write $(F,A) \sqsubseteq (G,A)$. Also, we say that the pairs (F,A) and (G,A) are *soft equal* if $(F,A) \sqsubseteq (G,A)$ and $(G,A) \sqsubseteq (F,A)$. Symbolically, we write $(F,A) = (G,A)$.

Definition 2.3 (see, for example, [18] and [26]) Let I be an arbitrary index set and $\{(F_i,A) : i \in I\} \subseteq \mathrm{SS}(X,A)$. The *soft union* of these soft sets is the soft set $(F,A) \in \mathrm{SS}(X,A)$, where the map $F : A \to \mathscr{P}(X)$ defined as follows: $F(p) = \cup\{F_i(p) : i \in I\}$, for every $p \in A$. Symbolically, we write

$$(F,A) = \sqcup\{(F_i,A) : i \in I\}.$$

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Definition 2.4 (see, for example, [18] and [26]) Let I be an arbitrary index set and $\{(F_i,A) : i \in I\} \subseteq \mathrm{SS}(X,A)$. The *soft intersection* of these soft sets is the soft set $(F,A) \in \mathrm{SS}(X,A)$, where the map $F : A \to \mathscr{P}(X)$ defined as follows: $F(p) = \cap\{F_i(p) : i \in I\}$, for every $p \in A$. Symbolically, we write

$$(F,A) = \sqcap\{(F_i,A) : i \in I\}.$$

Definition 2.5 (see, for example, [2] and [26]) Let $(F,A) \in \mathrm{SS}(X,A)$. The *soft complement* of (F,A) is the soft set $(H,A) \in \mathrm{SS}(X,A)$, where the map $H : A \to \mathscr{P}(X)$ defined as follows: $H(p) = X \setminus F(p)$, for every $p \in A$. Symbolically, we write

$$(H,A) = (F,A)^c.$$

Definition 2.6 (see [18]) The soft set $(F,A) \in \mathrm{SS}(X,A)$, where $F(p) = \emptyset$, for every $p \in A$ is called the *A-null soft set* of $\mathrm{SS}(X,A)$ and denoted by $\mathbf{0}_A$. The soft set $(F,A) \in \mathrm{SS}(X,A)$, where $F(p) = X$, for every $p \in A$ is called the *A-absolute soft set* of $\mathrm{SS}(X,A)$ and denoted by $\mathbf{1}_A$.

Definition 2.7 (see [26]) Let X and Y be two initial universe sets, A and B two sets of parameters, $f : X \to Y$, and $e : A \to B$. Then, by Φ_{fe} we denote the map from $\mathrm{SS}(X,A)$ to $\mathrm{SS}(Y,B)$ for which:

(1) If $(F,A) \in \mathrm{SS}(X,A)$, then the image of (F,A) under Φ_{fe}, denoted by $\Phi_{fe}(F,A)$, is the soft set $(G,B) \in \mathrm{SS}(Y,B)$ such that

$$G(p_Y) = \begin{cases} \bigcup\{f(F(p)) : p \in e^{-1}(\{p_Y\})\}, \\ \quad \text{if } e^{-1}(\{p_Y\}) \neq \emptyset, \\ \emptyset, \text{ if } e^{-1}(\{p_Y\}) = \emptyset, \end{cases}$$

for every $p_Y \in B$.

(2) If $(G,B) \in \mathrm{SS}(Y,B)$, then the *inverse image* of (G,B) under Φ_{fe}, denoted by $\Phi_{fe}^{-1}(G,B)$, is the soft set $(F,A) \in \mathrm{SS}(X,A)$ such that $F(p) = f^{-1}(G(e(p)))$, for every $p \in A$.

Proposition 2.8 (see Propositions 2.18 and 2.19 of [9]) Let $(F,A),(F_1,A) \in \mathrm{SS}(X,A)$, $(G,B),(G_1,B) \in \mathrm{SS}(Y,B)$. The following statements are true:

(1) If $(F,A) \sqsubseteq (F_1,A)$, then $\Phi_{fe}(F,A) \sqsubseteq \Phi_{fe}(F_1,A)$.
(2) If $(G,B) \sqsubseteq (G_1,B)$, then $\Phi_{fe}^{-1}(G,B) \sqsubseteq \Phi_{fe}^{-1}(G_1,B)$.
(3) $(F,A) \sqsubseteq \Phi_{fe}^{-1}(\Phi_{fe}(F,A))$.
(4) $\Phi_{fe}(\Phi_{fe}^{-1}(G,B)) \sqsubseteq (G,B)$.
(5) $\Phi_{fe}^{-1}((G,B)^c) = (\Phi_{fe}^{-1}(G,B))^c$.
(6) $\Phi_{fe}((F,A) \sqcup (F_1,A)) = \Phi_{fe}(F,A) \sqcup \Phi_{fe}(F_1,A)$.
(7) $\Phi_{fe}((F,A) \sqcap (F_1,A)) \sqsubseteq \Phi_{fe}(F,A) \sqcap \Phi_{fe}(F_1,A)$.
(8) $\Phi_{fe}^{-1}((G,B) \sqcup (G_1,B)) = \Phi_{fe}^{-1}(G,B) \sqcup \Phi_{fe}^{-1}(G_1,B)$.
(9) $\Phi_{fe}^{-1}((G,B) \sqcap (G_1,B)) = \Phi_{fe}^{-1}(G,B) \sqcap \Phi_{fe}^{-1}(G_1,B)$.

Definition 2.9 (see, for example, [26]) Let X be an initial universe set, A a set of parameters, and $\tau \subseteq \mathrm{SS}(X,A)$. We say that the family τ defines a *soft topology* on X if the following axioms are true:
(1) $\mathbf{0}_A, \mathbf{1}_A \in \tau$.
(2) If $(G,A),(H,A) \in \tau$, then $(G,A) \sqcap (H,A) \in \tau$.
(3) If $(G_i,A) \in \tau$ for every $i \in I$, then

$$\sqcup\{(G_i,A) : i \in I\} \in \tau.$$

The triplet (X,τ,A) is called a *soft topological space* or *soft space*. The members of τ are called *soft open sets* in X. Also, a soft set (F,A) is called *soft closed* if the complement $(F,A)^c$ belongs to τ. The family of soft closed sets is denoted by τ^c.

Definition 2.10 (see [9]) Let (X,τ,A) be a soft topological space, $a \in A$, and $x \in X$. We say that a soft set $(G,A) \in \tau$ is an *a-soft open neighborhood of x* in (X,τ,A) if $x \in G(a)$.

Proposition 2.11 (see [9]) Let (X,τ,A) be a soft topological space. Then, $(G,A) \in \tau$ if and only if for every $a \in A$ and $x \in G(a)$ there exists an *a*-soft open neighborhood $(G_{(a,x)},A)$ of x in (X,τ,A) such that $(G_{(a,x)},A) \sqsubseteq (G,A)$.

Definition 2.12 (see [24]) Let (X,τ,A) be a soft topological space. The *soft closure* $\mathrm{Cl}(F,A)$ of $(F,A) \in \mathrm{SS}(X,A)$ is the soft set

$$\sqcap\{(Q,A) \in \tau^c : (F,A) \sqsubseteq (Q,A)\}.$$

Remark. (See, for example, [26]) Let (X,τ,A) be a soft topological space and $(F,A),(G,A) \in \mathrm{SS}(X,A)$. We recall the following properties :
(1) $(F,A) \in \tau^c$ if and only if $\mathrm{Cl}(F,A) = (F,A)$.
(2) If $(F,A) \sqsubseteq (G,A)$, then $\mathrm{Cl}(F,A) \sqsubseteq \mathrm{Cl}(G,A)$.

Definition 2.13 (see [9]) Let (X,τ,A) be a soft topological space and $a \in A$. A point $x \in X$ is said to be an *a-cluster point* of $(F,A) \in \mathrm{SS}(X,A)$ if for every *a*-soft open neighborhood (G,A) of x we have $(F,A) \sqcap (G,A) \neq \mathbf{0}_A$. The set of all *a*-cluster points of (F,A) is denoted by $\mathrm{cl}(F,a)$.

Proposition 2.14 (see [9]) Let (X,τ,A) be a soft space and $(F,A) \in \mathrm{SS}(X,A)$. Then, $\mathrm{Cl}(F,A) = (R_{F,A},A)$, where the map $R_{F,A} : A \to \mathscr{P}(X)$ defined as follows: $R_{F,A}(p) = F(p) \cup \mathrm{cl}(F,p)$, for every $p \in A$.

Remark. Obviously, $F(p) \subseteq \mathrm{cl}(F,p)$, for every $p \in A$. Thus, in Proposition 2.14 we have $R_{F,A}(p) = \mathrm{cl}(F,p)$, for every $p \in A$.

Definition 2.15 (see, for example, [26]) Let (X,τ,A) be a soft topological space. A family $\mathscr{B} \subseteq \tau$ is called a *base* for (X,τ,A) if for every soft open set $(G,A) \neq \mathbf{0}_A$, there exist $(G_i,A) \in \mathscr{B}$, $i \in I$, such that

$$(G,A) = \sqcup\{(G_i,A) : i \in I\}.$$

Proposition 2.16 (see [9]) Let (X, τ, A) be a soft topological space. Then, a family $\mathscr{B} \subseteq \tau$ is a base for (X, τ, A) if and only if for every $a \in A$, $x \in X$, and every a-soft open neighborhood (G, A) of x there exists an a-soft open neighborhood $(G_{(a,x)}, A)$ of x such that $(G_{(a,x)}, A) \in \mathscr{B}$ and $(G_{(a,x)}, A) \sqsubseteq (G, A)$.

Definition 2.17 (see [9]) Let (X, τ_X, A) and (Y, τ_Y, B) be two soft topological spaces, $x \in X$, and $e : A \to B$. A map $f : X \to Y$ is called *soft e-continuous at the point x* if for every $a \in A$ and every $e(a)$-soft open neighborhood (G, B) of $f(x)$ in (Y, τ_Y, B) there exists an a-soft open neighborhood (F, A) of x in (X, τ_X, A) such that

$$\Phi_{fe}(F, A) \sqsubseteq (G, B).$$

If the map f is soft e-continuous at any point $x \in X$, then we say that the map f is *soft e-continuous*.

Proposition 2.18 (see [9]) Let (X, τ_X, A) and (Y, τ_Y, B) be two soft topological spaces and $e : A \to B$. Then, the following statements are equivalent:
(1) The map $f : X \to Y$ is soft e-continuous.
(2) $\Phi_{fe}^{-1}(G, B) \in \tau_X$, for every $(G, B) \in \tau_Y$.
(3) $\Phi_{fe}^{-1}(Q, B) \in \tau_X^c$, for every $(Q, B) \in \tau_Y^c$.
(4) $\Phi_{fe}(\mathrm{Cl}(F, A)) \sqsubseteq \mathrm{Cl}(\Phi_{fe}(F, A))$, for every $(F, A) \in \mathrm{SS}(X, A)$.

Proposition 2.19 (see [9]) Let (X, τ_X, A) and (Y, τ_Y, B) be two soft topological spaces, $\mathscr{B}_Y$ a base for (Y, τ_Y, B), and $e : A \to B$. Then, the following statements are equivalent:
(1) The map $f : X \to Y$ is soft e-continuous.
(2) $\Phi_{fe}^{-1}(G, B) \in \tau_X$, for every $(G, B) \in \mathscr{B}_Y$.

3 Separation axioms and convergence

Notation 3.1 Let $(F, A) \in \mathrm{SS}(X, A)$, $a \in A$, and $x \in X$. In what follows we write $x \in_a (F, A)$ (respectively, $x \notin_a (F, A)$) if and only if $x \in F(a)$ (respectively, $x \notin F(a)$).

Definition 3.2 A soft topological space (X, τ, A) is called a *soft T_0-space* if for every distinct points x, y of X and for every $a \in A$ there exists a soft open set (G, A) such that either $x \in_a (G, A)$ and $y \notin_a (G, A)$ or $x \notin_a (G, A)$ and $y \in_a (G, A)$.

Definition 3.3 A soft topological space (X, τ, A) is called a *soft T_1-space* if for every distinct points x, y of X and for every $a \in A$ there exists a soft open set (G, A) such that $x \in_a (G, A)$ and $y \notin_a (G, A)$.

Proposition 3.4 Let (X, τ, A) be a soft topological space. If for every $x \in X$ we have $(F_x, A) \in \tau^c$, where $F_x(p) = \{x\}$, for every $p \in A$, then (X, τ, A) is a soft T_1-space.

Proof. We suppose that for every $x \in X$ we have $(F_x, A) \in \tau^c$. We prove that the triplet (X, τ, A) is a soft

T_1-space. Let x and y be two distinct points of X and $a \in A$. Then, $(F_y, A) \in \tau^c$ and, therefore, $(F_y, A)^c \in \tau$. Since $x \in X \setminus \{y\} = X \setminus F_y(a)$, we have $x \in_a (F_y, A)^c$. Also, $y \notin_a (F_y, A)^c$. Thus, the triplet (X, τ, A) is a soft T_1-space.

Definition 3.5 A soft topological space (X, τ, A) is called a *soft T_2-space* if for every distinct points x, y of X and for every $a \in A$ there exist two soft open sets (G, A) and (H, A) such that $x \in_a (G, A)$, $y \in_a (H, A)$, and $G(a) \cap H(a) = \emptyset$.

Remark 3.6 Every soft T_2-space is a soft T_1-space and every soft T_1-space is a soft T_0-space.

Example 3.7 (1) Let $X = \{x, y, z\}$ and $A = \{0, 1\}$. We consider the following soft sets (F_i, A), $i = 1, 2, \ldots, 6$, over X defined as follows:

$$F_1(a) = \begin{cases} \{x\}, & \text{if } a = 0, \\ \{z\}, & \text{if } a = 1, \end{cases} \quad F_2(a) = \begin{cases} \{y\}, & \text{if } a = 0, \\ \{x\}, & \text{if } a = 1, \end{cases}$$

$$F_3(a) = \begin{cases} \{z\}, & \text{if } a = 0, \\ \{y\}, & \text{if } a = 1, \end{cases} \quad F_4(a) = \begin{cases} \{x,y\}, & \text{if } a = 0, \\ \{x,z\}, & \text{if } a = 1, \end{cases}$$

$$F_5(a) = \begin{cases} \{x,z\}, & \text{if } a = 0, \\ \{y,z\}, & \text{if } a = 1, \end{cases} \quad F_6(a) = \begin{cases} \{y,z\}, & \text{if } a = 0, \\ \{x,y\}, & \text{if } a = 1. \end{cases}$$

Then, the triplet (X, τ, A), where

$$\tau = \{(F_i, A) : i = 1, 2, \ldots, 6\} \cup \{\mathbf{0}_A, \mathbf{1}_A\}$$

is a soft T_2-space and, therefore, soft T_1-space and soft T_0-space.

(2) Let $X = \{1, 2, \ldots\}$, $A = \{0, 1, 2\}$, and

$$\tau = \{(G_n, A) : n \in \{1, 2, \ldots\}\} \cup \{\mathbf{0}_A, \mathbf{1}_A\},$$

such that $G_n : A \to \mathscr{P}(X)$, where

$$G_n(p) = \{n, n+1, \ldots\},$$

for every $p \in A$. The triplet (X, τ, A) is a soft topological space.

Let x and y be two distinct points of X and $a \in A$. We suppose that $y < x$. Then, $x \in \{x, x+1, \ldots\} = G_x(a)$ and $y \notin \{x, x+1, \ldots\}$. This means that $x \in_a (G_x, A)$ and $y \notin_a (G_x, A)$. Thus, the soft topological space (X, τ, A) is a soft T_0-space. Also, we observe that this soft topological space is neither soft T_1-space nor soft T_2-space.

(3) Let (X, t) be a topological space, A a nonempty set, and

$$\tau = \{(G_U, A) : U \in t\},$$

such that $G_U : A \to \mathscr{P}(X)$, where $G_U(p) = U$, for every $p \in A$. The triplet (X, τ, A) is a soft topological space. Then, the soft topological space (X, τ, A) is a soft T_i-space if and only if the topological space (X, t) is a T_i-space, $i \in \{0, 1, 2\}$.

Definition 3.8 A soft topological space (X, τ, A) is called a *soft T_3-space* if for every point $x \in X$, for every $a \in A$,

and for every soft closed set (Q,A) such that $x \notin_a (Q,A)$ there exist two soft open sets (G,A) and (H,A) such that $x \in_a (G,A)$, $Q(a) \subseteq H(a)$, and $G(a) \cap H(a) = \emptyset$.

Example 3.9 Let $X = \{x,y,z\}$ and $A = \{0,1\}$. We consider the following soft sets (G,A) and (H,A) over X defined as follows:

$$G(a) = \begin{cases} \{x\}, & \text{if } a = 0, \\ \{x,y\}, & \text{if } a = 1, \end{cases} \quad H(a) = \begin{cases} \{y,z\}, & \text{if } a = 0, \\ \{z\}, & \text{if } a = 1. \end{cases}$$

Then, the triplet (X,τ,A), where

$$\tau = \{\mathbf{0}_A, \mathbf{1}_A, (G,A), (H,A)\}$$

is a soft topological space. We observe that the soft topological space (X,τ,A) is a soft T_3-space but it is not a soft T_i-space for $i = 0,1,2$.

The proof of the following proposition is straightforward verification of the definitions.

Proposition 3.10 Let (X,τ,A) be a soft T_3-space such that for every $x \in X$ and every $a_0 \in A$ the soft set $(F_x^{a_0}, A)$ defined as follows:

$$F_x^{a_0}(a) = \begin{cases} \{x\}, & \text{if } a = a_0 \\ \emptyset, & \text{if } a \neq a_0, \end{cases}$$

is a soft closed set. Then, the soft topological space (X,τ,A) is a T_2-space.

Proposition 3.11 Let (X,τ,A) be a soft topological space. If for every $x \in X$, for every $a \in A$, and for every a-soft open neighborhood (F,A) of x there exists an a-soft open neighborhood (G,A) of x such that

$$x \in G(a) \subseteq \text{cl}(G,a) \subseteq F(a),$$

then (X,τ,A) is a soft T_3-space.

Proof. Let $x \in X$, $a \in A$, and $(Q,A) \in \tau^c$ such that $x \notin_a (Q,A)$. Then, $x \in_a (Q,A)^c$ and $(Q,A)^c \in \tau$. This means that the soft set $(Q,A)^c$ is an a-soft open neighborhood of x. By assumption there exists an a-soft open neighborhood (G,A) of x such that

$$x \in G(a) \subseteq \text{cl}(G,a) \subseteq X \setminus Q(a).$$

We consider the soft set $\text{Cl}(G,A)^c$. By Proposition 2.14 and Definition 2.5, $\text{Cl}(G,A)^c = (H,A)$, where $H(p) = X \setminus \text{cl}(G,p)$, for every $p \in A$. We observe that $(H,A) \in \tau$, $Q(a) \subseteq H(a)$, and $G(a) \cap H(a) = \emptyset$. Thus, the triplet (X,τ,A) is a soft T_3-space.

Definition 3.12 (see [24]) Let (X,τ,A) be a soft topological space and Y a nonempty subset of X. Then, the *soft subspace* (Y,τ_Y,A) defined as follows: $(G^Y,A) \in \tau_Y$ if and only if there exists $(G,A) \in \tau$ such that $G^Y(p) = Y \cap G(p)$, for every $p \in A$.

Remark. (see [24]) A soft set (F^Y,A) is soft closed if and only if there exists a soft closed set (F,A) such that $F^Y(p) = Y \cap F(p)$, for every $p \in A$.

Proposition 3.13 Let (X,τ,A) be a soft T_i-space, where $i = 0,1,2,3$. Then, every soft subspace (Y,τ_Y,A) of the soft space (X,τ,A) is a soft T_i-space.

Proof. (i=0) Let y_1 and y_2 be two distinct points of Y and $a \in A$. Since (X,τ,A) is a soft T_0-space, there exists $(G,A) \in \tau$ such that either $y_1 \in_a (G,A)$ and $y_2 \notin_a (G,A)$ or $y_1 \notin_a (G,A)$ and $y_2 \in_a (G,A)$. Without loss of generality we suppose that $y_1 \in_a (G,A)$ and $y_2 \notin_a (G,A)$. We consider the map $G^Y : A \to \mathscr{P}(Y)$ defined as follows: $G^Y(p) = Y \cap G(p)$, for every $p \in A$. Then, $(G^Y,A) \in \tau_Y$. We observe that $y_1 \in_a (G^Y,A)$ and $y_2 \notin_a (G^Y,A)$. Thus, the triplet (Y,τ_Y,A) is a soft T_0-space.

(i=1) Let y_1 and y_2 be two distinct points of Y and $a \in A$. Since (X,τ,A) is a soft T_1-space, there exists $(G,A) \in \tau$ such that $y_1 \in_a (G,A)$ and $y_2 \notin_a (G,A)$. We consider the map $G^Y : A \to \mathscr{P}(Y)$ defined as follows: $G^Y(p) = Y \cap G(p)$, for every $p \in A$. Then, $(G^Y,A) \in \tau_Y$, $y_1 \in_a (G^Y,A)$, and $y_2 \notin_a (G^Y,A)$. This means that the triplet (Y,τ_Y,A) is a soft T_1-space.

(i=2) Let y_1 and y_2 be two distinct points of Y and $a \in A$. Since (X,τ,A) is a soft T_2-space, there exist $(G,A) \in \tau$, $(H,A) \in \tau$ such that $x \in_a (G,A)$, $y \in_a (H,A)$, and $G(a) \cap H(a) = \emptyset$. We consider the maps $G^Y : A \to \mathscr{P}(Y)$ and $H^Y : A \to \mathscr{P}(Y)$ defined as follows: $G^Y(p) = Y \cap G(p)$ and $H^Y(p) = Y \cap H(p)$, for every $p \in A$. Then, $(G^Y,A) \in \tau_Y$, $(H^Y,A) \in \tau_Y$, $y_1 \in_a (G^Y,A)$, $y_2 \in_a (H^Y,A)$, and $G^Y(a) \cap H^Y(a) = \emptyset$. Thus, the triplet (Y,τ_Y,A) is a soft T_2-space.

(i=3) Let $y \in Y$, $a \in A$, and $(Q^Y,A) \in \tau_Y^c$ such that $y \notin_a (Q^Y,A)$. Then, there exists $(Q,A) \in \tau^c$ such that $Q^Y(p) = Y \cap Q(p)$, for every $p \in A$. Obviously, $y \notin_a (Q,A)$. Since (X,τ,A) is a soft T_3-space, there exist $(G,A) \in \tau$, $(H,A) \in \tau$ such that $y \in_a (G,A)$, $Q(a) \subseteq H(a)$, and $G(a) \cap H(a) = \emptyset$. We consider the maps $G^Y : A \to \mathscr{P}(Y)$ and $H^Y : A \to \mathscr{P}(Y)$ defined as follows: $G^Y(p) = Y \cap G(p)$ and $H^Y(p) = Y \cap H(p)$, for every $p \in A$. Then, $(G^Y,A) \in \tau_Y$, $(H^Y,A) \in \tau_Y$, $y \in_a (G^Y,A)$, $Q^Y(a) \subseteq H^Y(a)$, and $G^Y(a) \cap H^Y(a) = \emptyset$. Thus, the triplet (Y,τ_Y,A) is a soft T_3-space.

Definition 3.14 Let (X,τ,A) be a soft topological space. A point x of X is called *soft-limit* of a net $S = \{x_\lambda, \lambda \in \Lambda\}$ of X if for every $a \in A$ and for every a-soft open neighborhood (G,A) of x there exists $\lambda_0 \in \Lambda$ such that $x_\lambda \in_a (G,A)$ for every $\lambda \geq \lambda_0$. In this case we say that the net S *soft converges* to x. The set of all soft limits of the net S is denoted by s-$\lim(S)$. When the net S has exactly one soft limit x, then we write s-$\lim(S) = x$.

Proposition 3.15 Let (X,τ,A) be a soft T_2-space, $S = \{x_\lambda, \lambda \in \Lambda\}$ a net of X, and $x,y \in$ s-$\lim(S)$. Then, $x = y$.

Proof. We suppose that $x \neq y$ and let $a \in A$. By assumption there exist two soft open sets (G,A) and (H,A) such that $x \in_a (G,A)$, $y \in_a (H,A)$, and $G(a) \cap H(a) = \emptyset$. Since $x \in$

s-lim(S), there exists $\lambda_1 \in \Lambda$ such that $x_\lambda \in_a (G,A)$ for every $\lambda \geqslant \lambda_1$. Also, since $y \in$ s-lim(S), there exists $\lambda_2 \in \Lambda$ such that $x_\lambda \in_a (H,A)$ for every $\lambda \geqslant \lambda_2$. Let $\lambda_0 \in \Lambda$ such that $\lambda_0 \geqslant \lambda_1$ and $\lambda_0 \geqslant \lambda_2$. Then, for every $\lambda \geqslant \lambda_0$ we have $x_\lambda \in_a (G,A)$ and $x_\lambda \in_a (H,A)$, which is a contradiction.

Proposition 3.16 Let (X,τ,A) be a soft topological space, $(F,A) \in$ SS(X,A), and $a \in A$. If there exists a net $S = \{x_\lambda, \lambda \in \Lambda\}$ of X such that $x_\lambda \in_a (F,A)$, for every $\lambda \in \Lambda$, and $x \in$ s-lim(S), then $x \in_a$ Cl(F,A).

Proof. Let $S = \{x_\lambda, \lambda \in \Lambda\}$ be a net of X such that $x_\lambda \in_a (F,A)$, for every $\lambda \in \Lambda$. By Proposition 2.14, it suffices to prove that $x \in$ cl(F,a). Let (G,A) be an arbitrary a-soft open neighborhood of x. Since $x \in$ s-lim(S), there exists $\lambda_0 \in \Lambda$ such that $x_\lambda \in_a (G,A)$ for every $\lambda \geqslant \lambda_0$. Thus, $x_{\lambda_0} \in F(a) \cap G(a)$ and, therefore, $(F,A) \sqcap (G,A) \neq \mathbf{0}_A$. This means that $x \in$ cl(F,a).

Proposition 3.17 Let (X,τ_X,A) and (Y,τ_Y,A) be two soft spaces, $x \in X$, and e a map of A onto B. If the map $f : X \to Y$ is soft e-continuous at the point x, then for every net $S = \{x_\lambda, \lambda \in \Lambda\}$ of X which soft converges to x in (X,τ_X,A) we have that the net $\{f(x_\lambda), \lambda \in \Lambda\}$ of Y soft converges to $f(x)$ in (Y,τ_Y,B).

Proof. Let $f : X \to Y$ be a soft e-continuous map at x and $S = \{x_\lambda, \lambda \in \Lambda\}$ a net of X which soft converges to x in (X,τ_X,A). We prove that the net $\{f(x_\lambda), \lambda \in \Lambda\}$ of Y soft converges to $f(x)$ in (Y,τ_Y,B). Let $b \in B$ and (G,B) be a b-soft open neighborhood of $f(x)$ in (Y,τ_Y,B). Let $a \in e^{-1}(\{b\})$. Since f is soft e-continuous at the point x, there exists an a-soft open neighborhood (F,A) of x in (X,τ_X,A) such that $\Phi_{fe}(F,A) \sqsubseteq (G,B)$. Also, since S soft converges to x, there exists $\lambda_0 \in \Lambda$ such that $x_\lambda \in_a (F,A)$ for every $\lambda \geqslant \lambda_0$.

Let $\lambda \geqslant \lambda_0$. Then, $x_\lambda \in F(a)$ and, therefore,

$$f(x_\lambda) \in f(F(a)) \subseteq \bigcup\{f(F(p)) : p \in e^{-1}(\{b\})\}$$
$$\subseteq G(b).$$

Thus, $f(x_\lambda) \in_b (G,B)$. This means that the net

$$\{f(x_\lambda), \lambda \in \Lambda\}$$

of Y soft converges to $f(x)$ in (Y,τ_Y,B).

4 Cartesian product of two soft topological spaces

Definition 4.1 (see [23]) Let $(F,A) \in$ SS(X,A) and $(G,B) \in$ SS(Y,B). The *Cartesian product* of (F,A) and (G,B) is the soft set

$$(H,A \times B) \in SS(X \times Y, A \times B),$$

where the map $H : A \times B \to \mathscr{P}(X \times Y)$ defined as follows:

$$H(a,b) = F(a) \times G(b), \text{ for every } (a,b) \in A \times B.$$

Symbolically, we write $(H, A \times B) = (F,A) \otimes (G,B)$ and $H = F \times G$.

Definition 4.2 (see [23]) Let (X,τ_1,A_1) and (Y,τ_2,A_2) be two soft topological spaces. The soft topological space $(X \times Y, \tau, A_1 \times A_2)$, where τ is the collection of all soft unions of elements of

$$\{(F,A_1) \otimes (G,A_2) : (F,A_1) \in \tau_1, (G,A_2) \in \tau_2\},$$

is called soft product topological space over $X \times Y$. Symbolically, we write $\tau = \tau_1 \times \tau_2$.

Proposition 4.3 Let (X,τ_1,A_1) and (Y,τ_2,A_2) be two soft topological spaces, $e_1 : A_1 \times A_2 \to A_1$, where

$$e_1(a_1,a_2) = a_1,$$

for every $(a_1,a_2) \in A_1 \times A_2$, and $e_2 : A_1 \times A_2 \to A_2$, where

$$e_2(a_1,a_2) = a_2,$$

for every $(a_1,a_2) \in A_1 \times A_2$. The maps $pr_1 : X \times Y \to X$ and $pr_2 : X \times Y \to Y$, where $pr_1(x,y) = x$ and $pr_2(x,y) = y$, for every $(x,y) \in X \times Y$, are soft e_1-continuous and soft e_2-continuous, respectively.

Proof. We prove that the map pr_1 is soft e_1-continuous. By Proposition 2.18 it suffices to prove that $\Phi_{pr_1e_1}^{-1}(G,A_1) \in \tau_1 \times \tau_2$, for every $(G,A_1) \in \tau_1$. Let $(G,A_1) \in \tau_1$. Then, $\Phi_{pr_1e_1}^{-1}(G,A_1)$ is the soft set

$$(F, A_1 \times A_2) \in SS(X \times Y, A_1 \times A_2)$$

such that

$$F(a_1,a_2) = pr_1^{-1}(G(e_1(a_1,a_2)))$$
$$= pr_1^{-1}(G(a_1)) = G(a_1) \times Y,$$

for every $(a_1,a_2) \in A_1 \times A_2$. Thus,

$$\Phi_{pr_1e_1}^{-1}(G,A_1) = (F,A_1 \times A_2) = (G,A_1) \times \mathbf{1}_{A_2}.$$

Since $(G,A_1) \in \tau_1$ and $\mathbf{1}_{A_2} \in \tau_2$, we have

$$\Phi_{pr_1e_1}^{-1}(G,A_1) \in \tau_1 \times \tau_2.$$

This means that the map pr_1 is soft e_1-continuous.

Similarly, we can prove that the map pr_2 is soft e_2-continuous.

Proposition 4.4 Let (X,τ_1,A_1), (Y,τ_2,A_2) be two soft topological spaces, $(F,A_1) \in$ SS(X,A_1), and $(G,A_2) \in$ SS(Y,A_2). Then,

$$Cl((F,A_1) \otimes (G,A_2)) = Cl(F,A_1) \otimes Cl(G,A_2).$$

Proof. By Proposition 2.14 we have:
(1) $Cl(F,A_1) = (R_{F,A_1}, A_1)$, where the map $R_{F,A_1} : A_1 \to \mathscr{P}(X)$ defined as follows: $R_{F,A_1}(a_1) = $ cl(F,a_1), for every $a_1 \in A_1$.

(2) $\mathrm{Cl}(G,A_2) = (R_{G,A_2}, A_2)$, where the map $R_{G,A_2} : A_2 \to \mathscr{P}(Y)$ defined as follows: $R_{G,A_2}(a_2) = \mathrm{cl}(G,a_2)$, for every $a_2 \in A_2$.

(3) $\mathrm{Cl}((F,A_1) \otimes (G,A_2)) = (R, A_1 \times A_2)$, where the map

$$R : A_1 \times A_2 \to \mathscr{P}(X \times Y)$$

defined as follows:

$$R(a_1,a_2) = \mathrm{cl}(F \times G, (a_1,a_2)),$$

for every $(a_1,a_2) \in A_1 \times A_2$. We observe that

$$\mathrm{cl}(F \times G, (a_1,a_2)) = \mathrm{cl}(F,a_1) \times \mathrm{cl}(G,a_2).$$

Thus, for every $(a_1,a_2) \in A_1 \times A_2$ we have $R(a_1,a_2) = R_{F,A_1}(a_1) \times R_{G,A_2}(a_2)$. This means that $\mathrm{Cl}((F,A_1) \otimes (G,A_2)) = \mathrm{Cl}(F,A_1) \otimes \mathrm{Cl}(G,A_2)$.

Notation 4.5 Let (X,τ_1,A_1), (Y,τ_2,A_2), and (Z,τ_3,A_3) be three soft topological spaces.

(1) If $x \in X$, $y \in Y$, and $f : X \times Y \to Z$, then by f_x we denote the map of Y into Z defined by $f_x(y) = f(x,y)$ for every $y \in Y$. Also, by f^y we denote the map of X into Z defined by $f^y(x) = f(x,y)$ for every $x \in X$.

(2) If $a_1 \in A_1$, $a_2 \in A_2$, and $e : A_1 \times A_2 \to A_3$, then by e_{a_1} we denote the map of A_2 into A_3 defined by $e_{a_1}(a_2) = e(a_1,a_2)$ for every $a_2 \in A_2$. Also, by e^{a_2} we denote the map of A_1 into A_3 defined by $e^{a_2}(a_1) = e(a_1,a_2)$ for every $a_1 \in A_1$.

Proposition 4.6 Let (X,τ_1,A_1), (Y,τ_2,A_2), and (Z,τ_3,A_3) be three soft topological spaces, $e : A_1 \times A_2 \to A_3$, and $f : X \times Y \to Z$ a soft e-continuous map. Then, the following statements are true:

(1) For every $x \in X$ and $a_1 \in A_1$ the map $f_x : Y \to Z$ is soft e_{a_1}-continuous.

(2) For every $y \in Y$ and $a_2 \in A_2$ the map $f^y : X \to Z$ is soft e^{a_2}-continuous.

Proof. We prove only the statement (1). Let $y \in Y$, $a_2 \in A_2$, and (G,A_3) be an $e_{a_1}(a_2)$-soft open neighborhood of $f_x(y)$ in (Z,τ_3,A_3). It suffices to prove that there exists an a_2-soft open neighborhood (G_2,A_2) of y in (Y,τ_2,A_2) such that

$$\Phi_{f_x e_{a_1}}(G_2,A_2) \sqsubseteq (G,A_3).$$

Since the map $f : X \times Y \to Z$ is soft e-continuous at the point (x,y), by Proposition 2.19 and Definition 4.2, there exist an a_1-soft open neighborhood (G_1,A_1) of x in (X,τ_1,A_1) and an a_2-soft open neighborhood (G_2,A_2) of y in (Y,τ_2,A_2) such that

$$\Phi_{fe}((G_1,A_1) \otimes (G_2,A_2)) \sqsubseteq (G,A_3).$$

Let

$$\Phi_{f_x e_{a_1}}(G_2,A_2) = (H,A_3)$$

and

$$\Phi_{fe}((G_1,A_1) \otimes (G_2,A_2)) = (H',A_3).$$

Then, for every $a_3 \in A_3$ we have

$$H(a_3) = \begin{cases} \bigcup\{f_x(G_2(b)) : b \in e_{a_1}^{-1}(\{a_3\})\}, \\ \text{if } e_{a_1}^{-1}(\{a_3\}) \neq \emptyset, \\ \emptyset, \text{ if } e_{a_1}^{-1}(\{a_3\}) = \emptyset \end{cases}$$

and

$$H'(a_3) = \begin{cases} \bigcup\{f(G_1(a) \times G_2(b)) : (a,b) \in e^{-1}(\{a_3\})\}, \\ \text{if } e^{-1}(\{a_3\}) \neq \emptyset, \\ \emptyset, \text{ if } e^{-1}(\{a_3\}) = \emptyset. \end{cases}$$

Let $a_3 \in A_3$. We prove that

$$H(a_3) \subseteq G(a_3).$$

If $e_{a_1}^{-1}(\{a_3\}) = \emptyset$, then $H(a_3) = \emptyset \subseteq G(a_3)$. We suppose that $e_{a_1}^{-1}(\{a_3\}) \neq \emptyset$ and $b \in e_{a_1}^{-1}(\{a_3\})$. Then, $(a_1,b) \in e^{-1}(\{a_3\})$. It suffices to prove that

$$f_x(G_2(b)) \subseteq G(a_3).$$

Indeed, we have

$$f_x(G_2(b)) = \{f(x,y) : y \in G_2(b)\},$$

$x \in G_1(a_1)$, and $f(G_1(a_1) \times G_2(b)) \subseteq H'(a_3) \subseteq G(a_3)$. Therefore, for every $y \in G_2(b)$ we have

$$f(x,y) \in f(G_1(a_1) \times G_2(b)) \subseteq G(a_3).$$

This means that

$$H(a_3) = \bigcup\{f_x(G_2(b)) : b \in e_{a_1}^{-1}(\{a_3\})\} \subseteq G(a_3),$$

for every $a_3 \in A_3$. Thus,

$$\Phi_{f_x e_{a_1}}(G_2,A_2) = (H,A_3) \sqsubseteq (G,A_3).$$

5 Soft θ-topology: θ-open, θ-closed soft sets and soft θ-continuity

Definition 5.1 Let (X,τ,A) be a soft topological space and $a \in A$. A point $x \in X$ is said to be an *a-θ-interior point* of $(F,A) \in \mathrm{SS}(X,A)$ if there exists an a-soft open neighborhood (G,A) of x such that $\mathrm{Cl}(G,A) \sqsubseteq (F,A)$. The set of all *$a$-$\theta$-interior points* of (F,A) is denoted by $\mathrm{int}_\theta(F,a)$.

We observe that $\mathrm{int}_\theta(F,p) \subseteq F(p)$, for every $p \in A$.

Definition 5.2 Let (X,τ,A) be a soft topological space and $(F,A) \in \mathrm{SS}(X,A)$. The *$\theta$-interior* $\mathrm{Int}_\theta(F,A)$ of (F,A) is the soft set $(R_{F,A}, A)$, where the map $R_{F,A} : A \to \mathscr{P}(X)$ defined as follows: $R_{F,A}(p) = \mathrm{int}_\theta(F,p)$, for every $p \in A$.

We observe that $\mathrm{Int}_\theta(F,A) \sqsubseteq (F,A)$.

Definition 5.3 (1) Let $(F_i, A) \in SS(X, A)$, $i \in I$. Then, by $\cap \{F_i : i \in I\}$ and $\cup \{F_i : i \in I\}$ we denote the maps from A to $\mathscr{P}(X)$ for which

$$\cap \{F_i : i \in I\}(p) = \cap \{F_i(p) : i \in I\}$$

and

$$\cup \{F_i : i \in I\}(p) = \cup \{F_i(p) : i \in I\},$$

for every $p \in A$.

(2) Let $(F, A) \in SS(X, A)$. Then by F^c we denote the map from A to $\mathscr{P}(X)$ for which $F^c(p) = X \setminus F(p)$, for every $p \in A$. Obviously, $(F, A)^c = (F^c, A)$.

Remark. We observe that for the soft sets

$$(\cap \{F_i : i \in I\}, A) \quad \text{and} \quad (\cup \{F_i : i \in I\}, A)$$

of Definition 5.3(1) we have the following properties:

(i) $(\cap \{F_i : i \in I\}, A) = \sqcap \{(F_i, A) : i \in I\}$.

(ii) $(\cup \{F_i : i \in I\}, A) = \sqcup \{(F_i, A) : i \in I\}$.

(iii) If $(G, A) \in SS(X, A)$, then

$$(\cap \{F_i : i \in I\}, A) \sqcup (G, A) = (\cap \{F_i \cup G : i \in I\}, A).$$

(iv) If $(G, A) \in SS(X, A)$, then

$$(\cup \{F_i : i \in I\}, A) \sqcap (G, A) = (\cup \{F_i \cap G : i \in I\}, A).$$

Proposition 5.4 The following statements are true:

(1) If $(F_1, A), (F_2, A) \in SS(X, A)$ and $p \in A$, then

$$\text{int}_\theta(F_1, p) \cap \text{int}_\theta(F_2, p) = \text{int}_\theta(F_1 \cap F_2, p).$$

(2) If $(F_i, A) \in SS(X, A)$, $i \in I$ and $p \in A$, then

$$\cup \{\text{int}_\theta(F_i, p) : i \in I\} \subseteq \text{int}_\theta(\cup \{F_i : i \in I\}, p).$$

Proof. (1) We prove that

$$\text{int}_\theta(F_1, p) \cap \text{int}_\theta(F_2, p) \subseteq \text{int}_\theta(F_1 \cap F_2, p).$$

Let $x \in \text{int}_\theta(F_1, p) \cap \text{int}_\theta(F_2, p)$. Then, there exist two p-soft open neighborhoods (G_1, A) and (G_2, A) of x such that $\text{Cl}(G_1, A) \sqsubseteq (F_1, A)$ and $\text{Cl}(G_2, A) \sqsubseteq (F_2, A)$. The soft set $(G_1, A) \sqcap (G_2, A)$ is a p-soft open neighborhood of x and

$$\text{Cl}((G_1, A) \sqcap (G_2, A)) \sqsubseteq \text{Cl}(G_1, A) \sqcap \text{Cl}(G_2, A)$$
$$\sqsubseteq (F_1, A) \sqcap (F_2, A)$$
$$= (F_1 \cap F_2, A).$$

Thus, $x \in \text{int}_\theta(F_1 \cap F_2, p)$.

Conversely, we prove that

$$\text{int}_\theta(F_1 \cap F_2, p) \subseteq \text{int}_\theta(F_1, p) \cap \text{int}_\theta(F_2, p).$$

Let $x \in \text{int}_\theta(F_1 \cap F_2, p)$. Then, there exists a p-soft open neighborhood (G, A) of x such that

$$\text{Cl}(G, A) \sqsubseteq (F_1 \cap F_2, A).$$

Therefore, $\text{Cl}(G, A) \sqsubseteq (F_1, A)$ and $\text{Cl}(G, A) \sqsubseteq (F_2, A)$. Thus, $x \in \text{int}_\theta(F_1, p) \cap \text{int}_\theta(F_2, p)$.

(2) Let $x \in \cup \{\text{int}_\theta(F_i, p) : i \in I\}$. Then, there exists $i \in I$ such that $x \in \text{int}_\theta(F_i, p)$. Therefore, there exists a p-soft open neighborhood (G_i, A) of x such that

$$\text{Cl}(G_i, A) \sqsubseteq (F_i, A) \sqsubseteq (\cup \{F_i : i \in I\}, A).$$

Thus, $x \in \text{int}_\theta(\cup \{F_i : i \in I\}, p)$.

Corollary 5.5 The following statements are true:

(1) $\text{Int}_\theta(F_1, A) \sqcap \text{Int}_\theta(F_2, A) = \text{Int}_\theta(F_1 \cap F_2, A)$.

(2) $\sqcup \{\text{Int}_\theta(F_i, A) : i \in I\} \sqsubseteq \text{Int}_\theta(\cup \{F_i : i \in I\}, A)$.

Definition 5.6 A soft set (F, A) is said to be θ-open if $\text{Int}_\theta(F, A) = (F, A)$.

We observe that a soft set (F, A) is θ-open if and only if $\text{int}_\theta(F, p) = F(p)$, for every $p \in A$.

Proposition 5.7 The family τ_θ of all θ-open soft sets defines a soft topology on X.

Proof. (1) We have $\mathbf{0}_A, \mathbf{1}_A \in \tau_\theta$. Indeed, we observe that $\text{Int}_\theta(\mathbf{0}_A) = \mathbf{0}_A$ and $\text{Int}_\theta(\mathbf{1}_A) = \mathbf{1}_A$.

(2) Let $(G, A), (H, A) \in \tau_\theta$. We prove that $(G, A) \sqcap (H, A) \in \tau_\theta$. It suffices to prove that $(G \cap H, A) \in \tau_\theta$. Indeed, $\text{Int}_\theta(G, A) = (G, A)$ and $\text{Int}_\theta(H, A) = (H, A)$. By Corollary 5.5(1) we have

$$(G \cap H, A) = (G, A) \sqcap (H, A)$$
$$= \text{Int}_\theta(G, A) \sqcap \text{Int}_\theta(H, A)$$
$$= \text{Int}_\theta(G \cap H, A).$$

(3) Let $(G_i, A) \in \tau_\theta$ for every $i \in I$. We prove that $\sqcup \{(G_i, A) : i \in I\} \in \tau_\theta$. It suffices to prove that $(\cup \{G_i : i \in I\}, A) \in \tau_\theta$, that is

$$\text{Int}_\theta(\cup \{G_i : i \in I\}, A) = (\cup \{G_i : i \in I\}, A).$$

Indeed, for every $i \in I$ we have $\text{Int}_\theta(G_i, A) = (G_i, A)$. By Corollary 5.5(2) we have

$$(\cup \{G_i : i \in I\}, A) = \sqcup \{(G_i, A) : i \in I\}$$
$$= \sqcup \{\text{Int}_\theta(G_i, A) : i \in I\}$$
$$\sqsubseteq \text{Int}_\theta(\cup \{G_i : i \in I\}, A).$$

Also,

$$\text{Int}_\theta(\cup \{G_i : i \in I\}, A) \sqsubseteq (\cup \{G_i : i \in I\}, A).$$

Therefore, $\text{Int}_\theta(\cup \{G_i : i \in I\}, A) = (\cup \{G_i : i \in I\}, A)$.

Proposition 5.8 The following statements are true:

(1) $\tau_\theta \subseteq \tau$.

(2) $\tau \supseteq \tau_\theta \supseteq (\tau_\theta)_\theta \supseteq ((\tau_\theta)_\theta)_\theta \supseteq \cdots$.

(3) $\text{Int}_\theta(F, A) \in \tau$, for every $(F, A) \in SS(X, A)$.

(4) $\text{Int}_\theta(F, A) = \sqcup \{(G, A) \in \tau : \text{Cl}(G, A) \sqsubseteq (F, A)\}$, for every $(F, A) \in SS(X, A)$.

Proof. (1) Let $(G, A) \in \tau_\theta$. Then, $\text{Int}_\theta(G, A) = (G, A)$. We prove that $(G, A) \in \tau$. By Proposition 2.11 it suffices to prove that for every $a \in A$ and $x \in G(a)$ there exists an

a-soft open neighborhood $(G_{(a,x)}, A)$ of x in (X, τ, A) such that $(G_{(a,x)}, A) \sqsubseteq (G, A)$. Let $a \in A$ and $x \in G(a)$. Then, $x \in \mathrm{int}_\theta(G, a)$. Therefore, there exists an a-soft open neighborhood $(G_{(a,x)}, A)$ of x such that $\mathrm{Cl}(G_{(a,x)}, A) \sqsubseteq (G, A)$. Thus, $(G_{(a,x)}, A) \sqsubseteq (G, A)$.

(2) Follows by (1) and Proposition 5.7.

(3) Let $(F, A) \in \mathrm{SS}(X, A)$. We prove that $\mathrm{Int}_\theta(F, A) \in \tau$. By Proposition 2.11 it suffices to prove that for every $a \in A$ and $x \in \mathrm{int}_\theta(F, a)$ there exists an a-soft open neighborhood $(G_{(a,x)}, A)$ of x in (X, τ, A) such that

$$(G_{(a,x)}, A) \sqsubseteq \mathrm{Int}_\theta(F, A).$$

Let $a \in A$ and $x \in \mathrm{int}_\theta(F, a)$. Then, there exists an a-soft open neighborhood $(G_{(a,x)}, A)$ of x such that

$$\mathrm{Cl}(G_{(a,x)}, A) \sqsubseteq (F, A).$$

Obviously, $G_{(a,x)}(p) \subseteq \mathrm{int}_\theta(F, p)$, for every $p \in A$. Thus, $(G_{(a,x)}, A) \sqsubseteq \mathrm{Int}_\theta(F, A)$.

(4) Follows by (3) and Definitions 5.1 and 5.2

Remark. In general, $\mathrm{Int}_\theta(F, A) \notin \tau_\theta$.

Example 5.9 (1) Let (X, τ, A) be the soft topological space of Example 3.7(2). Then, $\tau_\theta = \{\mathbf{0}_A, \mathbf{1}_A\} \neq \tau$.

(2) Let (X, τ, A) be the soft topological space of Example 3.9. We consider the soft set (F, A) over X defined as follows:

$$F(p) = \begin{cases} \{x, z\}, & \text{if } a = 0, \\ \{x, y\}, & \text{if } a = 1. \end{cases}$$

Then, $\mathrm{Int}_\theta(F, A) = (G, A) \in \tau$. Since $(G, A) \neq (F, A)$, we have $\mathrm{Int}_\theta(F, A) \notin \tau_\theta$.

Definition 5.10 Let (X, τ, A) be a soft topological space and $a \in A$. A point $x \in X$ is said to be an a-θ-*cluster point* of $(F, A) \in \mathrm{SS}(X, A)$ if for every a-soft open neighborhood (G, A) of x we have $(F, A) \sqcap \mathrm{Cl}(G, A) \neq \mathbf{0}_A$. The set of all a-θ-cluster points of (F, A) is denoted by $\mathrm{cl}_\theta(F, a)$.

We observe that $F(p) \subseteq \mathrm{cl}_\theta(F, p)$, for every $p \in A$.

Definition 5.11 Let (X, τ, A) be a soft topological space and $(F, A) \in \mathrm{SS}(X, A)$. The θ-*closure* of (F, A) is the soft set $\mathrm{Cl}_\theta(F, A) = (R_{F,A}, A)$, where the map $R_{F,A} : A \to \mathscr{P}(X)$ defined as follows: $R_{F,A}(p) = \mathrm{cl}_\theta(F, p)$, for every $p \in A$.

We observe that $(F, A) \sqsubseteq \mathrm{Cl}_\theta(F, A)$ and $\mathrm{Cl}(F, A) \sqsubseteq \mathrm{Cl}_\theta(F, A)$.

Definition 5.12 A soft set (F, A) is said to be θ-closed if $\mathrm{Cl}_\theta(F, A) = (F, A)$. The family of all θ-closed soft sets is denoted by τ_θ^c.

We observe that a soft set (F, A) is θ-closed if and only if $\mathrm{cl}_\theta(F, p) = F(p)$, for every $p \in A$.

The proof of the following proposition is straightforward verification of the Proposition 5.7 and Propositions 2.10 and 2.12 of [9].

Proposition 5.13 Let (X, τ, A) be a soft topological space. The family τ_θ^c has the following properties:
(1) $\mathbf{0}_A, \mathbf{1}_A \in \tau_\theta^c$.
(2) If $(Q, A), (R, A) \in \tau_\theta^c$, then $(Q, A) \sqcup (R, A) \in \tau_\theta^c$.
(3) If $(Q_i, A) \in \tau_\theta^c$ for every $i \in I$, then

$$\sqcap \{(Q_i, A) : i \in I\} \in \tau_\theta^c.$$

Proposition 5.14 The following statements are true:
(1) If $(F_1, A), (F_2, A) \in \mathrm{SS}(X, A)$ and $p \in A$, then

$$\mathrm{cl}_\theta(F_1, p) \cup \mathrm{cl}_\theta(F_2, p) = \mathrm{cl}_\theta(F_1 \cup F_2, p).$$

(2) If $(F_i, A) \in \mathrm{SS}(X, A)$, $i \in I$ and $p \in A$, then

$$\mathrm{cl}_\theta(\cap \{F_i : i \in I\}, p) \subseteq \cap \{\mathrm{cl}_\theta(F_i, p) : i \in I\}.$$

Proof. (1) We prove that

$$\mathrm{cl}_\theta(F_1, p) \cup \mathrm{cl}_\theta(F_2, p) \subseteq \mathrm{cl}_\theta(F_1 \cup F_2, p).$$

Let $x \in \mathrm{cl}_\theta(F_1, p) \cup \mathrm{cl}_\theta(F_2, p)$ and (G, A) be a p-soft open neighborhood of x. We must prove that $(F_1 \cup F_2, A) \sqcap \mathrm{Cl}(G, A) \neq \mathbf{0}_A$. Without loss of generality we suppose that $x \in \mathrm{cl}_\theta(F_1, p)$. Then, $(F_1, A) \sqcap \mathrm{Cl}(G, A) \neq \mathbf{0}_A$. Therefore,

$$(F_1 \cup F_2, A) \sqcap \mathrm{Cl}(G, A) \neq \mathbf{0}_A.$$

Thus, $x \in \mathrm{cl}_\theta(F_1 \cup F_2, p)$.

Conversely, we prove that

$$\mathrm{cl}_\theta(F_1 \cup F_2, p) \subseteq \mathrm{cl}_\theta(F_1, p) \cup \mathrm{cl}_\theta(F_2, p).$$

Let $x \in \mathrm{cl}_\theta(F_1 \cup F_2, p)$ and (G, A) be a p-soft open neighborhood of x. Then,

$$(F_1 \cup F_2, A) \sqcap \mathrm{Cl}(G, A) \neq \mathbf{0}_A.$$

Therefore,

$$(F_1, A) \sqcap \mathrm{Cl}(G, A) \neq \mathbf{0}_A$$

or

$$(F_2, A) \sqcap \mathrm{Cl}(G, A) \neq \mathbf{0}_A.$$

Thus, $x \in \mathrm{cl}_\theta(F_1, p) \cup \mathrm{cl}_\theta(F_2, p)$.

(2) Let $x \in \mathrm{cl}_\theta(\cap \{F_i : i \in I\}, p)$ and (G, A) be a p-soft open neighborhood of x. Then, $(\cap \{F_i : i \in I\}, A) \sqcap \mathrm{Cl}(G, A) \neq \mathbf{0}_A$. Therefore, $(F_i, A) \sqcap \mathrm{Cl}(G, A) \neq \mathbf{0}_A$, for every $i \in I$. Thus, $x \in \cap \{\mathrm{cl}_\theta(F_i, p) : i \in I\}$.

Corollary 5.15 The following statements are true:
(1) $\mathrm{Cl}_\theta(F_1, A) \sqcup \mathrm{Cl}_\theta(F_2, A) = \mathrm{Cl}_\theta(F_1 \cup F_2, A)$.
(2) $\mathrm{Cl}_\theta(\cap \{F_i : i \in I\}, A) \sqsubseteq \sqcap \{\mathrm{Cl}_\theta(F_i, A) : i \in I\}$.

Proposition 5.16 Let (X, τ, A) be a soft space, $p \in A$, and $(F, A) \in \mathrm{SS}(X, A)$. Then, $\mathrm{cl}_\theta(F^c, p) = X \setminus \mathrm{int}_\theta(F, p)$.

Proof. We prove that $\mathrm{cl}_\theta(F^c, p) \subseteq X \setminus \mathrm{int}_\theta(F, p)$. Let $x \in \mathrm{cl}_\theta(F^c, p)$. Then, for every p-soft open neighborhood (G, A) of x we have $(F, A)^c \sqcap \mathrm{Cl}(G, A) \neq \mathbf{0}_A$. We suppose

that $x \in \text{int}_\theta(F,p)$. Then, there exists a p-soft open neighborhood (G,A) of x such that $\text{Cl}(G,A) \sqsubseteq (F,A)$. Therefore, $(F,A)^c \sqcap \text{Cl}(G,A) = \mathbf{0}_A$, which is a contradiction. Thus, $x \in X \setminus \text{int}_\theta(F,p)$.

Conversely, we prove that $X \setminus \text{int}_\theta(F,p) \subseteq \text{cl}_\theta(F^c,p)$. Let $x \in X \setminus \text{int}_\theta(F,p)$ and (G,A) be a p-soft open neighborhood of x. We must prove that

$$(F,A)^c \sqcap \text{Cl}(G,A) \neq \mathbf{0}_A.$$

Since $x \notin \text{int}_\theta(F,p)$, the relation $\text{Cl}(G,A) \sqsubseteq (F,A)$ is not true. Therefore, there exists $p_1 \in A$ such that

$$\text{cl}(G,p_1) \not\subseteq F(p_1).$$

This means that $(X \setminus F(p_1)) \cap \text{cl}(G,p_1) \neq \emptyset$ and, therefore,

$$(F,A)^c \sqcap \text{Cl}(G,A) \neq \mathbf{0}_A.$$

Thus, $x \in \text{cl}_\theta(F^c,p)$.

Corollary 5.17 Let (X,τ,A) be a soft topological space and $(F,A) \in \text{SS}(X,A)$. Then, the following statements are true:
(1) $\text{Cl}_\theta((F,A)^c) = \text{Cl}_\theta(F^c,A) = (\text{Int}_\theta(F,A))^c$.
(2) The soft set (F,A) is θ-open if and only if the soft set $(F,A)^c$ is θ-closed.

By Proposition 5.8 and Corollary 5.17 we have the following proposition.

Proposition 5.18 The following statements are true:
(1) $\tau_\theta^c \subseteq \tau^c$.
(2) $\tau^c \supseteq \tau_\theta^c \supseteq (\tau_\theta^c)_\theta^c \supseteq ((\tau_\theta^c)_\theta^c)_\theta^c \supseteq \dots$
(3) $\text{Cl}_\theta(F,A) \in \tau^c$, for every $(F,A) \in \text{SS}(X,A)$.

Remark. In general, $\text{Cl}_\theta(F,A) \notin \tau_\theta^c$.

Example 5.19 Let (X,τ,A) be the soft topological space of Example 3.9. We consider the soft set (F,A) over X defined as follows:

$$F(a) = \begin{cases} \{x\}, & \text{if } a = 0, \\ \{y\}, & \text{if } a = 1. \end{cases}$$

Then, $\text{Cl}_\theta(F,A) = (G,A) \in \tau^c$. Since $(G,A) \neq (F,A)$, we have $\text{Cl}_\theta(F,A) \notin \tau_\theta^c$.

Definition 5.20 Let (X,τ,A) be a soft topological space and $(F,A) \in \text{SS}(X,A)$. The θ-boundary of (F,A) is the soft set $\text{Bd}_\theta(F,A) = (R_{F,A},A)$, where the map $R_{F,A} : A \to \mathscr{P}(X)$ defined as follows: $R_{F,A}(p) = \text{cl}_\theta(F,p) \cap \text{cl}_\theta(F^c,p)$, for every $p \in A$.

Proposition 5.21 Let (X,τ,A) be a soft topological space and $(F,A) \in \text{SS}(X,A)$. Then, the following statements are true:
(1) $\text{Bd}_\theta(F,A) = \text{Cl}_\theta(F,A) \sqcap \text{Cl}_\theta(F^c,A)$.
(2) $\text{Bd}_\theta(F,A) \in \tau^c$.
(3) $\text{Bd}_\theta(F,A) = \text{Cl}_\theta(F,A) \sqcap (\text{Int}_\theta(F,A))^c$.
(4) $\text{Cl}_\theta(F,A) = \text{Int}_\theta(F,A) \sqcup \text{Bd}_\theta(F,A)$.
(5) $\text{Bd}_\theta(F,A) = \text{Bd}_\theta(F^c,A)$.

Proof. (1) Is straightforward verification of the Definition 5.20.
(2) Follows by relation (1) and Propositions 5.13 and 5.18(3).
(3) Follows by relation (1) and Corollary 5.17(1).
(4) We have
$\text{Int}_\theta(F,A) \sqcup \text{Bd}_\theta(F,A) =$
$\text{Int}_\theta(F,A) \sqcup (\text{Cl}_\theta(F,A) \sqcap \text{Cl}_\theta(F^c,A)) =$
$(\text{Int}_\theta(F,A) \sqcup \text{Cl}_\theta(F,A)) \sqcap (\text{Int}_\theta(F,A) \sqcup \text{Cl}_\theta(F^c,A)) =$
$\text{Cl}_\theta(F,A) \sqcap ((\text{Cl}_\theta(F^c,A))^c \sqcup \text{Cl}_\theta(F^c,A)) =$
$\text{Cl}_\theta(F,A) \sqcap \mathbf{1}_A = \text{Cl}_\theta(F,A)$.
(5) Is straightforward verification of the Definition 5.20.

Definition 5.22 Let (X,τ,A) be a soft topological space, $a \in A$, and $x \in X$. We say that a soft set (F,A) is an a-θ-soft open neighborhood of x in (X,τ,A) if $x \in F(a)$ and $(F,A) \in \tau_\theta$.

Proposition 5.23 Let (X,τ,A) be a soft topological space. Then, $(G,A) \in \tau_\theta$ if and only if for every $p \in A$ and $x \in G(p)$ there exists a p-θ-soft open neighborhood $(G_{(p,x)},A)$ of x in (X,τ,A) such that $(G_{(p,x)},A) \sqsubseteq (G,A)$.

Proof. Let $(G,A) \in \tau_\theta$. Then, for every $p \in A$ and $x \in G(p)$ we consider the soft set $(G_{(p,x)},A)$, where $G_{(p,x)} = G$. Obviously, $(G_{(p,x)},A)$ is a p-θ-soft open neighborhood of x.

Conversely, we suppose that for every $p \in A$ and $x \in G(p)$ there exists a p-θ-soft open neighborhood $(G_{(p,x)},A)$ of x in (X,τ,A) such that $(G_{(p,x)},A) \sqsubseteq (G,A)$. We prove that $(G,A) \in \tau_\theta$. We consider the set

$$I = \{(p,x) : p \in A, \ x \in G(p)\}.$$

Then,

$$(G,A) = \sqcup\{(G_{(p,x)},A) : (p,x) \in I\}.$$

Also, for every $p \in A$ and $x \in G(p)$ we have

$$\text{Int}_\theta(G_{(p,x)},A) = (G_{(p,x)},A).$$

Thus, by Corollary 5.5(2) we have
$(G,A) = \sqcup\{(G_{(p,x)},A) : (p,x) \in I\}$
$= \sqcup\{\text{Int}_\theta(G_{(p,x)},A) : (p,x) \in I\}$
$\sqsubseteq \text{Int}_\theta(\cup\{G_{(p,x)} : (p,x) \in I\},A) = \text{Int}_\theta(G,A)$.

Also, $\text{Int}_\theta(G,A) \sqsubseteq (G,A)$. This means that $\text{Int}_\theta(G,A) = (G,A)$ and, therefore, $(G,A) \in \tau_\theta$.

Proposition 5.24 Let (X,τ,A) be a soft topological space. Then, $(G,A) \in \tau_\theta$ if and only if for every $p \in A$ and $x \in G(p)$ there exists a p-soft open neighborhood $(G_{(p,x)},A)$ of x such that $\text{Cl}(G_{(p,x)},A) \sqsubseteq (G,A)$.

Proof. Let $(G,A) \in \tau_\theta$, $p \in A$, and $x \in G(p)$. Then, $(G,A) = \text{Int}_\theta(G,A)$ and, therefore,

$$x \in G(p) = \text{int}_\theta(G,p).$$

Thus, there exists a p-soft open neighborhood $(G_{(p,x)},A)$ of x such that $\text{Cl}(G_{(p,x)},A) \sqsubseteq (G,A)$.

Conversely, we suppose that for every $p \in A$ and $x \in G(p)$ there exists a p-soft open neighborhood $(G_{(p,x)}, A)$ of x such that $\mathrm{Cl}(G_{(p,x)}, A) \sqsubseteq (G, A)$. We prove that $(G, A) \in \tau_\theta$. It suffices to prove that $\mathrm{Int}_\theta(G, A) = (G, A)$ or equivalently, $\mathrm{int}_\theta(G, p) = G(p)$, for every $p \in A$. Let $p \in A$ and $x \in G(p)$. Then, $\mathrm{Cl}(G_{(p,x)}, A) \sqsubseteq (G, A)$ and, therefore, $x \in \mathrm{int}_\theta(G, p)$. Thus, $G(p) \subseteq \mathrm{int}_\theta(G, p)$. Also, $\mathrm{int}_\theta(G, p) \subseteq G(p)$. This means that $\mathrm{int}_\theta(G, p) = G(p)$.

Definition 5.25 Let (X, τ_X, A) and (Y, τ_Y, B) be two soft topological spaces, $x \in X$, and $e : A \to B$. A map $f : X \to Y$ is called *soft e-θ-continuous at the point x* if for every $a \in A$ and every $e(a)$-soft open neighborhood (G, B) of $f(x)$ in (Y, τ_Y, B) there exists an a-soft open neighborhood (F, A) of x in (X, τ_X, A) such that

$$\Phi_{fe}(\mathrm{Cl}(F, A)) \sqsubseteq \mathrm{Cl}(G, B).$$

If the map f is soft e-θ-continuous at any point $x \in X$, then we say that the map f is *soft e-θ-continuous*.

Proposition 5.26 Let (X, τ_X, A) and (Y, τ_Y, B) be two soft topological spaces and $e : A \to B$. If a map $f : X \to Y$ is soft e-continuous, then f is soft e-θ-continuous.

Proof. Let $x \in X$, $a \in A$, and (G, B) be an $e(a)$-soft open neighborhood of $f(x)$ in (Y, τ_Y, B). Since the map $f : X \to Y$ is soft e-continuous at x, there exists an a-soft open neighborhood (F, A) of x in (X, τ_X, A) such that

$$\Phi_{fe}(F, A) \sqsubseteq (G, B).$$

By Proposition 2.18 we have

$$\Phi_{fe}(\mathrm{Cl}(F, A)) \sqsubseteq \mathrm{Cl}(\Phi_{fe}(F, A)).$$

Thus,

$$\Phi_{fe}(\mathrm{Cl}(F, A)) \sqsubseteq \mathrm{Cl}(G, B)$$

and, therefore, the map f is soft e-θ-continuous at the point x.

Example 5.27 Let $X = \{1, 2, \ldots\}$, $A = \{0, 1, 2\}$,

$$\tau_1 = \{(G_n, A) : n \in \{1, 2, \ldots\}\} \cup \{\mathbf{0}_A, \mathbf{1}_A\},$$

where $G_n(p) = \{n, n+1, \ldots\}$, for every $p \in A$, and

$$\tau_2 = \{\mathbf{0}_A, \mathbf{1}_A\}.$$

Then, the triplets (X, τ_1, A) and (X, τ_2, A) are soft topological spaces. We consider the map $f : (X, \tau_1, A) \to (X, \tau_2, A)$ such that $f(x) = 3$, for every $x \in X$ and the map $e : A \to A$ such that $e(a) = a$, for every $a \in A$. Since $(\tau_1)_\theta = \{\mathbf{0}_A, \mathbf{1}_A\}$, the map f is soft e-θ-continuous. But the map f is not soft e-continuous.

Proposition 5.28 Let (X, τ_X, A) and (Y, τ_Y, B) be two soft spaces, $x \in X$, $e : A \to B$, and $f : X \to Y$. If for every $a \in A$ and every $e(a)$-soft open neighborhood (G, B) of $f(x)$ in (Y, τ_Y, B) there exists an a-θ-soft open neighborhood

(F, A) of x in (X, τ_X, A) such that $\Phi_{fe}(F, A) \sqsubseteq \mathrm{Cl}(G, B)$, then f is soft e-θ-continuous at x.

Proof. Let $a \in A$ and (G, B) be an $e(a)$-soft open neighborhood of $f(x)$. Then, there exists an a-θ-soft open neighborhood (F, A) of x in (X, τ_X, A) such that $\Phi_{fe}(F, A) \sqsubseteq \mathrm{Cl}(G, B)$. By Proposition 5.24 there exists an a-soft open neighborhood $(F_{(a,x)}, A)$ of x such that $\mathrm{Cl}(F_{(a,x)}, A) \sqsubseteq (F, A)$. Thus, by Proposition 2.8,

$$\Phi_{fe}(\mathrm{Cl}(F_{(a,x)}, A)) \sqsubseteq \Phi_{fe}(F, A) \sqsubseteq \mathrm{Cl}(G, B)$$

and, therefore, the map f is soft e-θ-continuous at the point x.

Proposition 5.29 Let (X, τ_X, A) and (Y, τ_Y, B) be two soft topological spaces, $e : A \to B$, and $f : X \to Y$. If $\Phi_{fe}^{-1}(\mathrm{Cl}(G, B)) \in (\tau_X)_\theta$, for every $(G, B) \in \tau_Y$, then f is soft e-θ-continuous.

Proof. Let $x \in X$. By Proposition 5.28 it suffices to prove that for every $a \in A$ and every $e(a)$-soft open neighborhood (G, B) of $f(x)$ in (Y, τ_Y, B) there exists an a-θ-soft open neighborhood (F, A) of x in (X, τ_X, A) such that $\Phi_{fe}(F, A) \sqsubseteq \mathrm{Cl}(G, B)$. Let $a \in A$ and (G, B) be an $e(a)$-soft open neighborhood of $f(x)$ in (Y, τ_Y, B). Then, $f(x) \in G(e(a))$ or

$$x \in f^{-1}(G(e(a))) \subseteq f^{-1}(\mathrm{cl}(G, e(a))).$$

We set $(F, A) = \Phi_{fe}^{-1}(\mathrm{Cl}(G, B))$. This means that $F(p) = f^{-1}(\mathrm{cl}(G, e(p)))$, for every $p \in A$. Thus,

$$x \in f^{-1}(\mathrm{cl}(G, e(a))) = F(a).$$

Since $(F, A) \in (\tau_X)_\theta$, the soft set (F, A) is an a-θ-soft open neighborhood of x in (X, τ_X, A). By Proposition 2.8,

$$\Phi_{fe}(F, A) = \Phi_{fe}(\Phi_{fe}^{-1}(\mathrm{Cl}(G, B)) \sqsubseteq \mathrm{Cl}(G, B).$$

Therefore, the map $f : X \to Y$ is soft e-θ-continuous at the point x.

Corollary 5.30 Let (X, τ_X, A) and (Y, τ_Y, B) be two soft topological spaces, $e : A \to B$, and $f : X \to Y$. If $\Phi_{fe}^{-1}(Q, B) \in (\tau_X)_\theta^c$, for every $(Q, B) \in \tau_Y^c$, then f is soft e-θ-continuous.

Proof. By Proposition 5.29 it suffices to prove that

$$\Phi_{fe}^{-1}(\mathrm{Cl}(G, B)) \in (\tau_X)_\theta,$$

for every $(G, B) \in \tau_Y$. Let $(G, B) \in \tau_Y$. Then, $(G, B)^c \in \tau_Y^c$. By Proposition 2.8 we have

$$\Phi_{fe}^{-1}((G, B)^c) = (\Phi_{fe}^{-1}(G, B))^c.$$

Since $\Phi_{fe}^{-1}((G, B)^c) \in (\tau_X)_\theta^c$, we have $\Phi_{fe}^{-1}(G, B) \in (\tau_X)_\theta$.

Proposition 5.31 If the map $f : X \to Y$ is soft e-θ-continuous, then

$$\Phi_{fe}(\mathrm{Cl}_\theta(F, A)) \sqsubseteq \mathrm{Cl}_\theta(\Phi_{fe}(F, A))$$

for every $(F,A) \in SS(X,A)$.

Proof. Let $\Phi_{fe}(Cl_\theta(F,A)) = (Q,B) \in SS(Y,B)$, where

$$Q(p_Y) = \begin{cases} \bigcup\{f(cl_\theta(F,p)) : p \in e^{-1}(\{p_Y\})\}, \\ \text{if } e^{-1}(\{p_Y\}) \neq \emptyset, \\ \emptyset, \text{ if } e^{-1}(\{p_Y\}) = \emptyset, \end{cases}$$

for every $p_Y \in B$ and $\Phi_{fe}(F,A) = (Q',B) \in SS(Y,B)$, where

$$Q'(p_Y) = \begin{cases} \bigcup\{f(F(p)) : p \in e^{-1}(\{p_Y\})\}, \\ \text{if } e^{-1}(\{p_Y\}) \neq \emptyset, \\ \emptyset, \text{ if } e^{-1}(\{p_Y\}) = \emptyset, \end{cases}$$

for every $p_Y \in B$. It suffices to prove that $(Q,B) \sqsubseteq Cl_\theta(Q',B)$ or equivalently $Q(p_Y) \subseteq cl_\theta(Q',p_Y)$, for every $p_Y \in B$. Let $p_Y \in B$. If $e^{-1}(\{p_Y\}) = \emptyset$, then $Q(p_Y) = cl_\theta(Q',p_Y) = \emptyset$. We suppose that $e^{-1}(\{p_Y\}) \neq \emptyset$. Let $y \in Q(p_Y)$. Then, there exists $p \in e^{-1}(\{p_Y\})$ such that $y \in f(cl_\theta(F,p))$. Let $y = f(x)$, where $x \in cl_\theta(F,p)$. We prove that $y \in cl_\theta(Q',p_Y)$. Let (G,B) be a p_Y-soft open neighborhood of y. It suffices to prove that $(Q',B) \sqcap Cl(G,B) \neq \mathbf{0}_B$. Since $p_Y = e(p)$ and the map f is soft e-θ-continuous at the point x, there exists a p-soft open neighborhood (H,A) of x in (X, τ_X, A) such that

$$\Phi_{fe}(Cl(H,A)) \sqsubseteq Cl(G,B). \qquad (1)$$

Since $x \in cl_\theta(F,p)$, we have $(F,A) \sqcap Cl(H,A) \neq \mathbf{0}_A$. Therefore, by Proposition 2.8 we have

$$\mathbf{0}_B \neq \Phi_{fe}((F,A) \sqcap Cl(H,A))$$
$$\sqsubseteq \Phi_{fe}(F,A) \sqcap \Phi_{fe}(Cl(H,A))$$
$$= (Q',B) \sqcap \Phi_{fe}(Cl(H,A)).$$

Thus, $(Q',B) \sqcap \Phi_{fe}(Cl(H,A)) \neq \mathbf{0}_B$ and, therefore, by relation (1) we have $(Q',B) \sqcap Cl(G,B) \neq \mathbf{0}_B$.

Corollary 5.32 If the map $f : X \to Y$ is soft e-θ-continuous, then

$$Cl_\theta(\Phi_{fe}^{-1}(G,B)) \sqsubseteq \Phi_{fe}^{-1}(Cl_\theta(G,B)),$$

for every $(G,B) \in SS(Y,B)$.

Proof. By Proposition 5.31 and Proposition 2.8 we have

$$\Phi_{fe}(Cl_\theta(\Phi_{fe}^{-1}(G,B))) \sqsubseteq Cl_\theta(\Phi_{fe}(\Phi_{fe}^{-1}(G,B)))$$
$$\sqsubseteq Cl_\theta(G,B).$$

Therefore,

$$Cl_\theta(\Phi_{fe}^{-1}(G,B)) \sqsubseteq \Phi_{fe}^{-1}(\Phi_{fe}(Cl_\theta(\Phi_{fe}^{-1}(G,B))))$$
$$\sqsubseteq \Phi_{fe}^{-1}(Cl_\theta(G,B)).$$

Proposition 5.33 If the map $f : X \to Y$ is soft e-θ-continuous, then

$$\Phi_{fe}^{-1}(G,B) \sqsubseteq Int_\theta(\Phi_{fe}^{-1}(Cl_\theta(G,B))),$$

for every $(G,B) \in \tau_Y$.

Proof. Let $\Phi_{fe}^{-1}(G,B) = (F,A) \in SS(X,A)$, where

$$F(p) = f^{-1}(G(e(p))), \text{ for every } p \in A$$

and $\Phi_{fe}^{-1}(Cl_\theta(G,B)) = (F',A) \in SS(X,A)$, where

$$F'(p) = f^{-1}(cl_\theta(G,e(p))), \text{ for every } p \in A.$$

We prove that

$$(F,A) \sqsubseteq Int_\theta(F',A)$$

or equivalently $F(p) \subseteq int_\theta(F',p)$, for every $p \in A$. Let $p \in A$ and $x \in F(p)$. It suffices to prove that $x \in int_\theta(F',p)$. We have $f(x) \in G(e(p))$. Since the map f is soft e-θ-continuous at the point x, there exists a p-soft open neighborhood (H,A) of x in (X, τ_X, A) such that

$$\Phi_{fe}(Cl(H,A)) \sqsubseteq Cl(G,B). \qquad (2)$$

We prove that $Cl(H,A) \sqsubseteq (F',A)$. Indeed, by relation (2) and Proposition 2.8 we have

$$Cl(H,A) \sqsubseteq \Phi_{fe}^{-1}(\Phi_{fe}(Cl(H,A))) \sqsubseteq \Phi_{fe}^{-1}(Cl(G,B))$$
$$\sqsubseteq \Phi_{fe}^{-1}(Cl_\theta(G,B)) = (F',A).$$

6 Questions

1. Is the converse of Propositions 3.11, 3.16, 3.17, and 5.33 true?

2. Let (X, τ, A) be a soft topological space. Under what conditions does the equality $\tau_\theta = \tau$ hold?

3. Let (X, τ, A) be a soft topological space. Under what conditions does the sequence $\tau, \tau_\theta, (\tau_\theta)_\theta, ((\tau_\theta)_\theta)_\theta, \ldots$ is eventually constant?

4. Find a soft topological space (X, τ, A) such that

$$\tau \supsetneqq \tau_\theta \supsetneqq (\tau_\theta)_\theta \supsetneqq ((\tau_\theta)_\theta)_\theta \supsetneqq \cdots.$$

7 Conclusion

The classical mathematical theories have their difficulties which are pointed out in [18] for the solution complicated problems in engineering and environment. To overcome these difficulties, Molodtsov [18] introduced the concept of soft set as a new mathematical tool. Furthermore, D. Pei and D. Miao [22] showed that soft sets are a class of special information systems.

Topology is an important area of mathematics with many applications in the domains of computer science and physical sciences. Soft topology is a relatively new and promising domain which can lead to the development of new mathematical models and innovative approaches

that will significantly contribute to the solution of complex problems in natural sciences.

In [9] for the soft set theory: new definitions, examples, new classes of soft sets, and properties for mappings between different classes of soft sets are introduced and studied. Moreover, the theory of soft topological spaces is investigated.

This paper continues the study of the theory of soft topological spaces. In section 2 we give known basic notions and results concerning the theory of soft sets and soft topological spaces. In section 3 we present separation axioms and we investigate the notion of convergence. In section 4 the Cartesian product of two soft topological spaces is studied. In section 5 we introduce and study notions concerning the soft θ-open sets, the soft θ-closed sets, and the soft θ-continuity. Finally, in section 6 we give some questions on soft topological spaces.

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