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System of Mixed Variational-Like Inclusions and J^{η}-Proximal Operator Equations in Banach Spaces

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Abstract: In this paper, we introduce and study a system of mixed variational-like inclusions and a system of J^{η}-proximal operator equations in Banach spaces which contains variational inequalities, variational inclusions, resolvent equations, system of variational inequalities and system of variational inclusions in the literature as special cases. It is established that the system of mixed variational-like inclusions is equivalent to fixed point problems. We also establish a relationship between system of mixed variational like inclusions and system of J^{η}-proximal operator equations. By applying the notion of J^{η}-proximal mapping, we prove the existence of solutions and the convergence of p-step iterative algorithm for the system of J^{η}-proximal operator equations in reflexive Banach spaces.

Keywords: System of mixed variational-like inclusions, J^{η}-proximal mapping, J^{η}-proximal operator equations, Algorithm.

1 Introduction

In recent past, variational inequality theory has emerged as one of the main branches of mathematical and engineering sciences. This theory provides us with a simple, natural, unified and general frame work to study a wide class of unrelated problems arising in fluid through porous media, elasticity, transportation, economics, optimization, regional, physical, structural and applied sciences, etc., see [1,2,7,9,14,17,22] and the references therein.

Generalizations of variational inequality problems which are called system of variational inequality problems were introduced and studied. Bianchi [8], Cohen and Chaplais [10], Pang [18] and Ansari and Yao [2] considered a system of scalar variational inequalities and Pang showed that the traffic equilibrium problem, the spatial equilibrium problem, the Nash equilibrium problem and the general equilibrium programming problem can be modeled as a system of variational inequalities. Ansari et al. [3] introduced and studied a system of vector equilibrium problems and a system of vector variational inequalities using a fixed point theorem. Allevi et al. [5] considered a system of generalized vector variational inequalities and established some existence results with relative pseudo-monotonicity. Verma [23,24] introduced and studied some systems of variational inequalities and developed some iterative algorithms for approximating the solutions in Hilbert spaces.

Fang and Huang [11,12] and Fang, Huang and Thompson [13] introduced and studied a new system of variational inclusions involving H-monotone operators, H-accretive operators, (H,η)-monotone operators, respectively. Peng and Zhu [19] introduced and studied some new systems of generalized mixed quasi-variational inclusions involving (H,η)-monotone operators. Very recently Peng [20] introduced a system of generalized mixed quasi-variational-like inclusions with (H,η)-accretive operators, i.e., a family of generalized mixed quasi-variational-like inclusions defined on a product of sets in Banach Spaces.

The resolvent operator technique for solving systems of variational inequalities and systems of variational inclusions are interesting and important. The resolvent operator technique is used to establish equivalence between mixed variational inequalities and resolvent equations. The resolvent equation technique is used to develop powerful and efficient numerical techniques for solving mixed variational inequalities and related optimization problems.

This paper is devoted to generalize the resolvent equations by introducing system of J^η-proximal operator equations in Banach Spaces. A relationship between system of mixed variational-like inclusions and system of J^η-proximal operator equations is established. We propose a p-step iterative algorithm for computing the approximate solutions which converge to the exact solutions of the system of J^η-proximal operator equations.

2 Formulation and Preliminaries

Throughout the paper, we assume that E is a real Banach space with its norm$\|.\|$, E^* is the topological dual of E, d is the metric induced by the norm $\|.\|$, $CB(E)$ (respectively, 2^E) is the family of all nonempty closed and bounded subsets (respectively, all nonempty subsets) of $E, D(.,.)$ is the Hausdorff metric on $CB(E)$defined by

$$D(A,B) = max\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(A,y)\},$$

where $d(x,B) = \inf_{y\in B} d(x,y)$ and $d(A,y) = \inf_{x\in A} d(x,y)$.

We also assume that $\langle .,.\rangle$ is the duality pairing between E and E^* and $\mathcal{F}: E \longrightarrow 2^{E^*}$ is the normalized duality mapping defined by

$$\mathcal{F}(x) = \{f \in E^*: \langle x,f\rangle = \|x\|\|f\|, \|x\| = \|f\|\}, \qquad \forall\, x \in E.$$

Definition 2.1 Let $M: E \longrightarrow CB(E)$ be a set-valued mapping, $J: E \longrightarrow E^*$, $\eta: E \times E \longrightarrow E$ and $g: E \longrightarrow E$ be three single-valued mappings.

i. M is said to be D-Lipschitz continuous with constant $\lambda_{DM} \geq 0$, if

$$D\big(M(x), M(y)\big) \leq \lambda_{DM}\|x-y\|, \ \forall\, x,y \in E;$$

ii. η is said to be J-strongly accretive with constant $\alpha > 0$, if

$$\langle J(x) - J(y), \eta(x,y)\rangle \geq \alpha\|x-y\|^2, \ \forall\, x,y \in E;$$

iii. g is said to be k-strongly accretive ($k \in (0,1)$), if there exists $j(x-y) \in \mathcal{F}(x-y)$ such that

$$\langle j(x-y), g(x-y)\rangle \geq k\|x-y\|^2, \ \forall\, x,y \in E;$$

iv. η is said to be Lipschitz continuous with constant $\tau > 0$, if

$$\|\eta(x,y)\| \leq \tau\|x-y\|,\ \forall\, x.y \in E;$$

where $\mathcal{F}:\ E \longrightarrow 2^{E^*}$ is the normalized duality mapping.

Definition 2.2 Let $\eta: E \times E \longrightarrow E$ and $\varphi: E \longrightarrow R \cup \{+\infty\}$. A vector $w^* \in E^*$ is called an η-subgradient of φ at $x \in dom\,\varphi$, if

$$\langle w^*, \eta(y,x)\rangle \leq \varphi(y) - \varphi(x), \qquad \forall y \in E.$$

Each φ can be associated with the following η-subdifferential mapping $\partial_\eta \varphi$ defined by

$$\partial_\eta \varphi(x) = \begin{cases} \{w^* \in E^*:\ \langle w^*, \eta(y,x)\rangle \leq \varphi(y) - \varphi(x), for\ all\ y \in E\}, & x \in dom\varphi \\ \emptyset, & x \notin dom\varphi. \end{cases}$$

Definition 2.3[6] Let E be a Banach space with the dual space E^*, $J: E \longrightarrow E^*$, $\eta: E \times E \longrightarrow E$ be the mappings and $\varphi: E \longrightarrow R \cup \{+\infty\}$ be a proper, η-subdifferentiable (may not be convex) functional. If for any given point $x^* \in E^*$ and $\rho > 0$, there is a unique point $x \in E$ satisfying

$$\langle Jx - x^*, \eta(y,x)\rangle + \rho\varphi(y) - \rho\varphi(x) \geq 0, \quad \forall y \in E;$$

then the mapping $x^* \longrightarrow x$, denoted by $J_\rho^{\partial_\eta \varphi}(x^*)$ is said to be J^η-proximal mapping of φ. We have $x^* - Jx \in \rho\partial_\eta \varphi(x)$, it follows that

$$J_\rho^{\partial_\eta \varphi}(x^*) = \left(J + \rho\partial_\eta \varphi\right)^{-1}(x^*).$$

Definition 2.4 A functional $f: E \times E \longrightarrow R \cup \{+\infty\}$ is said to be 0-diagonally quasi-concave (in short 0-DQCV) in y, if for any finite subset $\{x_1, \dots, x_n\} \subset E$ and for any $y = \sum_{i=1}^{n} \lambda_i x_i$ with $\lambda_i \geq 0$ and $\sum_{i=1}^{n} \lambda_i = 1$,

$$\min_{1 \leq i \leq n} f(x_i, y) \leq 0.$$

Theorem 2.1[6] Let E be a reflexive Banach space with the dual space E^*, $J: E \longrightarrow E^*$ be a mapping, $\eta: E \times E \longrightarrow E$ be Lipschitz continuous with constant $\tau > 0$, J-strongly accretive with constant $\alpha > 0$ such that $\eta(x,y) = -\eta(y,x)$ for all $x, y \in E$ and for any $x \in E$, the function $h(y,x) = \langle x^* - Jx, \eta(y,x)\rangle$ is 0-DQCV in y. Let $\varphi: E \longrightarrow R \cup \{+\infty\}$ be lower semicontinuous, η-subdifferentiable, proper functional which may not be convex. Then for any $\rho > 0$ and any $x^* \in E^*$, there exists a unique $x \in E$ such that

$$\langle Jx - x^*, \eta(y,x)\rangle + \rho\varphi(y) - \rho\varphi(x) \geq 0,\ \forall\, y \in E.$$

That is, $x = J_\rho^{\partial_\eta \varphi}(x^*)$ and so the J^η-proximal mapping of φ is well defined and $\frac{\tau}{\alpha}$ -Lipschitz continuous.

The mapping $\eta: E \times E \longrightarrow E$ satisfies four conditions in Theorem 2.1. For conditions 1-3, we have the following Matlab programming and condition 4 is shown separately.

Example 2.1 Let $E = R$ and $J = I$

function value= $\eta(x, y)$
if $abs(x * y) < 1/4$
value = $2 * \ x - 2 * y$;
elseif $abs(x * y) > 1/4$ & $abs(x * y) < 1/2$
value= $8 * abs(\ x * y) * (x - y)$;
elseif $abs(\ x * y) >= 1/2$
value $4 * (x - y)$;
end

Then it is easy to see that:
(1) $\langle \eta(x,y), x - y\rangle \geq 2|x - y|^2, \quad \forall x, y \in R,$ i.e., η is 2-strongly accretive;
(2) $\eta(x, y) = -\eta(y, x), \quad \forall x, y \in R;$
(3) $|\eta(x, y)| \leq 4|x - y|, \quad \forall x, y \in R,$ i.e., η is 4-Lipschitz continuous;
(4) We will show that for any $x \in R$, the function $\langle h(y, u) = x - u, \eta(y, u)\rangle = (x - u)\eta(y, u)$ is 0-DQCV in y.

Suppose that it is false, then there exists a finite set $\{y_1, y_2, \ldots \ldots, y_n\}$ and $u_0 = \sum_{i=1}^{n} \lambda_i y_i$ with $\lambda_i \geq 0$ and $\sum_{i=1}^{n} \lambda_i = 1$ such that for each $i = 1, \ldots \ldots, n$

$$0 < h(y_i, u_0) = \begin{cases} (x - u_0)(2y_i - 2\,u_0), & if\ |y_i u_0| < {}^1/_4, \\ (x - u_0)8|y_i u_0|(y_i - u_0), & {}^1/_4 \leq if\ |y_i u_0| < {}^1/_2, \\ 4(x - u_0)(y_i - u_0), & if\ {}^1/_2 \leq |y_i u_0|. \end{cases}$$

It follows that $(x - u_0)(2y_i - 2u_0) > 0$ for each $i = 1, 2, \ldots \ldots, n$, and hence we have

$$0 < \sum_{i=1}^{n} \lambda_i (x - u_0)(2y_i - 2\,u_0) = (x - u_0)(2\,u_0 - 2\,u_0) = 0$$

which is not possible. Hence $h(y, u)$ is 0-DQCV in y. Therefore, η satisfies all conditions in Theorem 2.1.

Proposition 2.1[21] Let E be a real Banach space and $\mathcal{F}: E \longrightarrow 2^{E^*}$ be a normalized duality mapping. Then, for any $x, y \in E$,

$$\|x - y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y)\rangle, \ \forall\, j(x + y) \in \mathcal{F}(x + y).$$

For $i = 1,2, \ldots, p$, Let $H_i, f_i: \prod_{j=1}^{p} E_j \longrightarrow E_i^*, g_i: E_i \longrightarrow E_i, \eta_i: E_i \times E_i \longrightarrow E_i$ be single-valued mappings and $M_i: E_i \longrightarrow CB(E_i), T_{1i}: E_1 \longrightarrow CB(E_i), T_{2i}: E_2 \longrightarrow CB(E_i), \ldots\ldots, T_{pi}: E_p \longrightarrow CB(E_i)$ be the set-valued mappings. Let $\varphi_i: E_i \longrightarrow R \cup \{+\infty\}$ be lower-semicontinuous functional on E_i (may not be convex) satisfying $g_i(x_i) \cap dom(\partial_{\eta_i}\varphi_i) \neq \emptyset$, where $\partial_{\eta_i}\varphi_i$ is η_i-subdifferential of φ_i. We consider the following system of mixed variational-like inclusions:

Find $(u_1, u_2, \ldots \ldots u_p, y_{11}, y_{12}, \ldots \ldots, y_{1p}, y_{21}, y_{22}, \ldots \ldots y_{2p}, y_{p1}, y_{p2}, \ldots \ldots y_{pp})$ such that for each $i = 1,2, \ldots \ldots p, x_i \in E_i, u_i \in M(x_i), y_{1i} \in T_{1i}(x_1), y_{2i} \in T_{2i}(x_2), \ldots \ldots, y_{pi} \in T_{pi}(x_p)$ and

$$\langle H_i(u_1, u_2, \ldots \ldots u_p) - f_i(y_{i1}, y_{i2}, \ldots \ldots, y_{ip}), \eta_i(a_i, g(x_i))\rangle \geq \varphi_i(g(x_i)) - \varphi_i(a_i), \ \forall a_i \in E_i. \quad (2.1)$$

Below are some special cases of problem (2.1).

(1) For $i = 1,2, \dots \dots p$ if $E_i = \mathcal{H}_i$ a Hilbert space and $g_i, H_i = I_i$ the identity mapping, M_i is a single-valued mapping, $f \equiv 0$, then problem (2.1) reduces to the following system of variational-like inclusions: to find $(x_1, x_2, \dots \dots x_p) \in \prod_{i=1}^{p} \mathcal{H}_i$ such that

$$\langle M_i\big((x_1, x_2, \dots \dots x_p), \eta_i(a_i, x_i)\big\rangle \geq \varphi_i(x_i) - \varphi_i(a_i),\ \forall a_i \in E_i. \tag{2.2}$$

(2) For $i = 1,2, \dots \dots p$, if $\eta_i(a_i, x_i) = a_i - x_i$, then problem (2.2) reduces to the following system of variational inclusions: to find $(x_1, x_2, \dots \dots x_p) \in \prod_{i=1}^{p} \mathcal{H}_i$ such that

$$\langle M_i\big(x_1, x_2, \dots \dots x_p\big), a_i - x_i\rangle \geq \varphi_i(x_i) - \varphi_i(a_i),\ \forall a_i \in E_i. \tag{2.3}$$

(3) For $i = 1,2, \dots \dots p$, if $\varphi_i = \delta_{k_i}(x_i)$, for all $x_i \in \mathcal{H}_i$, where $k_i \subset \mathcal{H}_i$ is a nonempty, closed and convex subset and δ_{k_i} denotes the indicator of k_i, then the problem (2.3) reduces to the following system of variational inequalities: to find $(x_1, x_2, \dots \dots x_p) \in \prod_{i=1}^{p} \mathcal{H}_i$ such that

$$\langle M_i\big(x_1, x_2, \dots \dots x_p\big), a_i - x_i\rangle \geq 0,\ \forall a_i \in K_i. \tag{2.4}$$

Problem (2.4) is introduces and studied in [2, 8, 9, 18]. For $p = 2$ Problems (2.2), (2.3) and (2.4) becomes the Problems (3.2), (3.3) and (3.4) of [13], respectively.

The following numerical example illustrates the idea of problem (2.1).

Example 2.2 Let $E_i = R$ for $i = 1, 2$ and let the pairing $\langle l, x\rangle$ denotes the value of l at x. Consider $H_1, H_2 : R \times R \to R, f_1, f_2 : R \times R \to R$, $g_i = I, i = 1, 2$ (the identity mapping), $\eta_1, \eta_2 : R \times R \to R$ be single-valued mappings and $M_1, M_2 : R \to CB(R); T_{11}, T_{12}, T_{21}, T_{22} : R \to CB(R)$, be the set-valued mappings. Let $\varphi_1, \varphi_2 : R \to R \cup \{\infty\}$ be lower-semicontinuous functionals on R (may not be convex). We take,

(i) $M_1(x_1) = M_2(x_2) = [0, 2\pi],\ \forall x_1, x_2 \in R$;

(ii) $H_1(u_1, u_2) = u_1 + \sin u_2, H_2(u_1, u_2) = u_1 + \cos u_2,\ \ \forall u_1 \in M_1(x_1), u_2 \in M_2(x_2) T_{11}(x_1) =$ $T_{12}(x_1) = T_{21}(x_2) = T_{22}(x_2) = [0,1], f_1(y_{11}, y_{12}) = -\left(1 + \frac{y_{12}}{2}\right)$, $f_2(y_{21}, y_{22}) = -\left(1 + \frac{y_{22}}{2}\right), \forall y_{11} \in T_{11}(x_1), y_{12} \in T_{12}(x_1), y_{21} \in T_{21}(x_2), y_{22} \in T_{22}(x_2)$;

(iii) $\eta_1(a_1, x_1) = a_1 - x_1, \eta_2(a_2, x_2) = a_2 - x_2,\ \forall a_1 \geq x_1, a_2 \geq x_2,\ a_1, a_2, x_1, x_2 \in R$;

(iv) $\varphi_1(x_1) = \begin{cases} 1 + x_1 & for\ x_1 \neq 2 \\ 2 & for\ x_1 = 2 \end{cases}$ and $\varphi_2(x_2) = \begin{cases} 1 + x_2 & for\ x_2 \neq 3 \\ 3 & for\ x_2 = 3 \end{cases}$

for all $a_1 \geq x_1, a_2 \geq x_2, a_1, a_2, x_1, x_2 \in R$ where both φ_1, φ_2 are lower-semicontinuous functionals and which are not convex.

Then the following system of mixed variational-like inclusions for $i = 1, 2$ of problem (2.1) is satisfied.

$$\langle u_1 + \sin u_2 - \left(-\left(1 + \frac{y_{12}}{2}\right)\right), a_1 - x_1\rangle \geq \varphi_1(x_1) - \varphi_1(a_1)$$

$$\langle u_1+\cos u_2-\left(-\left(1+\frac{y_{22}}{2}\right)\right),a_2-x_2\rangle\geq\varphi_2(x_2)-\varphi_2(a_2).$$

In connection with problem (2.1), we consider the following system of J^{η}-proximal operator equations:

Find $(z_1,z_2,\dots\dots,z_p,u_1,u_2,\dots\dots,u_p,y_{11},y_{12},\dots\dots,y_{1p},y_{21},y_{22},\dots\dots,y_{2p},\dots\dots,y_{p1},y_{p2},\dots\dots,y_{pp})$ such that for $i=1,2,\dots\dots,p$, $z_i\in E_i^*,x_i\in E_i,u_i\in M_i(x_i),y_{1i}\in T_{1i}(x_1),y_{2i}\in T_{2i}(x_2),\dots\dots,y_{pi}\in T_{pi}(x_p)$ such that

$$[H_i(u_1,u_2,\dots\dots,u_p)-f_i(y_{i1},y_{i2},\dots\dots,y_{ip})]+\rho_i^{-1}R_{\rho_i}^{\partial_{\eta_i}\varphi_i}(z_i)=0, \quad (2.5)$$

where $\rho_i>0$ is a constant, $R_{\rho_i}^{\partial_{\eta_i}\varphi_i}=I_i-J_i\left[J_{\rho_i}^{\partial_{\eta_i}\varphi_i}(z_i)\right]$, where $\left[J_i\left(J_{\rho_i}^{\partial_{\eta_i}\varphi_i}\right)\right](z_i)$, I_i is the identity mapping, $J_i:E_i\to E_i^*$ and $J_{\rho_i}^{\partial_{\eta_i}\varphi_i}(x_i^*)=\left(J_i+\rho_i\partial_{\eta_i}\varphi_i\right)^{-1}(x_i^*)$.

3 p-step Iterative algorithm and convergence result

We mention the following equivalence between the system of mixed variational-like inclusions and a fixed point problem.

Lemma 3.1 $(u_1,u_2,\dots\dots u_p,y_{11},y_{12},\dots\dots,y_{1p},y_{21},y_{22},\dots\dots y_{2p},y_{p1},y_{p2},\dots\dots y_{pp})$, where for each $i=1,2,\dots\dots p$, $x_i\in E_i,u_i\in M(x_i),y_{1i}\in T_{1i}(x_1),y_{2i}\in T_{2i}(x_2),\dots\dots,y_{pi}\in T_{pi}(x_p)$ is a solution of system of mixed variational-like inclusions (2.1) if and only if satisfies the following:

$$g_i(x_i)=J_{\rho_i}^{\partial_{\eta_i}\varphi_i}[J_i(g_i(x_i))-\rho_i[H_i(u_1,u_2,\dots\dots u_p)-f_i(y_{i1},y_{i2},\dots\dots,y_{ip})]] \quad (3.1)$$

Proof The proof follows directly from Definition 2.3.

Now we will show that the system of mixed variational-like inclusions is equivalent to the system of J^{η}-proximal operator equations.

Lemma 3.2 The system of mixed variational-like inclusions (2.1) has a solution $(u_1,u_2,\dots\dots u_p,y_{11},y_{12},\dots\dots,y_{1p},y_{21},y_{22},\dots\dots y_{2p},y_{p1},y_{p2},\dots\dots y_{pp})$, with $x_i\in E_i,u_i\in M(x_i),y_{1i}\in T_{1i}(x_1),y_{2i}\in T_{2i}(x_2),\dots\dots,y_{pi}\in T_{pi}(x_p)$ if and only if the system of J^{η}-proximal operator equations (2.5) has a solution $(z_1,z_2,\dots\dots z_p,u_1,u_2,\dots\dots u_p,y_{11},y_{12},\dots\dots,y_{1p},y_{21},y_{22},\dots\dots y_{2p},y_{p1},y_{p2},\dots\dots y_{pp})$ with $z_i\in E_i^*,x_i\in E_i,u_i\in M_i(x_i),y_{1i}\in T_{1i}(x_1),y_{2i}\in T_{2i}(x_2),\dots\dots,y_{pi}\in T_{pi}(x_p)$, where

$$g_i(x_i)=J_{\rho_i}^{\partial_{\eta_i}\varphi_i}(z_i)$$

and

$$z_i=J_i(g_i(x_i))-\rho_i[H_i(u_1,u_2,\dots\dots u_p)-f_i(y_{i1},y_{i2},\dots\dots,y_{ip})].$$

Proof Let $(u_1,u_2,\dots\dots u_p,y_{11},y_{12},\dots\dots,y_{1p},y_{21},y_{22},\dots\dots y_{2p},y_{p1},y_{p2},\dots\dots y_{pp})$ be a solution of system of mixed variational-like inclusions (2.1). Then by Lemma 3.1, it is a solution of following equation.

$$g_i(x_i)=J_{\rho_i}^{\partial_{\eta_i}\varphi_i}[J_i(g_i(x_i))-\rho_i[H_i(u_1,u_2,\dots\dots u_p)-f_i(y_{i1},y_{i2},\dots\dots,y_{ip})]]$$

Using the fact

$$R_{\rho_i}^{\partial_{\eta_i}\varphi_i}=\left[I_i-J_i\left(J_{\rho_i}^{\partial_{\eta_i}\varphi_i}\right)\right],$$

we have,

$$R_{\rho_i}^{\partial_{\eta_i}\varphi_i}[J_i(g_i(x_i)) - \rho_i[H_i(u_1,u_2,\ldots\ldots u_p) - f_i(y_{i1},y_{i2},\ldots\ldots,y_{ip})]]$$
$$= J_i(g_i(x_i)) - \rho_i[H_i(u_1,u_2,\ldots\ldots u_p) - f_i(y_{i1},y_{i2},\ldots\ldots,y_{ip})]$$
$$- J_i\left(J_{\rho_i}^{\partial_{\eta_i}\varphi_i}\{J_i(g_i(x_i)) - \rho_i[H_i(u_1,u_2,\ldots\ldots u_p) - f_i(y_{i1},y_{i2},\ldots\ldots,y_{ip})]\}\right)$$
$$= J_i(g_i(x_i)) - \rho_i[H_i(u_1,u_2,\ldots\ldots u_p) - f_i(y_{i1},y_{i2},\ldots\ldots,y_{ip})] - J_i(g_i(x_i)$$
$$= -\rho_i[H_i(u_1,u_2,\ldots\ldots u_p) - f_i(y_{i1},y_{i2},\ldots\ldots,y_{ip})]$$

which implies that

$$[H_i(u_1,u_2,\ldots\ldots u_p) - f_i(y_{i1},y_{i2},\ldots\ldots,y_{ip})] + \rho_i^{-1}R_{\rho_i}^{\partial_{\eta_i}\varphi_i}(z_i) = 0,$$

with $z_i = J_i(g_i(x_i)) - \rho_i[H_i(u_1,u_2,\ldots\ldots u_p) - f_i(y_{i1},y_{i2},\ldots\ldots,y_{ip})$, i.e., $(z_1,z_2,\ldots\ldots z_p,u_1,u_2,\ldots\ldots u_p,y_{11},y_{12},\ldots\ldots,y_{1p},y_{21},y_{22},\ldots\ldots y_{2p},y_{p1},y_{p2},\ldots\ldots y_{pp})$ is the solution of the system of J^{η}-proximal operator equations (2.5).

Conversely, let $(z_1,z_2,\ldots\ldots z_p,u_1,u_2,\ldots\ldots u_p,y_{11},y_{12},\ldots\ldots,y_{1p},y_{21},y_{22},\ldots\ldots y_{2p},y_{p1},y_{p2},\ldots\ldots y_{pp})$ is the solution of the system of J^{η}-proximal operator equations (2.5), then

$$\rho_i[H_i(u_1,u_2,\ldots\ldots u_p) - f_i(y_{i1},y_{i2},\ldots\ldots,y_{ip})] = -R_{\rho_i}^{\partial_{\eta_i}\varphi_i}(z_i)$$
$$= J_i\left(J_{\rho_i}^{\partial_{\eta_i}\varphi_i}(z_i)\right) - z_i.$$

$$\rho_i[H_i(u_1,u_2,\ldots\ldots u_p) - f_i(y_{i1},y_{i2},\ldots\ldots,y_{ip})]$$
$$= J_i\left(J_{\rho_i}^{\partial_{\eta_i}\varphi_i}[J_i(g_i(x_i)) - \rho_i[H_i(u_1,u_2,\ldots\ldots u_p) - f_i(y_{i1},y_{i2},\ldots\ldots,y_{ip})]]\right)$$
$$- J_i(g_i(x_i)) + \rho_i[H_i(u_1,u_2,\ldots\ldots u_p) - f_i(y_{i1},y_{i2},\ldots\ldots,y_{ip})].$$

which implies that

$$J_i(g_i(x_i)) = J_i\left(J_{\rho_i}^{\partial_{\eta_i}\varphi_i}[J_i(g_i(x_i)) - \rho_i[H_i(u_1,u_2,\ldots\ldots u_p) - f_i(y_{i1},y_{i2},\ldots\ldots,y_{ip})]]\right)$$

and thus

$$g_i(x_i) = J_{\rho_i}^{\partial_{\eta_i}\varphi_i}[J_i(g_i(x_i)) - \rho_i[H_i(u_1,u_2,\ldots\ldots u_p) - f_i(y_{i1},y_{i2},\ldots\ldots,y_{ip})]]$$

i.e., $(u_1,u_2,\ldots\ldots u_p,y_{11},y_{12},\ldots\ldots,y_{1p},y_{21},y_{22},\ldots\ldots y_{2p},y_{p1},y_{p2},\ldots\ldots y_{pp})$ is the solution of the system of J^{η}-proximal operator equations (2.1).

Alternative Proof Let $z_i = J_i(g_i(x_i)) - \rho_i[H_i(u_1,u_2,\ldots\ldots u_p) - f_i(y_{i1},y_{i2},\ldots\ldots,y_{ip})]$ then from (3.1), we have

$$g_i(x_i) = J_{\rho_i}^{\partial_{\eta_i}\varphi_i}(z_i)$$

and

$$z_i = J_i\left[J_{\rho_i}^{\partial_{\eta_i}\varphi_i}(z_i)\right] - \rho_i[H_i(u_1,u_2,\ldots\ldots u_p) - f_i(y_{i1},y_{i2},\ldots\ldots,y_{ip})].$$

By using the fact that

$$J_i\left[J_{\rho_i}^{\partial_{\eta_i}\varphi_i}(z_i)\right] = \left[J_i\left(J_{\rho_i}^{\partial_{\eta_i}\varphi_i}\right)\right](z_i),$$

it follows that

$$[H_i(u_1,u_2,\ldots\ldots u_p) - f_i(y_{i1},y_{i2},\ldots\ldots,y_{ip})] + \rho_i^{-1}R_{\rho_i}^{\partial_{\eta_i}\varphi_i}(z_i) = 0,$$

which is the required system of J^{η}-proximal operator equations.

Now we invoke Lemma 3.1 and Lemma 3.2 to suggest the following p-step iterative algorithm for solving system of J^{η}-proximal operator equations (2.5).

Algorithm 3.1 For $i = 1,2,\ldots\ldots p$ and for any given $z_i^0 \in E_i^*, x_i^0 \in E_i, u_i^0 \in M_i(x_i^0), y_{1i}^0 \in T_{1i}(x_1^0), y_{2i}^0 \in T_{2i}(x_2^0)\ldots\ldots, y_{pi}^0 \in T_{pi}(x_p^0)$, let

$$z_i{}^1 = J_i(g_i(x_i{}^0)) - \rho_i\left[H_i(u_1^0, u_2{}^0, \ldots\ldots u_p{}^0) - f_i(y_{i1}{}^0, y_{i2}{}^0, \ldots\ldots, y_{ip}{}^0)\right].$$

Take $z_i^1 \in E_i^*, x_i^1 \in E_i$ such that $g_i(x_i{}^1) = J_{\rho_i}^{\partial\eta_i\varphi_i}(z_i^1)$.

Since, $u_i^0 \in M_i(x_i^0), y_1^0 \in T_{1i}(x_1^0), y_2^0 \in T_{2i}(x_2^0)\ldots\ldots, y_p^0 \in T_{pi}(x_p^0)$, by Nadler [14], there exists $u_i^1 \in M_i(x_i^1), y_{1i}^1 \in T_{1i}(x_1^1), y_{2i}^1 \in T_{2i}(x_2^1)\ldots\ldots, y_{pi}^1 \in T_{pi}(x_p^1)$, such that for each $i = 1,2,\ldots\ldots p$, we have

$\|u_i^0 - u_i^1\| \le (1+1)D\left(M_i(x_i^0), M_i(x_i^1)\right);$

$$\|y_{1i}^0 - y_{1i}^1\| \le (1+1)D\left(T_{1i}(x_1^0), T_{1i}(x_1^1)\right);$$
$$\|y_{2i}^0 - y_{2i}^1\| \le (1+1)D\left(T_{2i}(x_2^0), T_{2i}(x_2^1)\right);$$
$$\vdots$$
$$\|y_{pi}^0 - y_{pi}^1\| \le (1+1)D\left(T_{pi}(x_p^0), T_{pi}(x_p^1)\right).$$

Let $z_i^2 \in E_i^*, x_i^2 \in E_i$, such that

$z_i{}^2 = J_i(g_i(x_i{}^1)) - \rho_i\left[H_i(u_1^1, u_2{}^1, \ldots\ldots u_p{}^1) - f_i(y_{i1}{}^1, y_{i2}{}^1, \ldots\ldots, y_{ip}{}^1)\right]$

and $g_i(x_i{}^2) = J_{\rho_i}^{\partial\eta_i\varphi_i}(z_i^2)$, again by Nadler [15], there exists $u_i^2 \in M_i(x_i^2), y_{1i}^2 \in T_{1i}(x_1^2), y_{2i}^2 \in T_{2i}(x_2^2)\ldots\ldots, y_{pi}^2 \in T_{pi}(x_p^2)$, such that for each $i = 1,2,\ldots\ldots p$,

$\|u_i^1 - u_i^2\| \le (1+1)D\left(M_i(x_i^1), M_i(x_i^2)\right);$

$$\|y_{1i}^1 - y_{1i}^2\| \le (1+1)D\left(T_{1i}(x_1^1), T_{1i}(x_1^2)\right);$$
$$\|y_{2i}^1 - y_{2i}^2\| \le (1+1)D\left(T_{2i}(x_2^1), T_{2i}(x_2^2)\right);$$
$$\vdots$$
$$\|y_{pi}^1 - y_{pi}^2\| \le (1+1)D\left(T_{pi}(x_p^1), T_{pi}(x_p^2)\right).$$

Continuing the above process inductively, we can obtain the following p-step iterative algorithm for solving system of J^{η}-proximal operator equations.

For $i = 1,2,\ldots\ldots p, z_i^0 \in E_i^*, x_i^0 \in E_i$, compute the sequences $\{z_i^n\}, \{x_i^n\}, \{u_i^n\}, \{y_{1i}^n\}, \{y_{2i}^n\}\ldots\ldots\{y_{pi}^n\}$ by the following p-step iterative schemes:

$$g_i(x_i{}^n) = J_{\rho_i}^{\partial\eta_i\varphi_i}(z_i^n), \tag{3.2}$$
$$u_i^n \in M_i(x_i^n): \quad \|u_i^n - u_i^{n-1}\| \le \left(1+\frac{1}{n}\right)D\left(M_i(x_i^n), M_i(x_i^{n-1})\right), \tag{3.3}$$
$$y_{1i}^n \in T_{1i}(x_1^n): \quad \|y_{1i}^n - y_{1i}^{n-1}\| \le \left(1+\frac{1}{n}\right)D\left(T_{1i}(x_1^n), T_{1i}(x_1^{n-1})\right), \tag{3.4}$$

$$y_{2i}^n \in T_{2i}(x_2^n): \quad \|y_{2i}^n - y_{2i}^{n-1}\| \le \left(1+\frac{1}{n}\right) D\left(T_{2i}(x_2^n), T_{1i}(x_2^{n-1})\right), \tag{3.5}$$

$$\vdots$$

$$y_{pi}^n \in T_{pi}(x_1^n): \quad \|y_{pi}^n - y_{pi}^{n-1}\| \le \left(1+\frac{1}{n}\right) D\left(T_{pi}(x_p^n), T_{pi}(x_p^{n-1})\right), \tag{3.6}$$

And

$$z_i^{n+1} = J_i(g_i(x_i^n)) - \rho_i[H_i(u_1^n, u_2^n, \ldots \ldots u_p^n) - f_i(y_{i1}^n, y_{i2}^n, \ldots \ldots, y_{ip}^n)], \tag{3.7}$$

where $n = 0,1,2, \ldots \ldots$ and $\rho_i > 0$ is a constant.

Theorem 3.1 For $i = 1,2, \ldots \ldots p$, let E_i is a reflexive Banach space and E_i^* be its dual. Suppose $\eta_i: E_i \times E_i \to E_i$ be Lipschitz continuous with constants $\tau_i > 0$ such that $\eta_i(x_1, x_2) = -\eta_i(x_2, x_1)$ for all $x_1, x_2 \in E_i$, J_i-strongly accretive with constant $\alpha_i > 0$ and for any $x_1 \in E_i$, the function $h_i(x_2, x_1) = \langle x_1^* - J_i x, \eta_i(x_2, x_1)\rangle$ is 0-DQCV in x_2. Let $J_i: E_i \to E_i^*$ be λ_{J_i}-Lipschitz continuous, $g_i: E_i \to E_i$ is λ_{g_i}-Lipschitz continuous and k_i-strongly accretive, $H_i: \prod_{j=1}^p E_j \to E_i^*$ is λ_{H_i}-Lipschitz continuous, $M_i: E_i \to CB(E_i)$ $(j = 1,2, \ldots \ldots p)$ is λ_{M_i}-D-Lipschitz continuous and $f_i: \prod_{j=1}^p E_j \to E_i^*$ is $\lambda_{f_{ij}}$-Lipschitz continuous in the j^{th}-argument. Let $T_{1i}: E_1 \to CB(E_i)$, $T_{2i}: E_2 \to CB(E_i), \ldots \ldots, T_{pi}: E_p \to CB(E_i)$ be $\lambda_{T_{1i}}$-D-Lipschitz continuous, $\lambda_{T_{2i}}$-D-Lipschitz continuous,....., $\lambda_{T_{pi}}$-D-Lipschitz continuous, respectively. Suppose $\varphi_i: E_i \longrightarrow R \cup \{+\infty\}$ be lower-semicontinuous, η_i-subdifferentiable, proper functional on E_i satisfying $f_i(E_i) \cap dom(\partial_{\eta_i}\varphi_i) \neq \emptyset$ and if there exists a constant $\rho_i > 0$ such that

$$\begin{cases} \frac{\lambda_{J_1}\lambda_{g_1} + \rho_1(\lambda_{H_1} + \lambda_{M_1}) + \rho_1(\lambda_{f_{11}}\lambda_{T_{11}})\tau_1}{\alpha_1\sqrt{1+2k_1}} < 1, \\ \frac{\lambda_{J_2}\lambda_{g_2} + \rho_2(\lambda_{H_2} + \lambda_{M_2}) + \rho_2(\lambda_{f_{22}}\lambda_{T_{22}})\tau_2}{\alpha_2\sqrt{1+2k_2}} < 1, \\ \vdots \\ \frac{\lambda_{J_p}\lambda_{g_p} + \rho_p(\lambda_{H_p} + \lambda_p) + \rho_p(\lambda_{f_{pp}}\lambda_{T_{pp}})\tau_p}{\alpha_p\sqrt{1+2k_p}} < 1. \end{cases} \tag{3.8}$$

Then the problem (2.5) admits a solution $(z_1, z_2, \ldots \ldots z_p, u_1, u_2, \ldots \ldots u_p, y_{11}, y_{12}, \ldots \ldots, y_{1p}, y_{21}, y_{22}, \ldots \ldots y_{2p}, y_{p1}, y_{p2}, \ldots \ldots y_{pp})$, and the sequences $\{z_1^n\}, \{z_2^n\}, \ldots, \{z_p^n\}, \{x_i^n\}, \{x_1^n\}, \{x_2^n\}, \ldots, \{x_p^n\}, \{u_1^n\}, \{u_2^n\}, \ldots, \{u_p^n\}, \{y_{11}^n\}, \{y_{12}^n\}, \ldots, \{y_{1p}^n\}$, $\{y_{21}^n\}, \{y_{22}^n\}, \ldots, \{y_{2p}^n\}, \ldots . \{y_{p1}^n\}, \{y_{p2}^n\}, \ldots \{y_{pp}^n\}$ generated by Algorithm 3.1, converge strongly to $z_1, z_2, \ldots z_p, x_1, x_2, \ldots x_p, u_1, u_2, \ldots u_p, y_{11}, y_{12}, \ldots y_{1p}, y_{21}, y_{22}, \ldots y_{2p}, \ldots, y_{p1}, y_{p2,\ldots,}y_{pp}$, respectively.

Proof From Algorithm 3.1, we have

$$\begin{aligned} \|z_i^{n+1} - z_i^n\| = {} & \|J_i(g_i(x_i^n)) - \rho_i[H_i(u_1^n, u_2^n, \ldots \ldots u_p^n) - f_i(y_{i1}^n, y_{i2}^n, \ldots \ldots, y_{ip}^n)] \\ & -\{J_i(g_i(x_i^{n-1})) - \rho_i[H_i(u_1^{n-1} u_2^{n-1} \ldots \ldots u_p^{n-1}) - f_i(y_{i1}^{n-1} y_{i2}^{n-1} \ldots \ldots, y_{ip}^{n-1})]\}\| \\ \le {} & \|J_i(g_i(x_i^n)) - J_i(g_i(x_i^{n-1}))\| + \rho_i\|H_i(u_1^n, u_2^n, \ldots \ldots u_p^n) \end{aligned}$$

$$-f_i(y_{i1}^n, y_{i2}^n, \ldots\ldots, y_{ip}^n) - [H_i(u_1^{n-1} u_2^{n-1} \ldots\ldots u_p^{n-1}) - f_i(y_{i1}^{n-1} y_{i2}^{n-1} \ldots\ldots, y_{ip}^{n-1})]$$
$$\leq \|J_i(g_i(x_i^n)) - J_i(g_i(x_i^{n-1}))\| + \rho_i \|H_i(u_1^n, u_2^n, \ldots\ldots u_p^n) - H_i(u_1^{n-1} u_2^{n-1} \ldots\ldots u_p^{n-1})\|$$
$$+\rho_i \|f_i(y_{i1}^n, y_{i2}^n, \ldots\ldots, y_{ip}^n) - f_i(y_{i1}^{n-1} y_{i2}^{n-1} \ldots\ldots, y_{ip}^{n-1})\|. \quad (3.9)$$

Since J_i is λ_{J_i}-Lipschitz continuous and g_i is λ_{g_i}-Lipschitz continuous, we have

$$\|J_i(g_i(x_i^n)) - J_i(g_i(x_i^{n-1}))\| \leq \lambda_{J_i} \lambda_{g_i} \|x_i^n - x_i^{n-1}\|. \quad (3.10)$$

Since H_i is λ_{H_i} -Lipschitz continuous and M_i is λ_{M_i}-D-Lipschitz continuous, we have

$$\begin{aligned}
&\|H_i(u_1^n, u_2^n, \ldots\ldots u_p^n) - H_i(u_1^{n-1} u_2^{n-1} \ldots\ldots u_p^{n-1})\| \\
&\leq \|H_i(u_1^n, u_2^n, \ldots\ldots u_p^n) - H_i(u_1^{n-1} u_2^n, \ldots\ldots u_p^n)\| \\
&\quad + \|H_i(u_1^{n-1} u_2^n, \ldots\ldots u_p^n) - H_i(u_1^{n-1} u_2^{n-1} u_3^n, \ldots\ldots u_p^n)\| \\
&\quad + \cdots \ldots\ldots\ldots\ldots\ldots\ldots \\
&\quad + \|H_i(u_1^{n-1} u_2^{n-1} \ldots\ldots, u_p^{n-1}, u_p^n) - H_i(u_1^{n-1} u_2^{n-1} \ldots\ldots u_p^{n-1})\| \\
&\leq \textstyle\sum_{i=1}^p \lambda_{H_i} \|u_i^n - u_i^{n-1}\| \\
&\leq \textstyle\sum_{i=1}^p \lambda_{H_i} \left(1 + \frac{1}{n}\right) D(M_i(x_i^n), M_i(x_i^{n-1}) \\
&\leq \textstyle\sum_{i=1}^p \lambda_{H_i} \left(1 + \frac{1}{n}\right) \lambda_{M_i} \|x_i^n - x_i^{n-1}\|. \quad (3.11)
\end{aligned}$$

It follows from the $\lambda_{f_{ij}}$-Lipschitz continuity of f_i in the j^{th}-argument and $\lambda_{T_{ij}}$-D-Lipschitz continuity of T_{ij} , that

$$\begin{aligned}
&\|f_i(y_{i1}^n, y_{i2}^n, \ldots\ldots, y_{ip}^n) - f_i(y_{i1}^{n-1} y_{i2}^{n-1} \ldots\ldots, y_{ip}^{n-1})\| \\
&\leq \|f_i(y_{i1}^n, y_{i2}^n, \ldots\ldots, y_{ip}^n) - f_i(y_{i1}^{n-1} y_{i2}^n, \ldots\ldots, y_{ip}^n)\| \\
&\quad + \|f_i(y_{i1}^{n-1} y_{i2}^n, \ldots\ldots, y_{ip}^n) - f_i(y_{i1}^{n-1} y_{i2}^{n-1} \ldots\ldots, y_{ip}^n)\| \\
&\quad + \|f_i(y_{i1}^n, y_{i2}^n, \ldots\ldots, y_{ip}^n) - f_i(y_{i1}^{n-1} y_{i2}^{n-1} \ldots\ldots, y_{ip}^{n-1})\| \\
&\quad + \cdots \ldots\ldots\ldots\ldots \\
&\quad + \|f_i(y_{i1}^{n-1} y_{i2}^{n-1}, \ldots\ldots, y_{ip}^n) - f_i(y_{i1}^{n-1} y_{i2}^{n-1} \ldots\ldots, y_{ip}^{n-1})\| \\
&\leq \textstyle\sum_{i=1}^p \lambda_{f_{ij}} \|y_{ij}^n - y_{ij}^{n-1}\| \\
&\leq \textstyle\sum_{i=1}^p \lambda_{f_{ij}} \left(1 + \frac{1}{n}\right) D(T_{ij}(x_i^n), T_{ij}(x_i^{n-1}) \\
&\leq \textstyle\sum_{i=1}^p \lambda_{f_{ij}} \left(1 + \frac{1}{n}\right) \lambda_{T_{ij}} \|x_i^n - x_i^{n-1}\|, \; i = 1,2, \ldots, p. \quad (3.12)
\end{aligned}$$

Using (3.10)-(3.12), (3.9) becomes

$$\|z_i^{n+1} - z_i^n\| \leq \left[\lambda_{J_i} \lambda_{g_i} + \rho_i \textstyle\sum_{i=1}^p \lambda_{H_i} \left(1 + \frac{1}{n}\right) \lambda_{M_i} + \rho_i \textstyle\sum_{j=1}^p \lambda_{f_{ij}} \left(1 + \frac{1}{n}\right) \lambda_{T_{ij}}\right] \times \|x_i^n - x_i^{n-1}\| \quad (3.13)$$

By using Theorem 2.1, Proposition 2.1, and k_i-strong accretiveness of g_i, we have

$$\begin{aligned}
\|x_i^n - x_i^{n-1}\|^2 &= \left\|J_{\rho i}^{\partial_{\eta_i} \varphi_i}(z_i^n) - J_{\rho i}^{\partial_{\eta_i} \varphi_i}(z_i^{n-1}) - [g_i(x_i^n) - x_i^n - (g_i(x_i^{n-1}) - x_i^{n-1})]\right\|^2 \\
&\leq \left\|J_{\rho i}^{\partial_{\eta_i} \varphi_i}(z_i^n) - J_{\rho i}^{\partial_{\eta_i} \varphi_i}(z_i^{n-1})\right\|^2 - 2\langle g_i(x_i^n) - x_i^n - (g_i(x_i^{n-1}) - x_i^{n-1}), J_i(x_i^n - x_i^{n-1})\rangle \\
&\leq \frac{\tau_i^2}{\alpha_i^2} \|z_i^n - z_i^{n-1}\|^2 - 2k_i \|x_i^n - x_i^{n-1}\|^2,
\end{aligned}$$

which implies that

$$\|x_i^n - x_i^{n-1}\|^2 \leq \left[\left(\frac{\tau_i^2}{\alpha_i^2}\right)/1 + 2k_i\right] \|z_i^n - z_i^{n-1}\|^2. \quad (3.14)$$

Applying (3.14), (3.13) becomes

$$\|z_i^{n+1}-z_i^n\| \le \frac{\left[\lambda_{J_i}\lambda_{g_i}+\rho_i\sum_{i=1}^{p}\lambda_{H_i}\left(1+\frac{1}{n}\right)\lambda_{M_i}+\rho_i\sum_{j=1}^{p}\lambda_{f_{ij}}\left(1+\frac{1}{n}\right)\lambda_{T_{ij}}\right]\tau_i}{\alpha_i\sqrt{1+2k_i}}\times\|z_i^n-z_i^{n-1}\|.$$

Thus, we have

$$\sum_{i=1}^{p}\|z_i^{n+1}-z_i^n\| \le \sum_{i=1}^{p}\frac{\left[\lambda_{J_i}\lambda_{g_i}+\rho_i\sum_{i=1}^{p}\lambda_{H_i}\left(1+\frac{1}{n}\right)\lambda_{M_i}+\rho_i\sum_{j=1}^{p}\lambda_{f_{ij}}\left(1+\frac{1}{n}\right)\lambda_{T_{ij}}\right]\tau_i}{\alpha_i\sqrt{1+2k_i}}\times\|z_i^n-z_i^{n-1}\|$$

$$= \frac{\left[\lambda_{J_1}\lambda_{g_1}+\rho_1\left(\lambda_{H_1}\left(1+\frac{1}{n}\right)\lambda_{M_1}\right)+\rho_1\left(\lambda_{f_{11}}\left(1+\frac{1}{n}\right)\lambda_{T_{11}}\right)\right]\tau_1}{\alpha_1\sqrt{1+2k_1}}$$

$$+\frac{\left[\lambda_{J_2}\lambda_{g_2}+\rho_2\left(\lambda_{H_2}\left(1+\frac{1}{n}\right)\lambda_{M_2}\right)+\rho_2\left(\lambda_{f_{22}}\left(1+\frac{1}{n}\right)\lambda_{T_{22}}\right)\right]\tau_2}{\alpha_2\sqrt{1+2k_2}}$$

$$+\cdots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$+\frac{\left[\lambda_{J_p}\lambda_{g_p}+\rho_p\left(\lambda_{H_p}\left(1+\frac{1}{n}\right)\lambda_{M_p}\right)+\rho_p\left(\lambda_{f_{pp}}\left(1+\frac{1}{n}\right)\lambda_{T_{pp}}\right)\right]\tau_p}{\alpha_p\sqrt{1+2k_p}}$$

$$\times\sum_{i=1}^{p}\|z_i^n-z_i^{n-1}\|$$

$$\le \theta_n\sum_{i=1}^{p}\|z_i^n-z_i^{n-1}\|, \tag{3.15}$$

where,

$$\theta_n = max\left\{\begin{array}{c}\frac{\left[\lambda_{J_1}\lambda_{g_1}+\rho_1\left(\lambda_{H_1}\left(1+\frac{1}{n}\right)\lambda_{M_1}\right)+\rho_1\left(\lambda_{f_{11}}\left(1+\frac{1}{n}\right)\lambda_{T_{11}}\right)\right]\tau_1}{\alpha_1\sqrt{1+2k_1}}\\ \frac{\left[\lambda_{J_2}\lambda_{g_2}+\rho_2\left(\lambda_{H_2}\left(1+\frac{1}{n}\right)\lambda_{M_2}\right)+\rho_2\left(\lambda_{f_{22}}\left(1+\frac{1}{n}\right)\lambda_{T_{22}}\right)\right]\tau_2}{\alpha_2\sqrt{1+2k_2}}\\ \vdots\\ \frac{\left[\lambda_{J_p}\lambda_{g_p}+\rho_p\left(\lambda_{H_p}\left(1+\frac{1}{n}\right)\lambda_{M_p}\right)+\rho_p\left(\lambda_{f_{pp}}\left(1+\frac{1}{n}\right)\lambda_{T_{pp}}\right)\right]\tau_p}{\alpha_p\sqrt{1+2k_p}}\end{array}\right\}$$

Let

$$\theta = max\left\{\begin{array}{c}\frac{\left[\lambda_{J_1}\lambda_{g_1}+\rho_1\left(\lambda_{H_1}\lambda_{M_1}\right)+\rho_1\left(\lambda_{f_{11}}\lambda_{T_{11}}\right)\right]\tau_1}{\alpha_1\sqrt{1+2k_1}}\\ \frac{\left[\lambda_{J_2}\lambda_{g_2}+\rho_2\left(\lambda_{H_2}\lambda_{M_2}\right)+\rho_2\left(\lambda_{f_{22}}\lambda_{T_{22}}\right)\right]\tau_2}{\alpha_2\sqrt{1+2k_2}}\\ \vdots\\ \frac{\left[\lambda_{J_p}\lambda_{g_p}+\rho_p\left(\lambda_{H_p}\lambda_{M_p}\right)+\rho_p\left(\lambda_{f_{pp}}\lambda_{T_{pp}}\right)\right]\tau_p}{\alpha_p\sqrt{1+2k_p}}\end{array}\right\}$$

Then $\theta_n \to \theta$ as $n \to \infty$. By condition (3.8), we know that $0 < \theta < 1$ and so (3.15) implies that $\{z_1^n\}, \{z_2^n\}, \ldots, \{z_p^n\}$ are all Cauchy sequences in E_i^*. Thus there exists $z_1, z_2, \ldots, z_p \in E_i^*$ such that $z_1^n \to z_1, z_2^n \to z_2, \ldots, z_p^n \to z_p$. From (3.14), it follows that $\{x_1^n\}, \{x_2^n\}, \ldots, \{x_p^n\}$ are all Cauchy sequences in E_i and thus there exists $x_1, x_2, \ldots, x_p \in E_i$ such that $x_1^n \to x_1, x_2^n \to x_2, \ldots, x_p^n \to x_p$. Infact, it follows from the λ_{M_i}-D-Lipschitz continuity of M_i, $\lambda_{T_{ij}}$-D-Lipschitz continuity of T_{1i}, $T_{2i}, \ldots, T_{pi}$ $(i = 1,2, \ldots, p)$, and from (3.3)-(3.6) that for $i = 1,2, \ldots, p$

$$\|u_i^n - u_i^{n-1}\| \le \left(1 + \frac{1}{n}\right) \lambda_{M_i} \|x_i^n - x_i^{n-1}\|, \tag{3.16}$$

$$\|y_{1i}^n - y_{1i}^{n-1}\| \le \left(1 + \frac{1}{n}\right) \lambda_{T_{1i}} \|x_1^n - x_1^{n-1}\|, \tag{3.17}$$

$$\|y_{2i}^n - y_{2i}^{n-1}\| \le \left(1 + \frac{1}{n}\right) \lambda_{T_{2i}} \|x_2^n - x_2^{n-1}\|, \tag{3.18}$$

$$\vdots$$

$$\|y_{pi}^n - y_{pi}^{n-1}\| \le \left(1 + \frac{1}{n}\right) \lambda_{T_{pi}} \|x_p^n - x_p^{n-1}\|. \tag{3.19}$$

From (3.16)-(3.19), we know that $\{u_i^n\}, \{y_{1i}^n\}, \{y_{2i}^n\} \ldots, \{y_{pi}^n\}$ are all Cauchy Sequences in E_i, thus there exists $u_i, y_{1i}, y_{2i}, \ldots, y_{pi} \in E_i$ such that $u_i^n \to u_i, y_{1i}^n \to y_{1i}, y_{2i}^n \to y_{2i}, \ldots y_{pi}^n \to y_{pi}$ as $n \to \infty$.

Next we show that $y_{1i} \in T_{1i}(x_1), y_{2i} \in T_{2i}(x_2), \ldots, y_{pi} \in T_{pi}(x_p)$,

$$\begin{aligned} d(y_{1i}, T_{1i}(x_1)) &\le \|y_{1i} - y_{1i}^n\| + d(y_{1i}^n, T_{1i}(x_1)) \\ &\le \|y_{1i} - y_{1i}^n\| + D(T_{1i}(x_1^n)), T_{1i}(x_1)) \\ &\le \|y_{1i} - y_{1i}^n\| + \lambda_{T_1} \|x_1^n - x_1\| \to 0, \end{aligned}$$

Since $T_{1i}(x_1)$ is closed, we have $y_{1i} \in T_{1i}(x_1)$, similarly we can show that $y_{2i} \in T_{2i}(x_2), \ldots, y_{pi} \in T_{pi}(x_p)$. Since J_i, g_i, H_i and f_i are all continuous and from the Algorithm 3.1, it follows that

$$z_i^{n+1} = J_i(g_i(x_i^n)) - \rho_i[H_i(u_1^n, u_2^n, \ldots u_p^n) - f_i(y_{i1}^n, y_{i2}^n, \ldots, y_{ip}^n)]$$
$$\to z_i = J_i(g_i(x_i)) - \rho_i[H_i(u_1, u_2, \ldots, u_p) - f_i(y_{i1}, y_{i2}, \ldots, y_{ip})] \text{ (as } n \to \infty)$$

and

$$J_{\rho i}^{\partial_{\eta_i} \varphi_i}(z_i^n) = g_i(x_i^n) \to g_i(x_i) = J_{\rho i}^{\partial_{\eta_i} \varphi_i}(z_i).$$

Then by Lemma 3.2, the required result follows.

4 Applications

Throughout this section, we assume that $E = \prod_{i=1}^p E_i$ and $E^i = \prod_{j=1, j \neq 1}^p E_j$ and we write $E = E^i \times E_i$. For each $x \in E, x_i \in E_i$, denotes the i^{th}- coordinate and $x^i \in E^i$, the projection of x onto $E^i$and hence we also write $x = (x^i, x_i)$.

In Problem (2.1), if we consider $\varphi_i = 0, f_i = 0, g_i = I_i, H_i = I_i, T_{1i} = T_{2i} = \cdots T_{pi} = 0$ and $M_i: E \to CB(E_i)$, then Problem (2.1) reduces to the following system of generalized variational-like inequalities:

Find $\bar{x} \in E$ such that for all $a_i \in E_i$, $\exists \bar{u}_i \in M(\bar{x})$ such that

$$\langle \overline{u_i}, \eta_i(x_i, a_i) \rangle \leq 0. \tag{4.1}$$

This problem is introduced by Ansari and Yao [4].

Let E_i is a finite dimensional Euclidean space R^{n_i} and $\psi: E \to R$, be a given function. Then the system of optimization problems is to find $\bar{x} \in E$ such that for $i = 1,2, \ldots \ldots p$

$$\psi_i(a) - \psi_i(\bar{x}) \geq 0, \forall a \in E. \tag{4.2}$$

We can choose $a \in E$ in such a way that $a_i = \bar{x}^i$, than (4.2) becomes Nash equilibrium Problem [16] which is to find $x \in E$ such that for $i = 1,2, \ldots \ldots p$

$$\psi_i(\bar{x}^i, a_i) - \psi_i(\bar{x}) \geq 0, \forall a \in E. \tag{4.3}$$

As an application of (4.1),Ansari and Yao [4] have shown that (4.1) is equivalent to the system of optimization Problems (4.2) under certain conditions, which includes Nash equilibrium problem [16] as a special case. We have noticed that no solution method of finding the solution of (4.1) and hence (4.2) and (4.3) is given in Ansari and Yao [4].In this paper we have studied a more general problem than (4.1),(4.2) and (4.3) and discussed a solution method for Problem (2.1).

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