Mathematical Science Letters
An International Journal

http://dx.doi.org/10.12785/msl/020203

Common fixed point for generalized weak contractions in metric spaces and application to nonlinear integral equations

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Received: 3 Jan 2013; Revised: 5 Apr 2013; Accepted: 6 Apr 2013
Published online: 1 May. 2013

Abstract: In this paper, we prove a common fixed point theorem for mappings satisfying a new generalization of weakly contractive condition in complete metric spaces. Our result extends and unifies some well known comparable results in the literature and is supported by an example. As an application, we give an existence and uniqueness result for the solution of integral equations.

Keywords: Fixed point, weak contraction, nonlinear integral equations.

1 Introduction and preliminaries

A mapping $T:X\to X$, where (X,d) is a metric, is said to be a contraction if there exists a constant $k\in(0,1)$ such that for all $x,y\in X$

$$d(Tx,Ty)\le kd(x,y).$$

Also we say that T is a φ-weak contraction if there exists a function $\varphi:[0,\infty)\to[0,\infty)$ such that φ is positive on $(0,\infty)$ and $\varphi(0)=0$, and

$$d(Tx,Ty)\le d(x,y)-\varphi(d(x,y)).$$

The weak contractive condition has played an important role in the study of existence of fixed points in several papers [see 1-3, 5-8, 11, 14-19]. The concept of the weak contraction was introduced by Alber and Guerre-Delabriere [3] in 1997. They defined such mappings for single-valued maps on Hilbert spaces and proved the existence of fixed point. Rhoades [17] has shown that this result is also valid in complete metric spaces.

Theorem 1.1 ([17]). *Let (X,d) be a nonempty complete metric space and let $T:X\to X$ be a φ-weak contraction on X. If φ is a continuous and nondecreasing function with $\varphi(t)>0$ for all $t>0$ and $\varphi(0)=0$, then T has a unique fixed point.*

Dutta and Choudhury [10] proved the following result as a new generalization of contraction principle.

Theorem 1.2 ([10]). *Let (X,d) be a nonempty complete metric space and let $T:X\to X$ be a self-mapping satisfying the inequality*

$$\psi(d(Tx,Ty))\le\psi(d(x,y))-\varphi(d(x,y)),$$

for all $x, y \in X$ *where* $\psi, \varphi : [0, \infty) \to [0, \infty)$ *are both continuous and monotone nondecreasing functions with* $\psi(t) = \varphi(t) = 0$ *if and only if* $t = 0$. *Then* T *has a unique fixed point.*

To generalize the preceding theorem Beg and Abbas [5] have shown that in Theorem 1.2 the mapping T can be replaced by a weakly contractive mapping with respect to g, that is,

$$\psi(d(Tx, Ty)) \le \psi(d(gx, gy)) - \varphi(d(gx, gy))$$

where ψ, φ are as same as in Theorem 1.2 and concluded that T and g have a coincidence point. In 2009, Altun and Turkoglu [4] proved a common fixed point theorem for weakly compatible mappings satisfying an implicit relation which generalizes some fixed point theorems.
Recently, the first author, Radenović and Roshan [2] proved some common fixed point results using the new concept, almost generalized (S, T)-contractive condition, in partially ordered metric space. Dorić [9] and Popescu [13] presented following nice results under some new control conditions which can be improved and unified by our result.

Theorem 1.3 ([9]). *Let* (X, d) *be a nonempty complete metric space and* $T : X \to X$ *be a mapping satisfying for all* $x, y \in X$

$$\psi(d(Sx, Ty)) \le \psi(M(x, y)) - \varphi(M(x, y)),$$

where M *is given by*

$$M(x, y) = \max\left\{d(x, y), d(Tx, x), d(Sy, y), \frac{1}{2}[d(y, Tx) + d(x, Sy)]\right\},$$

and

(a) $\psi : [0, \infty) \to [0, \infty)$ *is a continuous monotone nondecreasing function with* $\psi(t) = 0$ *if and only if* $t = 0$,
(b) $\varphi : [0, \infty) \to [0, \infty)$ *is a lower semi-continuous function with* $\varphi(t) = 0$ *if and only if* $t = 0$.
Then there exists the unique point $u \in X$ *such that* $u = Tu = Su$.

Theorem 1.4 ([13]). *Let* (X, d) *be a nonempty complete metric space and* $T : X \to X$ *be a mapping satisfying for all* $x, y \in X$

$$\psi(d(Tx, Ty)) \le \psi(M(x, y)) - \varphi(M(x, y)),$$

where M *is given by*

$$M(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2}[d(x, Ty) + d(Tx, y)]\right\},$$

and

(a) $\psi : [0, \infty) \to [0, \infty)$ *is a monotone nondecreasing function with* $\psi(t) = 0$ *if and only if* $t = 0$,

(b) $\varphi : [0, \infty) \to [0, \infty)$ *is a function with* $\varphi(t) = 0$ *if and only if* $t = 0$, *and* $\liminf_{n \to \infty} \varphi(a_n) > 0$ *if*

$\lim_{n \to \infty} a_n = a > 0$,

(c) $\varphi(a) > \psi(a) - \psi(a-)$ *for any* $a > 0$, *where* $\psi(a-)$ *is the left limit* ψ *at* a.

Then T *has a unique fixed point.*

Definition 1.5. Let (X,d) be a metric space, and let $S, F : X \to X$ be two single-valued functions. We say that S and F are *compatible* if

$$\lim_{n\to\infty} d(SFx_n, FSx_n) = 0$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} d(Fx_n, Sx_n) = 0.$

In the present work, we introduce a class of pairs of functions (ψ,φ) to establish a common fixed point theorem in a complete metric space X. As an application, we give an existence and uniqueness result for the solution of integral equations.

2 Main results

Let the functions $\varphi,\psi : [0,\infty) \to [0,\infty)$ satisfy the following conditions:

(a) ψ is a lower semi-continuous monotone nondecreasing function with $\psi(t) = 0$ if and only if $t = 0$,

(b) φ is a function with $\liminf_{n\to\infty} \varphi(a_n) < \psi(a)$ if $\lim_{n\to\infty} a_n = a > 0$ and $\liminf_{n\to\infty} \varphi(a_n) = 0$ if $\lim_{n\to\infty} a_n = 0$.

Moreover, we call the mappings hold the conditions above ψ-mapping and φ-mapping.

Theorem 2.1. *Let* (X,d) *be a nonempty complete metric space and suppose that* $S_i, T_i : X \to X$ *be single-valued functions for* $i = 1,2$ *with* $S_1X \subseteq S_2X$ *and* $T_1X \subseteq T_2X$ *such that for all* $x, y \in X$,

$$\psi(d(S_1x, T_1y)) \le \varphi(M(x,y)),$$

where ψ,φ *are defined as above and* M *is given by*

$$M(x,y) = \max\left\{d(T_2x, S_2y), d(T_2x, S_1x), d(S_2y, T_1y), \alpha[d(T_2x, T_1y) + d(S_2y, S_1x)]\right\},$$

for $\alpha \in [0, \frac{1}{2}]$. *If* S_1 *and* T_2 *are compatible,* S_2 *and* T_1 *are compatible, and if either* T_2 *or* S_2 *is continuous, then* S_1, S_2, T_1 *and* T_2 *have a unique common fixed point in* X.

Proof. Let x_0 be an arbitrary point in X. Since $S_1X \subseteq S_2X$ and $T_1X \subseteq T_2X$, we can choose $x_1, x_2 \in X$ such that $S_1x_0 = S_2x_1$ and $T_1x_1 = T_2x_2$. Define the sequence $\{x_n\}$ recursively as follows.

$$S_i x_{2n+(i-1)} = z_{2n},\ T_i x_{2n+i} = z_{2n+1}, \quad \text{for } i = 1,2.$$

Since

$$\psi(d(z_{2n}, z_{2n+1})) = \psi(d(S_1x_{2n}, T_1x_{2n+1}) \le \varphi(M(x_{2n}, x_{2n+1})),$$

if $d(z_{2n-1}, z_{2n}) < d(z_{2n}, z_{2n+1})$ for some $n \geq 1$, then we have

$$M(x_{2n}, x_{2n+1}) = d(z_{2n}, z_{2n+1}) > 0.$$

Using the condition (b) of ψ -mapping

$$\psi(d(z_{2n}, z_{2n+1})) \leq \varphi(d(z_{2n}, z_{2n+1})) < \psi(d(z_{2n}, z_{2n+1}))$$

which is a contradiction. So, we obtain

$$d(z_{2n}, z_{2n+1}) \leq d(z_{2n-1}, z_{2n}),$$

for all $n \geq 1$. Generally, we have

$$d(z_n, z_{n+1}) \leq d(z_{n-1}, z_n), \quad \psi(d(z_n, z_{n+1})) \leq \varphi(d(z_{n-1}, z_n)), \tag{2.1}$$

for all $n \geq 1$. Since the sequence $\{d(z_n, z_{n+1})\}$ is monotone non-increasing and bounded, so there exists $r \geq 0$ such that $\lim_{n\to\infty} d(z_n, z_{n+1}) = r$.

Suppose that $r > 0$. Then condition (b) as defined in this section and properties of ψ function together with (2.1) imply that

$$\psi(r) \leq \liminf_{n\to\infty} \psi(d(z_n, z_{n+1})) \leq \liminf_{n\to\infty} \varphi(d(z_{n-1}, z_n)) < \psi(r),$$

which is a contradiction unless $r = 0$. Therefore, $d(z_n, z_{n+1}) \to 0$ as $n \to \infty$.

Now we show that the sequence $\{z_n\}$ is a Cauchy sequence in X . Suppose, to the contrary, that there exists some $\varepsilon > 0$ such that for all $k \in \mathbb{N}$, there are $m(k), n(k) \in \mathbb{N}$ with $m(k) > n(k) \geq k$ satisfying:

(i) $m(k)$ is even and $n(k)$ is odd,

(ii) $d(z_{m(k)}, z_{n(k)}) \geq \varepsilon$.

Further, corresponding to $n(k)$, we can choose $m(k)$ in such a way that it is the smallest integer with $m(k) > n(k) \geq k$ and satisfying the conditions (i), (ii). This implies that $d(z_{m(k)-1}, z_{n(k)}) < \varepsilon$ for all $k \geq 1$. Using (ii), we obtain

$$\varepsilon \leq d(z_{m(k)}, z_{n(k)}) \leq d(z_{m(k)}, z_{m(k)-1}) + d(z_{m(k)-1}, z_{n(k)}) < d(z_{m(k)}, z_{m(k)-1}) + \varepsilon.$$

Letting $k \to \infty$, since $d(z_n, z_{n+1}) \to 0$ as $n \to \infty$

$$\lim_{k\to\infty} d(z_{m(k)}, z_{n(k)}) = \varepsilon.$$

Therefore, the condition (b) of ψ -mapping implies that

$$\psi(\epsilon) \le \liminf_{k\to\infty} \psi(d(z_{m(k)}, z_{n(k)})) \le \liminf_{k\to\infty} \varphi(d(z_{m(k)}, z_{n(k)})) < \psi(\epsilon)$$

which is a contradiction. Thus the sequence $\{z_n\}$ is a Cauchy sequence in X and so by completeness, there exists a $z \in X$ such that $z_n \to z$ as $n \to \infty$. Hence we get

$$d(T_i x_{2n+i}, z) \to 0,\ d(S_i x_{2n+(i-1)}, z) \to 0, \text{ as } n \to \infty,$$

for $i = 1, 2$. Assume T_2 is continuous. Then we have

$$d(T_2^2 x_{2n}, T_2 z) \to 0 \text{ and } d(T_2 S_1 x_{2n}, T_2 z) \to 0, \text{ as } n \to \infty.$$

Since S_1 and T_2 are compatible and $d(S_1 x_{2n}, T_2 x_{2n}) \to 0$ as $n \to \infty$, we have

$$d(S_1 T_2 x_{2n}, T_2 z) \le d(S_1 T_2 x_{2n}, T_2 S_1 x_{2n}) + d(T_2 S_1 x_{2n}, T_2 z),$$

which shows that $d(S_1 T_2 x_{2n}, T_2 z) \to 0$ as $n \to \infty$. On the other hand for any $n \in \mathbb{N}$,

$$\psi(d(S_1 T_2 x_{2n}, T_1 x_{2n+1})) \le \varphi(M(T_2 x_{2n}, x_{2n+1})).$$

Here we have the following possible cases.

(i) If $M(T_2 x_{2n}, x_{2n+1}) = d(T_2^2 x_{2n}, S_2 x_{2n+1})$, then we have

$$\psi(d(S_1 T_2 x_{2n}, T_1 x_{2n+1})) \le \varphi(d(T_2^2 x_{2n}, S_2 x_{2n+1})).$$

Suppose that $z \ne T_2 z$. Since

$$\lim_{n\to\infty} d(T_2^2 x_{2n}, S_2 x_{2n+1}) = \lim_{n\to\infty} d(S_1 T_2 x_{2n}, T_1 x_{2n+1}) = d(T_2 z, z),$$

we have

$$\begin{aligned} \psi(d(T_2 z, z)) &\le \liminf_{n\to\infty} \psi(d(S_1 T_2 x_{2n}, T_1 x_{2n+1})) \\ &\le \liminf_{n\to\infty} \varphi(d(T_2^2 x_{2n}, S_2 x_{2n+1})) \\ &< \psi(d(T_2 z, z)), \end{aligned}$$

which is a contradiction. Hence $T_2 z = z$.

(ii) If $M(T_2 x_{2n}, x_{2n+1}) = d(T_2^2 x_{2n}, S_1 T_2 x_{2n})$, since

$$\begin{aligned} d(S_1 T_2 x_{2n}, T_1 x_{2n+1}) &\le d(T_2^2 x_{2n}, S_2 x_{2n+1}) + d(T_2^2 x_{2n}, S_1 T_2 x_{2n}) + d(S_2 x_{2n+1}, T_1 x_{2n+1}) \\ &\le 3d(T_2^2 x_{2n}, S_1 T_2 x_{2n}), \end{aligned}$$

and $d(T_2^2 x_{2n}, S_1T_2x_{2n}) \to 0$ as $n \to \infty$, so $\lim_{n\to\infty} d(S_1T_2x_{2n}, T_1x_{2n+1}) = 0$. It follows that $d(T_2z, z) = 0$ as we know

$$\lim_{n\to\infty} d(S_1T_2x_{2n}, T_2z) = \lim_{n\to\infty} d(T_1x_{2n+1}, z) = 0.$$

(iii) Suppose that $M(T_2x_{2n}, x_{2n+1}) = d(S_2x_{2n+1}, T_1x_{2n+1})$, since

$$d(S_1T_2x_{2n}, T_1x_{2n+1}) \le 3d(S_2x_{2n+1}, T_1x_{2n+1}), \quad d(S_2x_{2n+1}, T_1x_{2n+1}) \to 0 \text{ as } n \to \infty,$$

so we have $\lim_{n\to\infty} d(S_1T_2x_{2n}, T_1x_{2n+1}) = 0$ which implies that $d(T_2z, z) = 0$.

(iv) Suppose that $M(T_2x_{2n}, x_{2n+1}) = \alpha[d(T_2^2x_{2n}, T_1x_{2n+1}) + d(S_2x_{2n+1}, S_1T_2x_{2n})]$ and $\alpha \ne 0$. If $d(T_2z, z) > 0$, then we get

$$\psi(d(T_2z, z)) \le \liminf_{n\to\infty} \psi(d(S_1T_2x_{2n}, T_1x_{2n+1}))$$

$$\le \liminf_{n\to\infty} \varphi(\alpha[d(T_2^2x_{2n}, T_1x_{2n+1}) + d(S_2x_{2n+1}, S_1T_2x_{2n})])$$

$$< \psi(2\alpha d(T_2z, z)),$$

which is a contradiction for $\alpha = \frac{1}{2}$. Let $0 < \alpha < \frac{1}{2}$. Since ψ is monotone nondecreasing function so following inequality holds.

$$d(T_2z, z) \le 2\alpha d(T_2z, z),$$

hence $\alpha \ge \frac{1}{2}$ which is a contradiction. So we have $T_2z = z$.

Following (i)-(iv), we get $T_2z = z$. For any $n \in \mathbb{N}$, $\psi(d(S_1z, T_1x_{2n+1})) \le \varphi(M(z, x_{2n+1}))$.

(v) If $M(z, x_{2n+1}) = d(T_2z, S_2x_{2n+1})$, then we have

$$d(T_2z, S_2x_{2n+1}) \to d(T_2z, z) = 0 \text{ as } n \to \infty.$$

Hence

$$\psi(d(S_1z, z)) \le \liminf_{n\to\infty} \psi(d(S_1z, T_1x_{2n+1})) \le \liminf_{n\to\infty} \varphi(d(T_2z, S_2x_{2n+1})) = 0.$$

So by property of ψ, we obtain that $S_1z = z$.

(vi) Suppose that $M(z, x_{2n+1}) = d(T_2z, S_1z)$. If $S_1z \ne z$ then we get

$$\psi(d(S_1z, z)) \le \liminf_{n\to\infty} \psi(d(S_1z, T_1x_{2n+1})) \le \liminf_{n\to\infty} \varphi(d(T_2z, S_1z)) < \psi(d(T_2z, S_1z)),$$

which is a contradiction. So we have $S_1z = z$.

(vii) If $M(z, x_{2n+1}) = d(S_2x_{2n+1}, T_1x_{2n+1})$, then by conditions (a) and (b) of ψ -mapping and φ -mapping we have the following.

$$\psi(d(S_1z, z)) \le \liminf_{n\to\infty} \psi(d(S_1z, T_1x_{2n+1})) \le \liminf_{n\to\infty} \varphi(d(S_2x_{2n+1}, T_1x_{2n+1})) = 0,$$

which implies that $S_1z = z$.

(viii) Suppose that $M(z, x_{2n+1}) = \alpha[d(T_2z, T_1x_{2n+1}) + d(S_2x_{2n+1}, S_1z)]$ and $\alpha \ne 0$. Since

$$\alpha[d(T_2z, T_1x_{2n+1}) + d(S_2x_{2n+1}, S_1z)] \to \alpha d(S_1z, z) \text{ as } n \to \infty,$$

if $d(Sz, z) > 0$ then we have

$$\begin{aligned}\psi(d(S_1z, z)) &\le \liminf_{n\to\infty} \psi(d(S_1z, T_1x_{2n+1}))\\ &\le \liminf_{n\to\infty} \varphi(\alpha[d(T_2z, T_1x_{2n+1}) + d(S_2x_{2n+1}, S_1z)])\\ &< \psi(\alpha d(S_1z, z)),\end{aligned}$$

whenever $0 < \alpha \le \frac{1}{2}$. Since ψ is monotone nondecreasing function, the following inequality holds.

$$d(S_1z, z) \le \alpha d(S_1z, z),$$

which shows a contradiction. Hence $S_1z = z$. Following (v)-(viii), we get $S_1z = z$.

Take $w \in X$ such that $S_2w = S_1z = z$. Then $T_1S_2w = T_1z$, and

$$\begin{aligned}\psi(d(z, T_1w)) &= \psi(d(S_1z, T_1w))\\ &\le \varphi(M(z, w))\\ &= \varphi(d(z, T_1w)).\end{aligned}$$

If $d(z, T_1w) > 0$, then by property of ψ function we get $\psi(d(z, T_1w)) < \psi(d(z, T_1w))$ which is a contradiction. Hence $T_1w = z$ and so $S_2T_1w = S_2z$. Since T_1, S_2 are compatible and $d(T_1w, S_2w) = 0$, we get $d(T_1z, S_2z) = d(T_1S_2w, S_2T_1w) = 0$, which implies that $T_1z = S_2z$. We also have

$$\begin{aligned}\psi(d(z, T_1z)) = \psi(d(S_1z, T_1z)) &\le \varphi(M(z, z))\\ &= \varphi(d(z, T_1z)).\end{aligned}$$

If $d(z,T_1z)>0$ then $\psi(d(z,T_1z))<\psi(d(z,T_1z))$ which is a contradiction. So we have $T_1z=z$. Hence z is a common fixed point of S_1,T_1,S_2 and T_2 with $S_1z=T_1z=T_2z=S_2z=z$.

To prove the uniqueness, let y be a common fixed point of S_1,T_1,S_2 and T_2 and $y\neq z$. Then we obtain

$$\psi(d(y,z))=\psi(d(S_1y,T_1z))\leq\varphi(M(y,z))=\varphi(d(y,z))<\psi(d(y,z)),$$

which is a contradiction. Hence z is the unique common fixed point of S_1,T_1,S_2 and T_2. Similarly, we can prove by continuity of S_2. □

Remark 2.2. We note that in Theorem 2.1 substituting $\psi(x)-\varphi(x)$ for $\varphi(x)$ implies the main result of Doric [9]. So the theorem above can be considered as a generalization of Theorem 1.3. Also comparing the conditions of Theorem 2.1 and the conditions of Theorem 1.4, we see that Theorem 2.1 extends and complements Theorem 1.4 as main result of Popescu [13].

We now give an example to illustrate the validity of Theorem 2.1 as follows.

Example 2.3. Let $X=[0,1]$ be endowed with the Euclidean metric $d(x,y)=|x-y|$ and let $T_1x=0$,

$$T_2x=\begin{cases}\ln(ex-x+1) & 0\leq x<\frac{1}{2};\\ \ln(\frac{e+1}{2}) & \frac{1}{2}\leq x\leq 1,\end{cases}$$

$$S_1x=\begin{cases}\frac{e^x-1}{e-1} & 0\leq x<\frac{1}{2};\\ 0 & \frac{1}{2}\leq x\leq 1,\end{cases}$$

$$S_2x=\begin{cases}0 & 0\leq x<\frac{1}{2};\\ 1 & \frac{1}{2}\leq x\leq 1.\end{cases}$$

It is easy to verify that S_1,S_2,T_1 and T_2 are mappings from X into itself and S_1,T_2 and S_2,T_1 are compatible. By taking $\psi(t)=1.001t$ and $\varphi(t)=t$ since

$$0\leq\ln(ex-x+1)-1.001\frac{e^x-1}{e-1}, \qquad \text{for } x\in[0,\frac{1}{2}),$$

and

$$\ln(ex-x+1)+1.001\frac{e^x-1}{e-1}\leq 1, \qquad \text{for } x\in[0,\frac{1}{2}),$$

we obtain

$$\psi(d(S_1x,T_1y)) \le \varphi(d(T_2x,S_2y)).$$

Hence $\psi(d(S_1x,T_1y)) \le \varphi(M(x,y))$ where $M(x,y)$ is defined as same as in Theorem 2.1. Obviously, all conditions of Theorem 2.1 are satisfied and S_1, S_2, T_1 and T_2 have a unique common fixed point on $X = [0,1]$.

As a consequence, for the case $T_2 = S_2 = I$ (identity mapping), we have the following result.

Corollary 2.4. Let (X,d) be a nonempty complete metric space and let $S,T : X \to X$ be two single-valued functions such that for all $x, y \in X$,

$$\psi(d(Sx,Ty)) \le \varphi(M(x,y)),$$

where ψ, φ are defined by (a), (b) and M is given by

$$M(x,y) = \max\{d(x,y), d(x,Sx), d(y,Ty), \alpha[d(x,Ty) + d(y,Sx)]\},\ \ 0 \le \alpha \le \frac{1}{2}.$$

Then S,T have a unique common fixed point in X .

3 Application

Consider the integral equations

$$u(t) = \int_a^b K_T(t,s,u(s))ds + g(t) \quad t \in I = [a,b],$$
$$u(t) = \int_a^b K_S(t,s,u(s))ds + g(t) \quad t \in I = [a,b]. \tag{3.1}$$

The purpose of this section is to give an existence theorem for a solution of the integral equations (3.1) using Corollary 2.4. This application was inspired by [12]. Previously, we considered the space $C(I)$ as the class of all continuous real functions on I . We know that this space with the metric given by

$$d(u,v) = \sup_{t \in I} |u(t) - v(t)|, \qquad u,v \in C(I),$$

is a complete metric space. Now we will prove the following result.

Theorem 3.1. *Consider the integral equations (3.1). Suppose that the following hypotheses hold:*

(i) $K_T, K_S : I \times I \times \mathbb{R} \to \mathbb{R}$ *and* $g : [0,\infty) \to [0,\infty)$ *are continuous;*

(ii) there exist a continuous function $G : I \times I \to [0,\infty)$ *and a nondecreasing function* $\varphi : [0,\infty) \to [0,\infty)$ *such that* $\liminf_{n\to\infty} \varphi(a_n) < \alpha a$ *if* $\lim_{n\to\infty} a_n = a > 0$ *and* $\liminf_{n\to\infty} \varphi(a_n) = 0$ *if* $\lim_{n\to\infty} a_n = 0$ *for all* $\alpha > 1$ *and we also have*

$$|K_T(t,s,x)-K_S(t,s,y)|\le G(t,s)\varphi(|x-y|)$$

for all $t,s\in I$ *and* $x,y\in\mathbb{R}$ *;*

(iii) $\sup_{t\in I}\int_a^b G^2(t,s)ds<\frac{1}{b-a}$.

Then the integral equations (3.1) have a unique common solution u^* *in* $C(I)$.

Proof. Since $\sup_{t\in I}\int_a^b G^2(t,s)ds<\frac{1}{b-a}$ so there is an $\alpha>1$ such that

$$\sup_{t\in I}\int_a^b G^2(t,s)ds\le\frac{1}{\alpha^2(b-a)}. \tag{3.2}$$

Define $T,S:C(I)\to C(I)$ by

$$Tu(t)=\int_a^b K_T(t,s,u(s))ds+g(t),\quad t\in I=[a,b],$$

$$Su(t)=\int_a^b K_S(t,s,u(s))ds+g(t),\quad t\in I=[a,b].$$

Now from (ii), the Cauchy-Schwarz inequality and (3.2) we have for all $t\in[a,b]$

$$\begin{aligned}|Tu(t)-Sv(t)|&\le\int_a^b|K_T(t,s,u(s))-K_S(t,s,v(s))|ds\\&\le\int_a^b G(t,s)\varphi(|u(s)-v(s)|)ds\\&\le\sqrt{\int_a^b G^2(t,s)ds}\sqrt{\int_a^b(\varphi(u(s)-v(s)))^2ds}\\&\le\frac{1}{\alpha\sqrt{b-a}}\sqrt{b-a}\,\varphi(d(u,v))\\&=\frac{1}{\alpha}\varphi(d(u,v)).\end{aligned}$$

Hence we have $\alpha d(Tu,Sv)\le\varphi(d(u,v))$.

Now, by considering the function $\psi:[0,\infty)\to[0,\infty)$ defined by $\psi(t)=\alpha t$ we have the following

$$\psi(d(Tu,Sv))\le\varphi(d(u,v))\le\varphi(M(u,v)),$$

for all $u,v\in C(I)$. Now, all conditions of Corollary 2.4 are satisfied. Thus the conclusion follows. □

Acknowledgements: The authors thank to anonymous referee for his/her appreciation, valuable comments and suggestions.

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