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Stability Results in Intuitionistic Fuzzy Normed Spaces for a Cubic Functional Equation

M. Mursaleen and Khursheed J. Ansari*

Department of Mathematics, Aligarh Muslim University, Aligarh 202002, India

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Abstract: In this paper, we determine some stability results concerning the cubic functional equation

$$kf(x+ky)+f(kx-y)=\frac{k(k^2+1)}{2}[f(x+y)+f(x-y)]+(k^4-1)f(y),$$

where $k\geq 2$ is a fixed integer, in the setting of intuitionistic fuzzy normed spaces (IFNS). Further we study the intuitionistic fuzzy continuity through the existence of a certain solution of a fuzzy stability problem for approximately cubic functional equation.

Keywords: Intuitionistic fuzzy normed spaces, Cubic functional equation, Hyers-Ulam stability.

1. Introduction and preliminaries

Sometime in modeling applied problems there may be a degree of uncertainty in the parameters used in the model or some measurements may be imprecise. Due to such features, we are tempted to consider the study of functional equations in the fuzzy setting. The notion of fuzzy sets was first introduced by Zadeh [31] in 1965 which is a powerful hand set for modeling uncertainty and vagueness in various problems arising in the field of science and engineering. For the last four decades, fuzzy theory has become very active area of research and a lot of developments have been made in the theory of fuzzy sets to find the fuzzy analogues of the classical set theory. The notion of intuitionistic fuzzy norm (see [13,16–19,22,29]) is also useful one to deal with the inexactness and vagueness arising in modeling. There are many situations where the norm of a vector is not possible to find and the concept of intuitionistic fuzzy norm seems to be more suitable in such cases, that is, we can deal with such situations by modeling the inexactness through the intuitionistic fuzzy norm.

In 1940, S.M. Ulam [30] raised the following question. Under what conditions does there exist an additive mapping near an approximately addition mapping? The case of approximately additive functions was solved by D.H. Hyers [3] under certain assumption. In 1978, a generalized version of the theorem of Hyers for approximately linear mapping was given by Th.M. Rassias [26]. A number of mathematicians were attracted by the result of Th.M. Rassias. The stability concept that was introduced and investigated by Rassias is called the Hyers-Ulam-Rassias stability. During the last decades, the stability problems of several functional equations have been extensively investigated by a number of authors (c.f. [1,4–12,14,15,21,23–25,27,28]) and references therein.

Recently, Bae, Lee and Park [2] established some stability results for the functional equation

$$kf(x+ky)+f(kx-y)$$
$$=\frac{k(k^2+1)}{2}[f(x+y)+f(x-y)]+(k^4-1)f(y),$$

where $k\geq 2$ is a fixed integer, in the setting of non-Archimedean $\mathcal{L}$-fuzzy normed spaces.

In this paper, we determine some stability results concerning the above cubic functional equation in the setting of intuitionistic fuzzy normed spaces (IFNS). We also study the intuitionistic fuzzy continuity through the existence of a certain solution of a fuzzy stability problem for approximately cubic functional equation.

* Corresponding author e-mail: mursaleenm@gmail.com

In this section we recall some notations and basic definitions used in this paper.

Definition 1.2. A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is said to be a *continuous t-norm* if it satisfies the following conditions:
(a) $*$ is associative and commutative, (b) $*$ is continuous, (c) $a * 1 = a$ for all $a \in [0,1]$, (d) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for each $a, b, c, d \in [0,1]$.

Definition 1.3. A binary operation $\diamondsuit : [0,1] \times [0,1] \to [0,1]$ is said to be a *continuous t-conorm* if it satisfies the following conditions:
(a') $\diamondsuit$ is associative and commutative, (b') $\diamondsuit$ is continuous, (c') $a \diamondsuit 0 = a$ for all $a \in [0,1]$, (d') $a \diamondsuit b \leq c \diamondsuit d$ whenever $a \leq c$ and $b \leq d$ for each $a, b, c, d \in [0,1]$.

Using the notions of continuous t-norm and t-conorm, Saadati and Park [29] introduced the concept of intuitionistic fuzzy normed space as follows:

Definition 1.4. The five-tuple $(X, \mu, \nu, *, \diamondsuit)$ is said to be an *intuitionistic fuzzy normed spaces* (for short, IFNS) if X is a vector space, $*$ is a continuous t-norm, $\diamondsuit$ is a continuous t-conorm, and μ, ν are fuzzy sets on $X \times (0, \infty)$ satisfying the following conditions. For every $x, y \in X$ and $s, t > 0$
(i) $\mu(x,t) + \nu(x,t) \leq 1$, (ii) $\mu(x,t) > 0$, (iii) $\mu(x,t) = 1$ if and only if $x = 0$, (iv) $\mu(\alpha x, t) = \mu(x, \frac{t}{|\alpha|})$ for each $\alpha \neq 0$, (v) $\mu(x,t) * \mu(y,s) \leq \mu(x+y, t+s)$, (vi) $\mu(x, \cdot) : (0,\infty) \to [0,1]$ is continuous, (vii) $\lim_{t\to\infty} \mu(x,t) = 1$ and $\lim_{t\to 0} \mu(x,t) = 0$, $(viii)$ $\nu(x,t) < 1$, (ix) $\nu(x,t) = 0$ if and only if $x = 0$, (x) $\nu(\alpha x, t) = \nu(x, \frac{t}{|\alpha|})$ for each $\alpha \neq 0$, (xi) $\nu(x,t) \diamondsuit \nu(y,s) \geq \nu(x+y, t+s)$, (xii) $\nu(x, \cdot) : (0,\infty) \to [0,1]$ is continuous, $(xiii)$ $\lim_{t\to\infty} \nu(x,t) = 0$ and $\lim_{t\to 0} \nu(x,t) = 1$.
In this case (μ, ν) is called an *intuitionistic fuzzy norm*.

Example 1.1. Let $(X, \|.\|)$ be a normed space, $a * b = ab$ and $a \diamondsuit b = \min\{a+b, 1\}$ for all $a, b \in [0,1]$. For all $x \in X$ and every $t > 0$, consider

$$\mu(x,t) = \begin{cases} \frac{t}{t+\|x\|} & \text{if } t > 0 \\ 0 & \text{if } t \leq 0; \end{cases} \quad \text{and}$$

$$\nu(x,t) = \begin{cases} \frac{\|x\|}{t+\|x\|} & \text{if } t > 0 \\ 0 & \text{if } t \leq 0. \end{cases}$$

Then $(X, \mu, \nu, *, \diamondsuit)$ is an IFNS.

The concepts of convergence and Cauchy sequences in an intuitionistic fuzzy normed space are studied in [29].

Let $(X, \mu, \nu, *, \diamondsuit)$ be an IFNS. Then, a sequence $x = (x_k)$ is said to be *intuitionistic fuzzy convergent* to $L \in X$ if $\lim \mu(x_k - L, t) = 1$ and $\lim \nu(x_k - L, t) = 0$ for all $t > 0$. In this case we write $x_k \xrightarrow{IF} L$ as $k \to \infty$.

Let $(X, \mu, \nu, *, \diamondsuit)$ be an IFNS. Then, $x = (x_k)$ is said to be *intuitionistic fuzzy Cauchy sequence* if $\lim \mu(x_{k+p} - x_k, t) = 1$ and $\lim \nu(x_{k+p} - x_k, t) = 0$ for all $t > 0$ and $p = 1, 2, \cdots$.

Let $(X, \mu, \nu, *, \diamondsuit)$ be an IFNS. Then $(X, \mu, \nu, *, \diamondsuit)$ is said to be *complete* if every intuitionistic fuzzy Cauchy sequence in $(X, \mu, \nu, *, \diamondsuit)$ is intuitionistic fuzzy convergent in $(X, \mu, \nu, *, \diamondsuit)$.

2. Intuitionistic fuzzy stability

The functional equation

$$kf(x+ky) + f(kx-y)$$

$$= \frac{k(k^2+1)}{2}[f(x+y) + f(x-y)] + (k^4-1)f(y), \quad (2.1)$$

where $k \geq 2$ is a fixed integer, is called the *cubic functional equation*, since the function $f(x) = cx^3$ is its solution. Every solution of the cubic functional equation is said to be a *cubic mapping*.

We begin with a generalized Hyers-Ulam-Rassias type theorem in IFNS for the cubic functional equation.

Theorem 2.1. Let X be a linear space and let (Z, μ', ν') be an IFNS. Let $\varphi : X \times X \to Z$ be a function such that for some $\alpha > k^3$

$$\mu'\left(\varphi(\frac{x}{k}, 0), t\right) \geq \mu'(\varphi(x,0), \alpha t) \quad \text{and}$$

$$\nu'\left(\varphi(\frac{x}{k}, 0), t\right) \leq \nu'(\varphi(x,0), \alpha t), \quad (2.2)$$

and $\lim_{n\to\infty} \mu'\left(k^{3n}\varphi(\frac{x}{k^n}, \frac{y}{k^n}), t\right) = 1$ and $\lim_{n\to\infty} \nu'\left(k^{3n}\varphi(\frac{x}{k^n}, \frac{y}{k^n}), t\right) = 0$ for all $x, y \in X$ and $t > 0$. Let (Y, μ, ν) be an intuitionistic fuzzy Banach space and let $f : X \to Y$ be a φ-approximately cubic mapping in the sense that

$$\begin{aligned} &\mu\Big(kf(x+ky) + f(kx-y) - \tfrac{k(k^2+1)}{2}\Big[f(x+y) \\ &+ f(x-y)\Big] - (k^4-1)f(y), t\Big) \geq \mu'(\varphi(x,y), t), \\ &\nu\Big(kf(x+ky) + f(kx-y) - \tfrac{k(k^2+1)}{2}\Big[f(x+y) \\ &+ f(x-y)\Big] - (k^4-1)f(y), t\Big) \leq \nu'(\varphi(x,y), t) \end{aligned} \quad (2.3)$$

for all $t > 0$ and all $x, y \in X$. Then there exists a unique cubic mapping $g : X \to Y$ such that

$$\mu(g(x) - f(x), t) \geq \mu'(\varphi(x,0), \frac{(\alpha - k^3)t}{2}) \quad \text{and}$$

$$\nu(g(x)-f(x),t)\le \nu'(\varphi(x,0),\frac{(\alpha-k^3)t}{2}) \quad (2.4)$$

for all $x \in X$ and all $t > 0$.

Proof. Put y=0 in (2.3). Then for all $x \in X$ and $t > 0$

$$\mu(f(kx)-k^3f(x),t)\ge \mu'(\varphi(x,0),t)$$

which implies that

$$\left.\begin{array}{l}\mu\Big(k^3f(\frac{x}{k})-f(x),t\Big)\ge \mu'\Big(\varphi(\frac{x}{k},0),t\Big)\\ \ge \mu'(\varphi(x,0),\alpha t),\\ \nu\Big(k^3f(\frac{x}{k})-f(x),t\Big)\le \nu'\Big(\varphi(\frac{x}{k},0),t\Big)\\ \le \nu'(\varphi(x,0),\alpha t).\end{array}\right\} \quad (2.5)$$

Replacing x by x/k^n in (2.5), we get

$$\mu\Big(k^{3(n+1)}f(\frac{x}{k^{n+1}})-k^{3n}f(\frac{x}{k^n}),k^{3n}t\Big)$$

$$\ge \mu'\Big(\varphi(\frac{x}{k^n},0),\alpha t\Big)\ge \mu'(\varphi(x,0),\alpha^{n+1}t) \quad \text{and}$$

$$\nu\Big(k^{3(n+1)}f(\frac{x}{k^{n+1}})-k^{3n}f(\frac{x}{k^n}),k^{3n}t\Big)$$

$$\le \nu'\Big(\varphi(\frac{x}{k^n},0),\alpha t\Big)\le \nu'(\varphi(x,0),\alpha^{n+1}t).$$

Replacing t by t/α^{n+1}, we obtain

$$\left.\begin{array}{l}\mu\Big(k^{3(n+1)}f(\frac{x}{k^{n+1}})-k^{3n}f(\frac{x}{k^n}),\frac{k^{3n}t}{\alpha^{n+1}}\Big)\\ \ge \mu'(\varphi(x,0),t) \quad \text{and}\\ \nu\Big(k^{3(n+1)}f(\frac{x}{k^{n+1}})-k^{3n}f(\frac{x}{k^n}),\frac{k^{3n}t}{\alpha^{n+1}}\Big)\\ \le \nu'(\varphi(x,0),t).\end{array}\right\} \quad (2.6)$$

It follows from $k^{3n}f(\frac{x}{k^n})-f(x)=\sum_{j=0}^{n-1}\Big(k^{3(j+1)}f(\frac{x}{k^{j+1}})-k^{3j}f(\frac{x}{k^j})\Big)$ and (2.6) that

$$\left.\begin{array}{l}\mu\Big(k^{3n}f(\frac{x}{k^n})-f(x),\ \sum_{j=0}^{n-1}\frac{k^{3j}t}{\alpha^{j+1}}\Big)\\ \ge \prod_{j=0}^{n-1}\mu\Big(k^{3(j+1)}f(\frac{x}{k^{j+1}})-k^{3j}f(\frac{x}{k^j}),\frac{k^{3j}t}{\alpha^{j+1}}\Big)\\ \ge \mu'(\varphi(x,0),t) \quad \text{and}\\ \nu\Big(k^{3n}f(\frac{x}{k^n})-f(x),\ \sum_{j=0}^{n-1}\frac{k^{3j}t}{\alpha^{j+1}}\Big)\\ \le \prod_{j=0}^{n-1}\nu\Big(k^{3(j+1)}f(\frac{x}{k^{j+1}})-k^{3j}f(\frac{x}{k^j}),\frac{k^{3j}t}{\alpha^{j+1}}\Big)\\ \le \mu'(\varphi(x,0),t),\end{array}\right\} \quad (2.7)$$

for all $x \in X$, $t > 0$ and $n > 0$ where $\prod_{j=0}^{n-1}a_j = a_1 * a_2 * .... * a_n$, $\prod_{j=0}^{n-1}b_j = b_1\diamondsuit b_2\diamondsuit....\diamondsuit b_n$.

By replacing x with x/k^m in (2.7), we get

$$\mu\Big(k^{3(n+m)}f(\frac{x}{k^{n+m}})-k^{3m}f(\frac{x}{k^m}),\ \sum_{j=0}^{n-1}\frac{k^{3(j+m)}t}{\alpha^{j+m+1}}\Big)$$

$$\ge \mu'\Big(\varphi(\frac{x}{k^m},0),t\Big)\ge \mu'(\varphi(x,0),t) \quad \text{and}$$

$$\nu\Big(k^{3(n+m)}f(\frac{x}{k^{n+m}})-k^{3m}f(\frac{x}{k^m}),\ \sum_{j=0}^{n-1}\frac{k^{3(j+m)}t}{\alpha^{j+m+1}}\Big)$$

$$\le \nu'\Big(\varphi(\frac{x}{k^m},0),t\Big)\le \nu'(\varphi(x,0),t)$$

Thus,

$$\mu\Big(k^{3(n+m)}f(\frac{x}{k^{n+m}})-k^{3m}f(\frac{x}{k^m}),\ \sum_{j=m}^{n+m-1}\frac{k^{3j}t}{\alpha^{j+1}}\Big)$$

$$\ge \mu'(\varphi(x,0),t) \quad \text{and}$$

$$\nu\Big(k^{3(n+m)}f(\frac{x}{k^{n+m}})-k^{3m}f(\frac{x}{k^m}),\ \sum_{j=m}^{n+m-1}\frac{k^{3j}t}{\alpha^{j+1}}\Big)$$

$$\le \nu'(\varphi(x,0),t)$$

for all $x \in X$, $t > 0$, $m \ge 0$ and $n \ge 0$. Hence

$$\left.\begin{array}{l}\mu\Big(k^{3(n+m)}f(\frac{x}{k^{n+m}})-k^{3m}f(\frac{x}{k^m}),t\Big)\\ \ge \mu'\Big(\varphi(x,0),\frac{t}{\sum_{j=m}^{n+m-1}\frac{k^{3j}t}{\alpha^{j+1}}}\Big) \quad \text{and}\\ \nu\Big(k^{3(n+m)}f(\frac{x}{k^{n+m}})-k^{3m}f(\frac{x}{k^m}),t\Big)\\ \le \nu'\Big(\varphi(x,0),\frac{t}{\sum_{j=m}^{n+m-1}\frac{k^{3j}t}{\alpha^{j+1}}}\Big),\end{array}\right\} \quad (2.8)$$

for all $x \in X$, $t > 0$, $m \ge 0$ and $n \ge 0$.

Since $\alpha > k^3$ and $\sum_{j=0}^{\infty}(\frac{k^3}{\alpha}) < \infty$, the Cauchy criterion for convergence in IFNS shows that $\Big(k^{3n}f(\frac{x}{k^n})\Big)$ is a Cauchy sequence in (Y,μ,ν). Since (Y,μ,ν) is complete, this sequence converges to some point $g(x) \in Y$. Fix $x \in X$ and put $m = 0$ in (2.8) to obtain

$$\mu\Big(k^{3n}f(\frac{x}{k^n})-f(x),t\Big)\ge \mu'\Big(\varphi(x,0),\frac{t}{\sum_{j=0}^{n-1}\frac{k^{3j}}{\alpha^{j+1}}}\Big) \quad \text{and}$$

$$\nu\Big(k^{3n}f(\frac{x}{k^n})-f(x),t\Big)\le \nu'\Big(\varphi(x,0),\frac{t}{\sum_{j=0}^{n-1}\frac{k^{3j}}{\alpha^{j+1}}}\Big),$$

for all $t > 0$ and $n > 0$. Thus we obtain

$$\mu(g(x) - f(x), t) \geq \mu\left(g(x) - k^{3n} f(\frac{x}{k^n}), t/2\right)$$

$$*\mu\left(k^{3n} f(\frac{x}{k^n}) - f(x), t/2\right) \geq \mu'\left(\varphi(x, 0), \frac{t}{2\sum_{j=0}^{n-1} \frac{k^{3j}}{\alpha^{j+1}}}\right),$$

$$\nu(g(x) - f(x), t) \leq \nu\left(g(x) - k^{3n} f(\frac{x}{k^n}), t/2\right)$$

$$\diamond\nu\left(k^{3n} f(\frac{x}{k^n}) - f(x), t/2\right) \leq \nu'\left(\varphi(x, 0), \frac{t}{2\sum_{j=0}^{n-1} \frac{k^{3j}}{\alpha^{j+1}}}\right)$$

for large n. Taking the limit as $n \to \infty$ and using the definition of IFNS, we get

$$\mu(g(x) - f(x), t) \geq \mu'\left(\varphi(x, 0), \frac{(\alpha - k^3)t}{2}\right) \text{ and}$$

$$\nu(g(x) - f(x), t) \leq \nu'\left(\varphi(x, 0), \frac{(\alpha - k^3)t}{2}\right),$$

for all $x \in X$, $t > 0$. Replace x and y by x/k^n and y/k^n, respectively in (2.3), we have

$$\mu\Bigg(k^{3n} f(\frac{x + ky}{k^n}) + k^{3n} f(\frac{kx - y}{k^n}) - \frac{k(k^2 + 1)}{2} \left[k^{3n} f(\frac{x + y}{k^n}) + k^{3n} f(\frac{x - y}{k^n})\right] + k^{3n}(k^4 - 1) f(\frac{y}{k^n}), t\Bigg) \geq \mu'\left(\varphi(\frac{x}{k^n}, \frac{y}{k^n}), \frac{t}{k^{3n}}\right)$$

and

$$\nu\Bigg(k^{3n} f(\frac{x + ky}{k^n}) + k^{3n} f(\frac{kx - y}{k^n}) - \frac{k(k^2 + 1)}{2} \left[k^{3n} f(\frac{x + y}{k^n}) + k^{3n} f(\frac{x - y}{k^n})\right] + k^{3n}(k^4 - 1) f(\frac{y}{k^n}), t\Bigg) \leq \nu'\left(\varphi(\frac{x}{k^n}, \frac{y}{k^n}), \frac{t}{k^{3n}}\right)$$

for all $x, y \in X$ and all $t > 0$. Since

$$\lim_{n\to\infty} \mu'\left(k^{3n} \varphi(\frac{x}{k^n}, \frac{y}{k^n}), t\right) = 1 \text{ and}$$

$$\lim_{n\to\infty} \nu'\left(k^{3n} \varphi(\frac{x}{k^n}, \frac{y}{k^n}), t\right) = 0,$$

for all $x, y \in X$ and all $t > 0$. We observe that g fulfills (2.1). Therefore g is a cubic mapping.

To Prove the uniqueness of the cubic mapping g, assume that there exists a cubic mapping $h : X \to Y$ which satisfies (2.4). For fix $x \in X$, clearly $k^{3n} g(\frac{x}{k^n}) = g(x)$ and $k^{3n} h(\frac{x}{k^n}) = h(x)$ for all $n \in$. It follows from (2.4) that

$$\begin{aligned}\mu(g(x) - h(x), t) &= \mu\left(k^{3n} g(\frac{x}{k^n}) - k^{3n} h(\frac{x}{k^n}), t\right) \\ &\geq \mu\left(k^{3n} g(\frac{x}{k^n}) - k^{3n} f(\frac{x}{k^n}), \frac{t}{2}\right) \\ &* \mu\left(k^{3n} f(\frac{x}{k^n}) - k^{3n} h(\frac{x}{k^n}), \frac{t}{2}\right) \\ &\geq \mu'\left(\varphi(\frac{x}{k^n}, 0), \frac{(\alpha - k^3)t}{2k^{3n}}\right) \\ &\geq \mu'\left(\varphi(x, 0), \frac{\alpha^n(\alpha - k^3)t}{2k^{3n}}\right)\end{aligned}$$

and similarly

$$\nu(g(x) - h(x), t) \leq \nu'\left(\varphi(x, 0), \frac{\alpha^n(\alpha - k^3)t}{2k^{3n}}\right).$$

Since $\lim_{n\to\infty} \frac{\alpha^n(\alpha - k^3)}{2k^{3n}} = \infty$ as $\alpha > k^3$, we get

$$\lim_{n\to\infty} \mu'\left(\varphi(x, 0), \frac{\alpha^n(\alpha - k^3)t}{2k^{3n}}\right) = 1,$$

and

$$\lim_{n\to\infty} \nu'\left(\varphi(x, 0), \frac{\alpha^n(\alpha - k^3)t}{2k^{3n}}\right) = 0.$$

Therefore

$$\mu(g(x) - h(x), t) = 1 \text{ and } \nu(g(x) - h(x), t) = 0,$$

for all $t > 0$. Hence $g(x) = h(x)$.

This completes the proof.

In the following theorem we consider $0 < \alpha < k^3$.

Theorem 2.2. Let X be a linear space and let (Z, μ', ν') be an IFNS. Let $\varphi : X \times X \to Z$ be a function such that for some $0 < \alpha < k^3$

$$\mu'(\varphi(kx, 0), t) \geq \mu'(\alpha\varphi(x, 0), t) \text{ and}$$

$$\nu'(\varphi(kx, 0), t) \leq \nu'(\alpha\varphi(x, 0), t),$$

and $\lim_{n\to\infty} \mu'(\varphi(k^n x, k^n y), k^{3n} t) = 1$ and $\lim_{n\to\infty} \nu'(\varphi(k^n x, k^n y), k^{3n} t) = 0$ for all x, y in X and $t > 0$. Let (Y, μ, ν) be an intuitionistic fuzzy Banach space and let $f : X \to Y$ be a φ-approximately cubic mapping in the sense that

$$\left.\begin{aligned}&\mu\Big(kf(x + ky) + f(kx - y) - \tfrac{k(k^2+1)}{2}\Big[f(x + y) + f(x - y)\Big] - (k^4 - 1)f(y), t\Big) \geq \mu'(\varphi(x, y), t), \text{ and} \\ &\nu\Big(kf(x + ky) + f(kx - y) - \tfrac{k(k^2+1)}{2}\Big[f(x + y) + f(x - y)\Big] - (k^4 - 1)f(y), t\Big) \leq \nu'(\varphi(x, y), t)\end{aligned}\right\}$$

for all $t > 0$ and all $x, y \in X$. Then there exists a unique cubic mapping $g : X \to Y$ such that

$$\mu(g(x) - f(x), t) \geq \mu'\left(\varphi(x, 0), \frac{(k^3 - \alpha)t}{2}\right) \text{ and}$$

$$\nu(g(x) - f(x), t) \leq \nu'\left(\varphi(x, 0), \frac{(k^3 - \alpha)t}{2}\right)$$

for all $x \in X$ and all $t > 0$.

Proof. The techniques are similar to that of Theorem 2.1. Hence we present a sketch of proof. Put $y = 0$ in (2.3), we get

$$\mu\left(\frac{f(kx)}{k^3} - f(x), t\right) \geq \mu'(\varphi(x, 0), t) \text{ and}$$

$$\nu\left(\frac{f(kx)}{k^3} - f(x), t\right) \leq \nu'(\varphi(x, 0), t),$$

for all $x \in X$, $t > 0$. Therefore

$$\mu\left(\frac{f(k^{n+1}x)}{k^3} - f(k^n x), t\right) \geq \mu'\left(\varphi(x, 0), \frac{t}{\alpha^n}\right) \text{ and}$$

$$\nu\left(\frac{f(k^{n+1}x)}{k^3} - f(k^n x), t\right) \leq \nu'\left(\varphi(x, 0), \frac{t}{\alpha^n}\right),$$

for all $x \in X$ and $t > 0$. For each $x \in X, n \geq 0, m \geq 0$ and $t > 0$, we can deduce

$$\left.\begin{aligned} &\mu\left(\frac{f(k^{n+m}x)}{k^{3(n+m)}} - \frac{f(k^m x)}{k^{3m}}, t\right) \\ &\geq \mu'\left(\varphi(x, 0), \frac{t}{\sum_{j=m}^{n+m-1} \frac{\alpha^j}{k^{3(j+1)}}}\right) \text{ and} \\ &\nu\left(\frac{f(k^{n+m}x)}{k^{3(n+m)}} - \frac{f(k^m x)}{k^{3m}}, t\right) \\ &\leq \nu'\left(\varphi(x, 0), \frac{t}{\sum_{j=m}^{n+m-1} \frac{\alpha^j}{k^{3(j+1)}}}\right) \end{aligned}\right\} \quad (2.9)$$

for all $x \in X$, $t > 0$, $m \geq 0$ and $n \geq 0$. Thus, $\left(\frac{f(k^n x)}{k^3 n}\right)$ is a Cauchy sequence in intuitionistic fuzzy Banach space. Therefore, there is a function $g : X \to Y$ defined by $g(x) = \lim_{n\to\infty} \frac{f(k^n x)}{k^{3n}}$. (2.9) with $m = 0$ implies

$$\mu(g(x) - f(x), t) \geq \mu'\left(\varphi(x, 0), \frac{(k^3 - \alpha)t}{2}\right) \text{ and}$$

$$\nu(g(x) - f(x), t) \leq \nu'\left(\varphi(x, 0), \frac{(k^3 - \alpha)t}{2}\right)$$

for all $x \in X$ and all $t > 0$.

This completes the proof.

Example 2.3. Let X be a Hilbert space and Z be a normed space. Denote by (μ, ν) and (μ', ν') the intuitionistic fuzzy norms given as in Example 1.1 on X and Z, respectively. Let $\varphi : X \times X \to Z$ be defined by $\varphi(x, y) = ((k-1)^2\|kx - y\|^2 + k^4\|y\|^2)z_\circ$, where $z_\circ$ is a fixed unit vector in Z. Define $f : X \to X$ by $f(x) = \|x\|^2 x + \|x\|^2 x_\circ$ for some unit vector $x_\circ \in X$. Then

$$\mu\left(kf(x + ky) + f(kx - y) - \frac{k(k^2 + 1)}{2}\left[f(x + y)\right.\right.$$

$$\left.\left.+ f(x - y)\right] - (k^4 - 1)f(y), t\right)$$

$$= \frac{t}{t + \|(k-1)(kx - y)^2 - k^4 y^2\|}$$

$$\geq \frac{t}{t + (k-1)\|kx - y\|^2 + k^4\|y\|^2}$$

$$\geq \frac{t}{t + (k-1)^2\|kx - y\|^2 + k^4\|y\|^2} = \mu'(\varphi(x, y), t)$$

and

$$\nu\left(kf(x+ky)+f(kx-y) - \frac{k(k^2 + 1)}{2}\left[f(x+y)+f(x-y)\right]\right.$$

$$\left. - (k^4 - 1)f(y), t\right) \leq \nu'(\varphi(x, y), t).$$

Also

$$\mu'(\varphi(kx, 0), t) = \frac{t}{t + \|\varphi(kx, 0)\|}$$

$$= \frac{t}{t + \|(k-1)^2(k^2 x)^2\|} = \mu'(k^2\varphi(x, 0), t)$$

and $\nu'(\varphi(kx, 0), t) = \nu'(k^2\varphi(x, 0), t)$. Thus,

$$\lim_{n\to\infty} \mu'(\varphi(k^n x, k^n y), k^{3n} t)$$

$$= \lim_{n\to\infty} \frac{k^{3n} t}{k^{3n} t + k^{2n}[\|(k-1)(kx - y)\|^2 + \|k^2 y\|^2]} = 1$$

and

$$\lim_{n\to\infty} \nu'(\varphi(k^n x, k^n y), k^{3n} t)$$

$$= \lim_{n\to\infty} \frac{k^{5n}[\|(k-1)(kx - y)\|^2 + \|k^2 y\|^2]}{k^{3n} t + k^{2n}[\|(k-1)(kx - y)\|^2 + \|k^2 y\|^2]} = 0.$$

Hence, conditions of Theorem 2.2 for $\alpha = k^2$ are fulfilled. Therefore, there is a unique cubic mapping $g : X \to Y$ such that

$$\mu(g(x) - f(x), t) \geq \mu'(\varphi(x, 0), k^2 t) \text{ and}$$

$$\nu(g(x) - f(x), t) \leq \nu'(\varphi(x, 0), k^2 t).$$

This completes the proof.

3. Intuitionistic fuzzy continuity

Recently, the intuitionistic fuzzy continuity is discussed in [20]. In this section, we establish some interesting results of continuous approximately cubic mappings.

Definition 3.1. Let $f : \mathbb{R} \to X$ be a function, where is endowed with the Euclidean topology and X is an intuitionistic fuzzy normed space equipped with intuitionistic fuzzy norm (μ, ν). Then, f is called *intuitionistic fuzzy continuous* at a point $s_\circ \in \mathbb{R}$ if for all $\epsilon > 0$ and all $0 < \alpha < 1$ there exists $\delta > 0$ such that for each s with $0 < |s - s_\circ| < \delta$

$$\mu(f(sx)-f(s_\circ x), \epsilon) \geq \alpha \text{ and } \nu(f(sx)-f(s_\circ x), \epsilon) \leq 1-\alpha.$$

Theorem 3.2. Let X be a normed space and (Z, μ', ν') be an IFNS. Let (Y, μ, ν) be an intuitionistic fuzzy Banach space and $f : X \to Y$ be a (p, q)-approximately cubic mapping in the sense that for some p, q and some $z_\circ \in Z$

$$\mu\bigg(kf(x+ky)+f(kx-y)-\frac{k(k^2+1)}{2}\Big[f(x+y)+f(x-y)\Big]$$

$$-(k^4-1)f(y), t\bigg) \geq \mu'((\|x\|^p + \|y\|^q)z_\circ, t),$$

$$\nu\bigg(kf(x+ky)+f(kx-y)-\frac{k(k^2+1)}{2}\Big[f(x+y)+f(x-y)\Big]$$

$$-(k^4-1)f(y), t\bigg) \leq \nu'((\|x\|^p + \|y\|^q)z_\circ, t),$$

for all $x, y \in X$ and all $t > 0$. If $p, q < 3$, then there exists a unique cubic mapping $g : X \to Y$ such that

$$\mu(g(x) - f(x), t) \geq \mu'\left(\|x\|^p z_\circ, \frac{(n^3 - n^p)t}{2}\right) \text{ and}$$

$$\nu(g(x) - f(x), t) \leq \nu'\left(\|x\|^p z_\circ, \frac{(n^3 - n^p)t}{2}\right), \quad (3.1)$$

for all $x \in X$ and all $t > 0$. Furthermore, if for some $x \in X$ and all $n \in \mathbb{N}$, the mapping $h : \mathbb{R} \to Y$ defined by $h(s) = f(k^n sx)$ is intuitionistic fuzzy continuous. Then the mapping $s \mapsto g(sx)$ from $\mathbb{R}$ to Y is intuitionistic fuzzy continuous.

Proof. If we define $\varphi : X \times X \to Z$ by $\varphi(x, y) = (\|x\|^p + \|y\|^q)z_\circ$. Existence and uniqueness of the cubic mapping g satisfying (3.1) are deduced from Theorem 2.2 (see Example 2.3). Note that for each $x \in X, t \in \mathbb{R}$ and $n \in \mathbb{N}$, we have

$$\left.\begin{aligned}
&\mu\left(g(x) - \tfrac{f(k^n x)}{k^{3n}}, t\right) = \mu\left(\tfrac{g(k^n x)}{k^{3n}} - \tfrac{f(k^n x)}{k^{3n}}, t\right)\\
&= \mu(g(k^n x) - f(k^n x), k^{3n} t)\\
&\geq \mu'\left(k^{np}\|x\|^p z_\circ, \tfrac{k^{3n}(n^3-n^p)t}{2}\right)\\
&= \mu'\left(\|x\|^p z_\circ, \tfrac{k^{3n}(n^3-n^p)t}{2k^{np}}\right) \text{ and}\\
&\nu\left(g(x) - \tfrac{f(k^n x)}{k^{3n}}, t\right) \leq \nu'\left(\|x\|^p z_\circ, \tfrac{k^{3n}(n^3-n^p)t}{2k^{np}}\right).
\end{aligned}\right\} \quad (3.2)$$

Fix $x \in X$ and $s_\circ \in \mathbb{R}$. Given $\epsilon > 0$ and $0 < \alpha < 1$. From (3.2) it follows that

$$\mu\left(g(sx) - \frac{f(k^n sx)}{k^{3n}}, t\right) \geq \mu'\left(\|x\|^p z_\circ, \frac{k^{3n}(n^3 - n^p)t}{2|s|^p k^{np}}\right)$$

$$\geq \mu'\left(\|x\|^p z_\circ, \frac{k^{3n}(n^3 - n^p)t}{2(1+|s_\circ|)^p k^{np}}\right),$$

$$\nu\left(g(sx) - \frac{f(k^n sx)}{k^{3n}}, t\right) \leq \nu'\left(\|x\|^p z_\circ, \frac{k^{3n}(n^3 - n^p)t}{2|s|^p k^{np}}\right)$$

$$\leq \nu'\left(\|x\|^p z_\circ, \frac{k^{3n}(n^3 - n^p)t}{2(1+|s_\circ|)^p k^{np}}\right),$$

for all $|s - s_\circ| < 1$ and $s \in \mathbb{R}$. Since $\lim_{n\to\infty} \frac{k^{3n}(n^3-n^p)t}{2(1+|s_\circ|)^p k^{np}} = \infty$, there exists $n_\circ \in \mathbb{N}$ such that

$$\mu\left(g(sx) - \frac{f(k^{n_\circ} sx)}{k^{3n_\circ}}, \frac{\epsilon}{3}\right) \geq \alpha \text{ and}$$

$$\nu\left(g(sx) - \frac{f(k^{n_\circ} sx)}{k^{3n_\circ}}, \frac{\epsilon}{3}\right) \leq 1 - \alpha$$

for all $|s - s_\circ| < 1$ and $s \in \mathbb{R}$. By the intuitionistic fuzzy continuity of the mapping $t \to f(k^{n_\circ} tx)$, there exists $\delta < 1$ such that for each s with $0 < |s - s_\circ| < \delta$, we have

$$\mu\left(\frac{f(k^{n_\circ} sx)}{k^{3n_\circ}} - \frac{f(k^{n_\circ} s_\circ x)}{k^{3n_\circ}}, \frac{\epsilon}{3}\right) \geq \alpha \text{ and}$$

$$\nu\left(\frac{f(k^{n_\circ} sx)}{k^{3n_\circ}} - \frac{f(k^{n_\circ} s_\circ x)}{k^{3n_\circ}}, \frac{\epsilon}{3}\right) \leq 1 - \alpha.$$

It follows that

$$\mu(g(sx) - g(s_\circ x), \epsilon) \geq \mu\left(g(sx) - \frac{f(k^{n_\circ} sx)}{k^{3n_\circ}}, \frac{\epsilon}{3}\right)$$

$$*\mu\left(\tfrac{f(k^{n_\circ} sx)}{k^{3n_\circ}} - \tfrac{f(k^{n_\circ} s_\circ x)}{k^{3n_\circ}}, \tfrac{\epsilon}{3}\right) * \mu\left(\tfrac{f(k^{n_\circ} s_\circ x)}{k^{3n_\circ}} - g(s_\circ x), \tfrac{\epsilon}{3}\right) \geq \alpha$$

and $\nu(g(sx) - g(s_\circ x), \epsilon) \leq 1 - \alpha$, for each s with $0 < |s - s_\circ| < \delta$. Hence, the mapping $s \mapsto g(sx)$ is intuitionistic fuzzy continuous.

This completes the proof.

In the following theorem we prove a result similar to Theorem 3.2, for the case $p, q > 3$.

Theorem 3.3. Let X be a normed space and (Z, μ', ν') be an IFNS. Let (Y, μ, ν) be an intuitionistic fuzzy Banach space and $f : X \to Y$ be a (p, q)-approximately cubic mapping in the sense that for some p, q and some $z_\circ \in Z$

$$\mu\Big(kf(x+ky)+f(kx-y)-\frac{k(k^2+1)}{2}\Big[f(x+y)+f(x-y)\Big]$$
$$-(k^4-1)f(y), t\Big) \geq \mu'((\|x\|^p + \|y\|^q)z_\circ, t)$$

and

$$\nu\Big(kf(x+ky)+f(kx-y)-\frac{k(k^2+1)}{2}\Big[f(x+y)+f(x-y)\Big]$$
$$-(k^4-1)f(y), t\Big) \leq \nu'((\|x\|^p + \|y\|^q)z_\circ, t),$$

for all $x, y \in X$ and all $t > 0$. If $p, q > 3$, there exists a unique cubic mapping $g : X \to Y$ such that

$$\mu(g(x) - f(x), t) \geq \mu'\left(\|x\|^p z_\circ, \frac{(n^p - n^3)t}{2}\right) \quad \text{and}$$

$$\nu(g(x) - f(x), t) \leq \nu'\left(\|x\|^p z_\circ, \frac{(n^p - n^3)t}{2}\right), \qquad (3.3)$$

for all $x \in X$ and all $t > 0$. Furthermore, if for some $x \in X$ and all $n \in \mathbb{N}$, the mapping $h : \mathbb{R} \to Y$ defined by $h(s) = f(k^n sx)$ is intuitionistic fuzzy continuous. Then the mapping $s \mapsto g(sx)$ from $\mathbb{R}$ to Y is intuitionistic fuzzy continuous.

Proof. If we define $\varphi : X \times X \to Z$ by $\varphi(x, y) = (\|x\|^p + \|y\|^q)z_\circ$. Then

$$\mu'(\varphi(\frac{x}{k}, 0), t) = \mu'(\|x\|^p z_\circ, k^p t) \quad \text{and}$$

$$\nu'(\varphi(\frac{x}{k}, 0), t) = \nu'(\|x\|^p z_\circ, k^p t),$$

for all $x \in X$ and $t > 0$. Since $p > 3$, we have $\alpha = k^p > k^3$. By Theorem (2.1), there exists a unique cubic mapping g which satisfies (3.3). Rest of the proof can be done on the same lines as in Theorem 3.2.

This completes the proof.

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M. Mursaleen is a full professor of mathematics in the Department of Mathematics, Aligarh Muslim University, India. He has visited a number of countries, e.g. USA, UK, Hungary, Switzerland, Turkey, Serbia, Thailand, Libya, Jordan, Egypt, UAE and Saudi Arabia, under various academic programmes. He is referee and editor of many mathematical journals of international repute. His research interests are in the areas of Sequence Spaces, Summability, Approximation Theory, Functional Equations, Fixed Point Theory, and Measures of Noncompactness; and has published about 180 research papers in many reputed scientific journals and guided successfully 11 Ph.D. students.

Khursheed J. Ansari received the Master degree in Mathematics from Banaras Hindu University, Varanasi, India. Currently he has joined the Ph. D. Program in the Department of Mathematics, Aligarh Muslim University, Aligarh, India, in September 2012 under the supervision of Prof. M. Mursaleen. His research interests are in the stability of functional equations and wavelets.