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Positive Periodic Solutions of Singular Systems for First Order Difference Equations

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Abstract: We establish the existence of one or more than one positive periodic solutions of singular systems of first order difference equations $\Delta\mathbf{x}(k) = -\mathbf{a}(k)\mathbf{x}(k) + \lambda\mathbf{b}(k)\mathbf{f}(\mathbf{x}(k))$. The proof of our results is based on the Krasnoselskii fixed point theorem in a cone.

Keywords: Periodic solutions, singular first order, functional difference equations, Kranoselskii fixed point theorem.

1 Introduction

Let $\mathbb{R} = (-\infty,\infty), \mathbb{R}_+ = [0,\infty)$, $\mathbb{R}^n_+ = \prod_{i=1}^n \mathbb{R}_+$, for any $\mathbf{x} = (x_1,\ldots,x_n) \in \mathbb{R}^n_+$. In this paper, we investigate the existence and multiplicity of positive solutions of singular first-order for non-autonomous systems of difference equations

$$\Delta\mathbf{x}(k) = -\mathbf{a}(k)\mathbf{x}(k) + \lambda\mathbf{b}(k)\mathbf{f}(\mathbf{x}(k)), \tag{1}$$

where $\mathbf{a}(k) = \operatorname{diag}[a_1(k),a_2(k),\ldots,a_n(k)]$, $\mathbf{b}(k) = \operatorname{diag}[b_1(k),b_2(k),\ldots,b_n(k)]$, $\mathbb{Z}$ is the set of integers, $\omega \in \mathbb{N}$ is a fixed integer, $\lambda > 0$ and a_i, b_i are $\omega-$periodic and continuous with $0 < a_i(k) < 1$ for all $k \in [0,\omega-1]$ and $f_i \in C(\mathbb{R}^n_+\setminus\{\mathbf{0}\},(0,\infty))$ for $i = 1,\ldots,n$. Here $\Delta\mathbf{x}(k) = x(k+1) - x(k)$, for $k \in Z$.

The existence of positive solutions for differential and difference equations has been studied extensively in recent years. Some appropriate references would be [1,3,4,8,9, 16,15,17,14,11]. To our knowledge, there are few works on the existence results of positive solutions of the type problem (1), see for example [17,12,13,7]. However those results do not deal with singular problems.

Agarwal and O'Regan [1] provided some results on solutions of singular first order differential equations. Chu and Nieto [2] showed the existence of periodic solutions for singular first order differential equations with impulses based on a nonsingular alternative of Leray. The results in [1,2] for first ordr differential equations deal with a single equation. Motivated by the work of Wang [16], we will establish the existence of one or more than one positive periodic solutions for the following first-order non-autonomous singular systems

$$x_i'(t) = -a_i(t)x_i(t) + \lambda b_i(t)f_i(x_1(t),\cdots,x_n(t)),\ i = 1,\cdots,n, \tag{2}$$

where $\lambda > 0$ is a positive paramater. We will obtain the discrete analogue of (2) and thus generalize the work of Mohamed et. al [10] to systems. The proof of our result is based on the well-known Kranoselskii fixed point theorem [5].

2 Preliminaries

Let X be the set of all real $\omega-$periodic sequences $\mathbf{x} : \mathbb{Z} \to \mathbb{R}^n_+$.

$$X = \left\{\mathbf{x} : [0,\omega] \to \mathbb{R}^n_+ : \mathbf{x}(k+\omega) = \mathbf{x}(k), k \in \mathbb{Z}\right\}.$$

Endowed with the maximum norm $\|\mathbf{x}\| = \sum_{i=1}^n |x_i|$ where $|x_i| = \max_{k\in\mathbb{Z}} |x_i(k)|$, $\mathbf{x} = (x_1,x_2\ldots,x_n)^T$. Then X is a Banach space. First we make assumptions for the problem (1).

(H1) $a_i : \mathbb{Z} \to (0,1), \sum_{i=0}^{\omega-1} b_i > 0$ are continuous and ω-periodic such that, $a_i(k) = a_i(k+w)$, $b_i(k) = b_i(k+w)$ for $i = 1,2,\ldots,n$ where ω is a constant denoting the common period of the systems.

(H2) $f_i : \mathbb{R}^n_+\setminus\{\mathbf{0}\} \to (0,\infty)$ is continuous, where $i = 1,\ldots,n$.

We now state the Kranoselskii Fixed Point Theorem [5].

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Lemma 1. *Let X be a Banach space, and let $K \subset X$ be a cone in X. Assume Ω_1, Ω_2 are open subsets of X with $0 \in \Omega_1, \bar{\Omega}_1 \subset \Omega_2$, and let*

$$T : K \cap (\bar{\Omega}_2 \setminus \Omega_1) \to K$$

be a completely continuous operator such that either

(i) $\|Tx\| \le \|x\|, x \in K \cap \partial\Omega_1$ and $\|Tx\| \ge \|x\|, x \in K \cap \partial\Omega_2$; or

(ii) $\|Tx\| \ge \|x\|, x \in K \cap \partial\Omega_1$ and $\|Tx\| \le \|x\|, x \in K \cap \partial\Omega_2$;

Then T has a fixed point in $T : K \cap (\bar{\Omega}_2 \setminus \Omega_1) \to K$.

Lemma 2. *Assume (H1), (H2) hold. If $\mathbf{x} \in X$, then $\mathbf{x}$ is a solution of (1) if and only if*

$$x_i(k) = \sum_{s=0}^{\omega-1} G_i(k,s)\lambda b_i(s) f_i(\mathbf{x}(s)), \quad k,s \in [0,\omega], i = 1,\dots,n$$

where

$$G_i(k,s) = \frac{\prod_{r=s+1}^{k+\omega-1}(1-a_i(r))}{1-\prod_{r=0}^{\omega-1}(1-a_i(r))}, \quad k,s \in [0,\omega-1], i = 1,\dots,n.$$

Note that the denominator in $G_i(k,s)$ is not zero since $0 < a_i(k) < 1$ for $k \in [0,\omega-1]$.

Proof. It is clear that (1) is equivalent to

$$x_i(k+1) = (1-a_i(k))x_i(k) + \lambda b_i(k) f_i(\mathbf{x}(k)) \quad i = 1,\dots,n.$$

and that it can be written as

$$\Delta\left(x_i(k)\prod_{r=0}^{k-1}(1-a_i(r))^{-1}\right) = \lambda\prod_{r=0}^{k}(1-a_i(r))^{-1} b_i(k) f_i(\mathbf{x}(k)).$$

By summing the above equation from $s = k$ to $s = k+\omega-1$ and since $x_i(k+\omega) = x_i(k)$, we have

$$x_i(k) = \left[\prod_{r=0}^{k+\omega-1}(1-a_i(r))^{-1} - \prod_{r=0}^{k-1}(1-a_i(r))^{-1}\right]^{-1} \lambda\sum_{k}^{k+\omega-1}\prod_{r=0}^{k}(1-a_i(r))^{-1} b_i(k) f_i(\mathbf{x}(k)).$$

It is clear that $G_i(k,s) = G_i(k+\omega, s+\omega)$ for all $(k,s) \in \mathbb{Z}^2$. A direct calculation shows that

$$m_i := \frac{\prod_{s=0}^{\omega-1}(1-a_i(s))}{1-\prod_{s=0}^{\omega-1}(1-a_i(s))} \le G_i(k,s) \le \frac{1}{1-\prod_{s=0}^{\omega-1}(1-a_i(s))} =: M_i.$$

Define $\sigma_i = \prod_{s=0}^{\omega-1}(1-a_i(s))$. Clearly for $i = 1,\dots,n, \sigma_i = \frac{m_i}{M_i} > 0$,

$$|x_i| = \max_{k\in[0,\omega-1]} |x_i(k)| \le M_i \sum_{k=0}^{\omega-1} \lambda b_i(k) f_i(\mathbf{x}(k)).$$

Therefore,

$$\mathbf{x}(k) \ge m_i \sum_{k=0}^{\omega-1} \lambda b_i(k) f_i(\mathbf{x}(k)) \ge \frac{m_i}{M_i}|x_i| = \sigma_i |x_i|,$$

for $i = 1,\dots,n$. Now we define a cone

$$K = \{\mathbf{x} = (x_1,\dots,x_n) \in X, k \in [0,\omega], x_i(k) \ge \frac{m_i}{M_i}|x_i| = \sigma_i|x_i|, \forall i = 1,\dots,n\}.$$

It is clear that K is a cone in X and $\min_{k\in[0,\omega]}\sum_{i=1}^{n}|x_i(k)| \ge \sigma\|\mathbf{x}\|$ for $\mathbf{x} = (x_1,x_2,\dots,x_n) \in K$. For $r > 0$, define $\Omega_r = \{\mathbf{x} \in K : \|\mathbf{x}\| < r\}$. Let $\mathbf{T} : K\setminus\{\mathbf{0}\} \to X$ be a map with components $(T_1,\dots,T_n)$:

$$T_i\mathbf{x}(k) = \sum_{s=0}^{\omega-1} G_i(k,s)\lambda b_i(s) f_i(\mathbf{x}(s)), \qquad i = 1,\dots,n. \quad (3)$$

where

$$G_i(k,s) = \frac{\prod_{r=s+1}^{k+\omega-1}(1-a_i(r))}{1-\prod_{r=0}^{\omega-1}(1-a_i(r))}, \qquad k,s \in [0,\omega-1]$$

$, i = 1,\dots,n$, satisfying

$$\frac{\sigma_i}{1-\sigma_i} \le G_i(k,s) \le \frac{1}{1-\sigma_i}, \qquad k \le s \le k+\omega.$$

We denote

$$\mathbf{T}\mathbf{x}(k) = (T_1\mathbf{x}(k),\dots,T_n\mathbf{x}(k))^T.$$

It is clear that $\mathbf{T}\mathbf{x}(k+w) = \mathbf{T}\mathbf{x}(k)$. Thus this implies that $\mathbf{T} : K\setminus\{\mathbf{0}\} \to X$. □

Lemma 3. $\mathbf{T}(K\setminus\{\mathbf{0}\}) \subset K$.

Proof: For any $\mathbf{x} = (x_1,\dots,x_n) \in K\setminus\{\mathbf{0}\}$, by (3) for all $k \in [0,\omega]$, where $i = 1,\dots,n$ we have

$$|T_i\mathbf{x}| = \max_{k\in[0,\omega-1]} |T_i\mathbf{x}(k)| \le M\sum_{s=0}^{\omega-1}\lambda|b_i(s) f_i(\mathbf{x}(s))|.$$

Therefore,

$$T_i\mathbf{x}(k) = \sum_{s=0}^{\omega-1} G_i(k,s)\lambda b_i(s) f_i(\mathbf{x}(s)) \ge m_i\sum_{s=0}^{\omega-1}\lambda|b_i(s) f_i(\mathbf{x}(s))| \ge \frac{m_i}{M_i}|T_i\mathbf{x}|.$$

Hence

$$T_i\mathbf{x}(k) \ge \sigma_i|T_i\mathbf{x}|, \qquad i = 1,\dots,n.$$

This implies that $\mathbf{T}(K\setminus\{\mathbf{0}\}) \subset K$. □

Lemma 4. *$\mathbf{T}(K\setminus\{\mathbf{0}\}) \subset K$ is completely continuous operator.*

Proof. Let $x_m(k), x_0(k) \in K\backslash\{\mathbf{0}\}$ with $x_m(k) \to x_0(k)$ as $m \to \infty$. From (3) and since $f(\xi)$ is continuous in ξ, as $m \to \infty$, we have

$$|T_i x_m(k) - T_i x_0(k)| \le M_i \sum_{s=0}^{\omega-1} |\lambda b_i(s)| |f_i(x_m(s)) - f_i(x_0(s))| \to 0, \quad i = 1, \ldots, n.$$

Hence $|T_i x_m(k) - T_i x_0(k)| \to 0$, it follows that the operator $\mathbf{T} = (T_1, \ldots, T_n)$ is continuous.

Further if $Y \subset K\backslash\{\mathbf{0}\}$ is a bounded set, then $\|\mathbf{x}\| \le C_1 = $ const for all $\mathbf{x} \in Y$. Set $C_2 = \max_{k\in[0,\omega-1]} \lambda b_i(s) f_i(\mathbf{x}(s)), \mathbf{x} \in Y$ then from (3) we get, for all $\mathbf{x} \in Y$

$$|T_i\mathbf{x}| \le M \sum_{s=0}^{\omega-1} \lambda |b_i(s)| |f_i(\mathbf{x}(s))| \le M\omega C_2, \quad i = 1, \ldots, n.$$

This shows that $\mathbf{T}(Y)$ is a bounded set in K. Since K is n-dimensional, $\mathbf{T}(Y)$ is relatively compact in K. Therefore $\mathbf{T}$ is a completely continuous operator. □

Now we introduce some notations that will be used in the next following lemmas. For $r > 0$, let

$$\sigma = \min_{i=1,\ldots,n} \{\sigma_i\}$$

,

$$C(r) = \max\left\{f(\mathbf{x}) : \mathbf{x} \in \mathbb{R}^n_+, \sigma r \le \|\mathbf{x}\| \le r\right\} > 0,$$

$$\Gamma = \sigma \sum_{i=1}^{n} m_i \sum_{s=0}^{\omega-1} b_i(s) > 0, \qquad \chi = \sum_{i=1}^{n} M_i \sum_{s=0}^{\omega-1} b_i(s) > 0.$$

Lemma 5. *Assume that (H1),(H2) hold. For any $\eta > 0$ and $\boldsymbol{x} = (x_1, \ldots, x_n) \in K\backslash\{\boldsymbol{0}\}$, if there exists a f_i such that $f_i(\boldsymbol{x}(k)) \ge \sum_{i=1}^{n} x_i(k)\eta$ for $k \in [0, \omega]$, then $\|\boldsymbol{Tx}\| \ge \lambda\Gamma\eta \|\boldsymbol{x}\|$.*

Proof. Since $\mathbf{x} \in K\backslash\{\mathbf{0}\}$ and $f_i(\mathbf{x}(k)) \ge \sum_{i=1}^{n} x_i(k)\eta$ for $k \in [0, \omega]$, we have

$$\begin{aligned}
\|\mathbf{Tx}\| &\ge \lambda \sum_{i=1}^{n} m_i \sum_{s=0}^{\omega-1} b_i(s) f_i(\mathbf{x}(s)) \\
&\ge \lambda \sum_{i=1}^{n} m_i \sum_{s=0}^{\omega-1} b_i(s) \sum_{i=1}^{n} x_i(k)\eta \\
&\ge \lambda \sum_{i=1}^{n} m_i \sum_{s=0}^{\omega-1} b_i(s) \sum_{i=1}^{n} \sigma_i |x_i| \eta \\
&\ge \lambda\sigma \sum_{i=1}^{n} m_i \sum_{s=0}^{\omega-1} b_i(s) \|\mathbf{x}\| \eta.
\end{aligned}$$

Thus $\|\mathbf{Tx}\| \ge \lambda\Gamma\eta \|\mathbf{x}\|$. □

Let $\hat{f}_i : [1, \infty) \to \mathbb{R}^n_+$ be the function given by

$$\hat{f}_i(\theta) = \max\left\{f_i(\mathbf{x}) : \mathbf{x} \in \mathbb{R}^n_+, \text{ and } 1 \le \| \mathbf{x} \| \le \theta\right\}, i = 1, \cdots, n.$$

It is easy to see that $\hat{f}_i(\theta)$ is a nondecreasing function on $[1, \infty)$. The following lemma is essentially the same as [5], Lemma 3.6 and [15], Lemma 2.8.

Lemma 6. *[16],[15] Assume (H1) holds. If $\lim_{\|\boldsymbol{x}\|\to\infty} \frac{f_i(\boldsymbol{x})}{\|\boldsymbol{x}\|}$ exists (which can be infinty), then $\lim_{\theta\to\infty} \frac{\hat{f}_i(\theta)}{\theta}$ exists and $\lim_{\theta\to\infty} \frac{\hat{f}_i(\theta)}{\theta} = \lim_{\|\boldsymbol{x}\|\to\infty} \frac{f_i(\boldsymbol{x})}{\|\boldsymbol{x}\|}$.*

Lemma 7. *Assume that (H1), (H2) holds. Let $r > \frac{1}{\sigma}$ and if there exists an $\varepsilon > 0$ such that $\hat{f}_i(r) \le \varepsilon r$, $\quad i = 1, \ldots, n$, then $\|\boldsymbol{Tx}\| \le \lambda\chi\varepsilon \|\boldsymbol{x}\|$ for $\boldsymbol{x} = (x_1, \ldots, x_n) \in \partial\Omega_r$.*

Proof. From the definition of $\mathbf{T}$, for $\mathbf{x} \in \partial\Omega_r$, we have

$$\begin{aligned}
\|\mathbf{Tx}\| &\le \lambda \sum_{i=1}^{n} M_i \sum_{s=0}^{\omega-1} b_i(s) f_i(\mathbf{x}(s)) \\
&\le \lambda \sum_{i=1}^{n} M_i \sum_{s=0}^{\omega-1} b_i(s) \hat{f}_i(r) \\
&\le \lambda\varepsilon \sum_{i=1}^{n} M_i \sum_{s=0}^{\omega-1} b_i(s) \|\mathbf{x}\| \\
&\le \lambda\varepsilon\chi \|\mathbf{x}\|.
\end{aligned}$$

This implies that $\|\mathbf{Tx}\| \le \lambda\varepsilon\chi \|\mathbf{x}\|$. □

In view of the definitions of $C(r)$, it follows that $0 < f_i(\mathbf{x}(k)) \le C(r)$ for $k \in [0, \omega]$, if $\mathbf{x} \in \partial\Omega_r$, $r > 0$. Thus it is easy to see that the following lemma can be shown in similar manners as in Lemma 7.

Lemma 8. *Assume (H1), (H2) hold. If $\boldsymbol{x} \in \partial\Omega_r, r > 0$, then $\|\boldsymbol{Tx}\| \le \lambda\chi C(r)$.*

Proof. From the definitions of $\mathbf{T}$ for $\mathbf{x} \in \partial\Omega_r$, we have

$$\begin{aligned}
\|\mathbf{Tx}\| &\le \lambda \sum_{i=1}^{n} M_i \sum_{s=0}^{\omega-1} b_i(s) f_i(\mathbf{x}(s)) \\
&\le \lambda \sum_{i=1}^{n} M_i \sum_{s=0}^{\omega-1} b_i(s) C(r) \\
&\le \lambda\chi C(r).
\end{aligned}$$

This implies that $\|\mathbf{Tx}\| \le \lambda\chi C(r)$. □

3 Main Results

Theorem 1. *Let (H1),(H2) hold. Assume that $\lim_{\|\boldsymbol{x}\|\to 0} f_i(\boldsymbol{x}) = \infty$ for $i = 1, \ldots, n$.*

(a) If $\lim_{\|\boldsymbol{x}\|\to\infty} \frac{f_i(\boldsymbol{x})}{\|\boldsymbol{x}\|} = 0, i = 1, \ldots, n$, then for all $\lambda > 0$, (1) has a positive periodic solution.

(b) If $\lim_{\|\boldsymbol{x}\|\to\infty} \frac{f_i(\boldsymbol{x})}{\|\boldsymbol{x}\|} = \infty, i = 1, \ldots, n$, then for all sufficiently small $\lambda > 0$, (1) has two positive periodic solution.

(c) There exists a $\lambda_0 > 0$ such that (1) has a positive periodic solution for $0 < \lambda < \lambda_0$.

Proof:
Part (a). From the assumptions, there is an $r_1 > 0$ such that

$$f_i(\mathbf{x}) \geq \eta \|\mathbf{x}\|, \qquad i = 1, \dots, n.$$

for $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n_+$ and $0 < \|\mathbf{x}\| \leq r_1$, where $\eta > 0$ is chosen so that

$$\lambda \Gamma \eta > 1$$

If $\mathbf{x} = (x_1, \dots, x_n) \in K \setminus \{\mathbf{0}\} \cap \partial \Omega_{r_1}$, then

$$f_i(\mathbf{x}(k)) \geq \sum_{i=1}^{n} x_i(k) \eta, \quad \text{for} \quad k \in [0,1],\ i = 1, \dots, n.$$

Lemma 5 implies that

$$\|\mathbf{Tx}\| \geq \lambda \Gamma \eta \|\mathbf{x}\| > \|\mathbf{x}\|, \quad \text{for} \quad \mathbf{x} \in K \setminus \{\mathbf{0}\} \cap \partial \Omega_{r_1}. \quad (4)$$

We now determine Ω_{r_2}. Since $\lim_{\|\mathbf{x}\| \to \infty} \frac{f_i(\mathbf{x})}{\|\mathbf{x}\|} = 0$, it follows from Lemma 6 that $\lim_{\theta \to \infty} \frac{\hat{f}_i(\theta)}{\theta} = 0, i = 1, \dots, n.$ Therefore there is an $r_2 > \max\left\{2r_1, \frac{1}{\sigma}\right\}$ such that

$$\hat{f}_i(r_2) \leq \varepsilon r_2$$

where the constant $\varepsilon > 0$ satisfies

$$\lambda \varepsilon \chi < 1$$

Thus, we have by Lemma 7 that

$$\|\mathbf{Tx}\| \leq \lambda \varepsilon \chi \|\mathbf{x}\| < \|\mathbf{x}\|, \quad \text{for} \quad \mathbf{x} \in K \setminus \{\mathbf{0}\} \cap \partial \Omega_{r_2}. \quad (5)$$

By Theorem 1 applied to (4) and (5), it follows that $\mathbf{T}$ has a fixed point in $K \setminus \{\mathbf{0}\} \cap (\bar{\Omega}_{r_2} \setminus \Omega_{r_1})$, which is the desired positive solution of (1). □
Part (b). Fix two numbers $0 < r_3 < r_4$, there exists a λ_0 such that

$$\lambda_0 < \frac{r_3}{\chi C(r_3)}, \quad \lambda_0 < \frac{r_4}{\chi C(r_4)}.$$

By Lemma 8, it implies that for $0 < \lambda < \lambda_0$

$$\|\mathbf{Tx}\| \leq \lambda \chi C(r_j) \leq \frac{r_j}{\chi C(r_j)} \chi C(r_j) = r_j = \|\mathbf{x}\|.$$

Thus,

$$\|\mathbf{Tx}\| < \|\mathbf{x}\| \ \text{ for } \ \mathbf{x} \in K \setminus \{\mathbf{0}\} \cap \partial \Omega_{r_j}, \quad (j = 3, 4). \quad (6)$$

On the other hand, in view of the assumptions $\lim_{\|\mathbf{x}\| \to \infty} \frac{f_i(\mathbf{x})}{\|\mathbf{x}\|} = \infty$ and $\lim_{\|\mathbf{x}\| \to 0} f_i(\mathbf{x}) = \infty$, there are positive numbers $0 < r_2 < r_3 < r_4 < \hat{H}$ such that

$$f_i(\mathbf{x}) \geq \eta \|\mathbf{x}\|, \quad i = 1, \dots, n$$

for $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n_+$ and $0 < \|\mathbf{x}\| \leq r_2$ or $\|\mathbf{x}\| \geq \hat{H}$ where $\eta > 0$ is chosen so that

$$\lambda \Gamma \eta > 1.$$

Thus if $\mathbf{x} = (x_1, \dots, x_n) \in K \setminus \{\mathbf{0}\} \cap \partial \Omega_{r_2}$, then

$$f_i(\mathbf{x}) \geq \eta \|\mathbf{x}\|, \quad i = 1, \dots, n.$$

Let $r_1 = \max\left\{2r_4, \frac{\hat{H}}{\sigma_i}\right\}$ if $\mathbf{x} \in K \setminus \{\mathbf{0}\} \cap \partial \Omega_{r_1}$, then

$$\min_{k \in [0,\omega]} \sum_{i=1}^{n} \mathbf{x}(k) \geq \sigma_i \|\mathbf{x}\| = \sigma_i r_1 \geq \hat{H},$$

which implies that

$$f_i(\mathbf{x}) \geq \eta \|\mathbf{x}\|, \quad i = 1, \dots, n.$$

Thus, by Lemma 5 implies that

$$\|\mathbf{Tx}\| \geq \lambda \Gamma \eta \|\mathbf{x}\| > \|\mathbf{x}\|, \ \ \mathbf{x} \in K \setminus \{\mathbf{0}\} \cap \partial \Omega_{r_1}, \quad (7)$$

and

$$\|\mathbf{Tx}\| \geq \lambda \Gamma \eta \|\mathbf{x}\| > \|\mathbf{x}\|, \ \ \mathbf{x} \in K \setminus \{\mathbf{0}\} \cap \partial \Omega_{r_2}. \quad (8)$$

It follows from Theorem 1 applied to (6), (7) and (8), $\mathbf{T}$ has two fixed points x_1 and x_2 such that $x_1 \in K \setminus \{\mathbf{0}\} \cap \bar{\Omega}_{r_3} \setminus \Omega_{r_2}$ and $x_2 \in K \setminus \{\mathbf{0}\} \cap \bar{\Omega}_{r_1} \setminus \Omega_{r_4}$, which are the desired distinct positive periodic solutions of (1) for $\lambda < \lambda_0$ satisfying

$$r_2 < \|x_1\| < r_3 < r_4 < \|x_2\| < r_1.$$

□
Part (c). Choose a number $r_1 = 1$. By Lemma 8 we infer that there exist a $\lambda_0 = \frac{r_1}{\chi C(r_1)} > 0$ such that

$$\|\mathbf{Tx}\| < \|\mathbf{x}\|, \quad \text{for} \quad \mathbf{x} \in K \setminus \{\mathbf{0}\} \cap \partial \Omega_{r_1}, \ \ 0 < \lambda < \lambda_0. \quad (9)$$

On the other hand, in view of assumption $\lim_{\|\mathbf{x}\| \to 0} f_i(\mathbf{x}) = \infty$, there exists a positive number $0 < r_2 < r_1$ such that $f_i(\mathbf{x}) \geq \eta \|\mathbf{x}\|$ for $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n_+$ and $0 < \|\mathbf{x}\| \leq r_2$, where $\eta > 0$ is chosen so that

$$\lambda \Gamma \eta > 1.$$

Thus, Lemma 5 implies that

$$\|\mathbf{Tx}\| \geq \lambda \Gamma \eta \|\mathbf{x}\| > \|\mathbf{x}\|, \quad \text{for} \quad \mathbf{x} \in K \setminus \{\mathbf{0}\} \cap \partial \Omega_{r_2}. \quad (10)$$

It follows from Theorem 1 applied to (9) and (10) that $\mathbf{T}$ has a fixed point in $K \setminus \{\mathbf{0}\} \cap \bar{\Omega}_{r_1} \setminus \Omega_{r_2}$. Consequently, (1) has a positive solution for $0 < \lambda < \lambda_0$. □

4 Application

Consider the following system of two equations

$$\Delta x(k) = -a_1(k)x(k) + \lambda b_1(k)\left(\sqrt{x^2(k) + y^2(k)}\right)^{-\alpha} + \lambda\left(\sqrt{x^2(k) + y^2(k)}\right)^{\beta}, \quad (11)$$

$$\Delta y(k) = -a_2(k)y(k) + \lambda b_2(k)\left(\sqrt{x^2(k) + y^2(k)}\right)^{-\alpha} + \lambda\left(\sqrt{x^2(k) + y^2(k)}\right)^{\beta}, \quad k \in Z. \quad (12)$$

with $\alpha, \beta > 0, a_i(k) > 0, b_i(k) > 0$ for $i = 1,2$ are ω-periodic. Note that

$$f_i(x(k),y(k)) = (\sqrt{x^2(k)+y^2(k)})^{-\alpha} + (\sqrt{x^2(k)+y^2(k)})^{\beta},$$

i=1,2. It is easy to verify that $a_i(k), b_i(k)$ satisfy the assumptions (H1) and (H2). Note that $\sqrt{x^2(k)+y^2(k)} \leq |x|+|y| \leq \sqrt{2}\sqrt{x^2(k)+y^2(k)}$. Thus

$$f_i(x(k),y(k)) \leq (|\,x\,|+|\,y\,|)^{-\alpha} + (|\,x\,|+|\,y\,|)^{\beta}$$

for $i = 1,2$. By Theorem 1,

$$\lim_{|x|+|y|\to 0} (|\,x\,|+|\,y\,|)^{-\alpha} + (|\,x\,|+|\,y\,|)^{\beta} = \infty.$$

(a)If $0 < \beta < 1$, then for all $\lambda > 0$, (11) has a positive periodic solution.

$$\lim_{|x|+|y|\to\infty} (|\,x\,|+|\,y\,|)^{-\alpha-1} + (|\,x\,|+|\,y\,|)^{\beta-1} = 0$$

(b)If $\beta > 1$, then for all sufficiently small $\lambda > 0$ (11) has two positive periodic solutions.

$$\lim_{|x|+|y|\to\infty} (|\,x\,|+|\,y\,|)^{-\alpha-1} + (|\,x\,|+|\,y\,|)^{\beta-1} = \infty.$$

The following Corollary is an application of our theorems. Assume that a_1, a_2 satisfy (H1). Let $\alpha > 0, \beta > 0, \lambda > 0$.

(a)If $0 < \beta < 1$, then for all $\lambda > 0$, (11) has a positive periodic solution.
(b)If $\beta > 1$, then, for all sufficiently small $\lambda > 0$, (11) has two positive periodic solutions.
(c)There exists a $\lambda_0 > 0$ such that (11) has a positive periodic solution for $0 < \lambda < \lambda_0$.

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