New Explicit Solutions for Troesch's Boundary Value Problem

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In this note, we consider a general problem

$$y'' = \lambda(x)\sinh(\mu(x)y),\ 0 < x < 1,$$

$$y(0) = 0, \quad y(1) = 1,$$

where $\lambda(x)$ and $\mu(x)$ are two functions. An explicit solution is obtained.

Keywords: Explicit solution, Troesch's equation, Lie point symmetry.

1   Introduction

Due to the development of systems of nonlinear ordinary differential equations, the boundary value problem of the nonlinear Troesch equation [1–4]:

$$y'' = \lambda\sinh(\lambda y),\ 0 < x < 1, \tag{1.1}$$

$$y(0) = 0,\ y(1) = 1, \tag{1.2}$$

where λ is a constant, has been given extensive attention in recent years numerically [5–8]. The analytic treatment was investigated in [3, 4], where the authors obtained the closed-form solution to this boundary value problem in terms of the Jacobian elliptic function as

$$y(x) = \frac{2}{\lambda}\sinh^{-1}\left\{\frac{y'(0)}{2}\ \text{sc}\left(\lambda x \middle| 1 - \frac{1}{4}y'^2(0)\right)\right\},$$

where $y'(0)$ is the derivative of y at 0 and given by the expression $y'(0) = 2\sqrt{1 - m}$ in which m is the solution of the transcendental equation

$$\frac{\sinh(\lambda/2)}{\sqrt{1 - m}} = \text{sc}(\lambda|m).$$

The Jacobian elliptic function [1, 2] is defined by

$$\text{sc}(\lambda|m) = \frac{\sin \phi}{\cos \phi},$$

where ϕ, λ and m are related through the integral

$$\lambda = \int_0^\phi \frac{1}{\sqrt{1 - m \sin^2 \theta}} d\theta.$$

One of the main problem of mathematics appears when $\lambda(x)$ and $\mu(x)$ are analytic functions and are added to the original equation of Troesch.

The new problem, incorporating the above functions, is

$$y'' = \lambda(x) \sinh(\mu(x)y), \ 0 < x < 1, \tag{1.3}$$

$$y(0) = 0, \quad y(1) = 1. \tag{1.4}$$

Such additional functions of x arise when there is propagation in an inhomogenious medium.

A question which arises naturally is under what conditions on the functions $\lambda(x)$ and $\mu(x)$ does the given boundary value problem (1.3)-(1.4) have an explicit solution?

2   The Explicit Solution

Let

$$u = \mu(x)y. \tag{2.1}$$

Substituting this into eq.(1.3) we get

$$\frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + q(x)\frac{du}{dx} + r(x)u = \frac{\lambda(x)}{2}\left(e^u - e^{-u}\right), \tag{2.2}$$

where

$$p(x) = \frac{1}{\mu(x)}, \quad q(x) = \left(\frac{1}{\mu(x)}\right)' \quad \text{and} \quad r(x) = \left(\frac{1}{\mu(x)}\right)''.$$

Multiplying both sides of (2.2) by

$$\xi(x) = \exp\left\{\int \frac{q(x)}{p(x)} dx\right\}$$

and taking into account $\xi'(x)p(x) = \xi(x)q(x)$, we obtain

$$(\xi p u')' + \xi r u = \frac{\xi \lambda}{2}\left(e^u - e^{-u}\right). \tag{2.3}$$

Assume that $r(x) = 0$, that is, $(1/\mu(x))'' = 0$. Eq.(2.3) becomes

$$(\xi p u')' = \frac{\xi \lambda}{2}\left(e^u - e^{-u}\right). \tag{2.4}$$

Let

$$u = \ln z. \tag{2.5}$$

Then eq.(2.4) may be written as

$$(\xi p)' \left(\frac{z'}{z} \right) + (\xi p) \left(\frac{z''}{z} - \left(\frac{z'}{z} \right)^2 \right) = \frac{\xi \lambda}{2} \left(z - \frac{1}{z} \right). \tag{2.6}$$

The transformation of $v = z'/z$ into eq.(2.6) leads to

$$(\xi p)' v + (\xi p) v' = \frac{\xi \lambda}{2} \left(z - \frac{1}{z} \right).$$

Since $z = z'/v$, we get

$$(\xi p)' v + (\xi p) v' = \frac{\xi \lambda}{2} \left(\frac{z'}{v} - \frac{z'}{vz^2} \right).$$

It follows that

$$v \left((\xi p)v \right)' = \frac{\xi \lambda}{2} \left(z' - \frac{z'}{z^2} \right). \tag{2.7}$$

To integrate (2.7) we set $w = \xi p v$ and, if we choose $\xi^2(x)p(x)\lambda(x) = a > 0$, where a is a constant, that is, $\lambda(x) = a\mu^3(x)$, because of $\xi(x) = 1/\mu(x)$ and $p(x) = 1/\mu(x)$, we obtain

$$w^2 = a \left(z + \frac{1}{z} \right) + b,$$

where b is an arbitrary constant of integration. Thus

$$w = \pm \sqrt{az + \frac{a}{z} + b}$$

and so

$$v = \frac{\pm 1}{\xi p} \sqrt{az + \frac{a}{z} + b}.$$

Hence

$$\frac{z'}{z} = \frac{\pm 1}{\xi p} \sqrt{az + \frac{a}{z} + b}.$$

It follows that

$$\frac{dz}{\sqrt{az^3 + bz^2 + az}} = \frac{\pm 1}{\xi(x)p(x)} dx. \tag{2.8}$$

Integrating both sides of eq.(2.8) we obtain

$$\frac{\Phi(z) F \left(\phi \mid m \right)}{\Psi(z)} = \pm \int \mu^2(x) dx + c, \tag{2.9}$$

where

$$\Phi(z) = iz^{\frac{3}{2}} \sqrt{2} \sqrt{\frac{2a/z + b + \sqrt{b^2 - 4a^2}}{b + \sqrt{b^2 - 4a^2}}} \sqrt{\frac{2a}{(b - \sqrt{b^2 - 4a^2})z} + 1},$$

$F(\phi \mid m)$ is the elliptic integral of the first kind,

$$\phi = i \sinh^{-1}\left(\frac{\sqrt{2}\sqrt{a/\left[b+\sqrt{b^2-4a^2}\right]}}{\sqrt{z}}\right),$$

$$m = \frac{b+\sqrt{b^2-4a^2}}{b-\sqrt{b^2-4a^2}}$$

and

$$\Psi(z) = \sqrt{\frac{a}{b+\sqrt{b^2-4a^2}}}\sqrt{z\left((z^2+1)a+bz\right)}.$$

The general solution of eq.(1.3) is given by $y = (1/\mu(x))\ln z$, where the implicit solution for z is given by (2.9).

Now, if we choose $b = 2a$ and from (2.8) we have

$$\frac{2\tan^{-1}(\sqrt{z})}{\sqrt{a}} = \pm\int\mu^2(x)dx + c,$$

then

$$z = \tan^2(\pm\frac{\sqrt{a}}{2}\int\mu^2(x)dx + c_1), \qquad (2.10)$$

where $c_1 = (1/2)\sqrt{ac}$.

In order to find the particular solution to the given boundary value problem, (1.3)-(1.4), we use the boundary conditions (1.4) and the substitution $y = (1/\mu(x))\ln z$ to get $z(0) = 1$ and $z(1) = e^{\mu_1}$, where $\mu_1 = \mu(1)$. Therefore with this choice the values of c_1 become

$$c_1 = n\pi + \frac{\pi}{4} \pm \frac{\sqrt{a}}{2}s(0), \; n = 0, 1, \ldots,$$

and

$$c_1 = \tan^{-1}\left(\exp\left[\frac{\mu_1}{2}\right]\right) \pm \frac{\sqrt{a}}{2}s(1),$$

where $s(x) = \int\mu^2(x)dx$, and we get in both of theses cases the following condition

$$\tan^{-1}\left(\exp\left[\frac{\mu_1}{2}\right]\right) = \pm\frac{\sqrt{a}}{2}\left(s(0)+s(1)\right) + n\pi + \frac{\pi}{4}.$$

Thus we have proved the following theorem

Theorem 2.1. *Let $\mu(x) > 0$ be continuously differentiable function such that $(1/\mu(x))'' = 0$ and let $\lambda(x) = a\mu^3(x)$, where $a > 0$ is a constant. Then the functions*

$$y = \frac{1}{\mu(x)}\ln\left(\tan^2\left(\pm\frac{\sqrt{a}}{2}\int\mu^2(x)dx + n\pi + \frac{\pi}{4} \pm \frac{\sqrt{a}}{2}s(0)\right)\right), \; n = 0, 1, \ldots,$$

solve the problem (1.3)-(1.4) if

$$\tan^{-1}\left(\exp\left[\frac{\mu_1}{2}\right]\right) = \pm\frac{\sqrt{a}}{2}\left(s(0)+s(1)\right) + n\pi + \frac{\pi}{4},$$

where $\mu_1 = \mu(1)$ and $s(x) = \int\mu^2(x)dx$.

Remark 2.1. The solutions obtained in Theorem 2.1 for the functions $\lambda(x)$ and $\mu(x)$ are based on the choice of the arbitrary constant of integration. In principle we can choose the other from eq.(2.8).

Below we present the general theory of the Lie symmetry analysis for (1.3)-(1.4).

3 General Theory of Lie Symmetries

The original equation of Troesch is autonomous and so possesses the Lie point symmetry ∂_x. The generalized equation does not possess any Lie point symmetries for arbitrary functions $\lambda(x)$ and $\mu(x)$. Consequently one asks the question. Is there any combination of functions $\lambda(x)$ and $\mu(x)$ for which there does exist a Lie point symmetry? If this is the case, there exists a transformation of the independent and dependent variables which renders the equation autonomous. The second-order equation can then be reduced to a quadrature and, maybe under some further restrictions, evaluated along the lines already indicated in the paper.

By definition [9, 10] a second-order differential equation $L(x, y, y', y'') = 0$ possesses a Lie point symmetry of the form, $\Gamma = \xi(x, y)\partial_x + \eta(x, y)\partial_y$, such that

$$\Gamma[\mu(x)y] = 0. \tag{3.1}$$

The advantage of this condition is that the coefficient functions can be taken to be simply functions of x and y. Also the action of the second extension of Γ, i.e. $\Gamma^{[2]}$ on L, is equal to zero, i.e.

$$\Gamma^{[2]} L(x, y, y', y'') = \left[\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \eta^{[1]}\frac{\partial}{\partial y'} + \eta^{[2]}\frac{\partial}{\partial y''}\right] L(x, y, y', y'') = 0 \tag{3.2}$$

with $\eta^{[2]}$ is given by

$$\eta^{[k]}(x, y, y', \ldots, y^{(k)}) = \frac{D\eta^{(k-1)}}{Dx} - y^{(k)}\frac{D\xi}{Dx}, \quad k = 1, 2, \ldots,$$

where $\eta^{[0]} = \eta$ and $\frac{D}{Dx}$ is the total derivative with respect to x, i.e.

$$\frac{D}{Dx} = \frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + \cdots + y^{(n+1)}\frac{\partial}{\partial y^{(n)}} + \cdots$$

In our case $L(x, y, y', y'')$ is defined by

$$L(x, y, y', y'') = -y'' + f(x, y), \tag{3.3}$$

where $f(x, y) = \lambda(x)\sinh(\mu(x)y)$, and the action of the operator $\Gamma^{[2]}$ on $L(x, y, y', y'')$ leads to a polynomial equation in y'. By equating coefficients of powers of y' one arrives at

$$\xi_{yy} = 0, \tag{3.4}$$

$$\eta_{yy} - 2\xi_{yx} = 0, \tag{3.5}$$

$$2\eta_{xy} - \xi_{xx} - 3f\xi_y = 0, \tag{3.6}$$

$$\eta_{xx} - \xi f_x - \eta f_y + \eta_y f - 2\xi_x f = 0. \tag{3.7}$$

Integrating eq.(3.4) and eq.(3.5) we get

$$\xi(x,y) = a(x)y + b(x), \ \eta(x,y) = a'(x)y^2 + c(x)y + d(x). \tag{3.8}$$

Substituting these expressions into eq.(3.6) we obtain

$$2c'(x) = b''(x), \ a(x) = 0. \tag{3.9}$$

Finally, after we substitute eq.(3.4) and eq.(3.5) into eq.(3.6)-eq.(3.7), we get

$$\xi(x,y) = b(x), \tag{3.10}$$

$$\eta(x,y) = c(x)y \tag{3.11}$$

and

$$c''y - \left((c - 2b')\lambda - b\lambda'\right)\sinh(\mu(x)y) - \lambda y\left(\mu'b + c\mu\right)\cosh(\mu(x)y) = 0. \tag{3.12}$$

From eq.(3.12) and by identification we get

$$c''(x) = 0, \quad \text{i.e.,} \quad c(x) = \alpha x + \beta, \tag{3.13}$$

$$\frac{\mu'(x)}{\mu(x)} = -\frac{c(x)}{b(x)}, \tag{3.14}$$

which is obtained by the first condition eq.(3.1), and

$$(c(x) - 2b'(x))\lambda(x) - b(x)\lambda'(x) = 0. \tag{3.15}$$

Hence

$$\frac{\lambda'(x)}{\lambda(x)} = \frac{c(x)}{b(x)} - 2\frac{b'(x)}{b(x)}. \tag{3.16}$$

The substitution of eq.(3.14) into eq.(3.16) gives

$$\lambda(x) = \frac{\Lambda_0}{\mu(x)b^2(x)}, \tag{3.17}$$

where Λ_0 is a constant. Differentiating eq.(3.9) and taking into account eq.(3.13) we obtain

$$b'''(x) = 0, \ \text{i.e.,} \ b(x) = Ax^2 + Bx + C, \tag{3.18}$$

where $A, \ B$ and C are constants.

Integrating eq.(3.14) and with the help of eq.(3.13) we get

$$\mu(x) = \mu_0 \exp\left\{ -\int_0^x \frac{\alpha s + \beta}{b(s)} ds \right\}. \tag{3.19}$$

If we summarize the previous calculations, the Lie point symmetry is of the form

$$\Gamma = b(x)\partial_x + c(x)y\partial_y, \tag{3.20}$$

where

$$\mu(x) = \mu_0 \exp\left\{ -\int_0^x \frac{\alpha s + \beta}{b(s)} ds \right\}, \tag{3.21}$$

$$b(x) = Ax^2 + Bx + C \tag{3.22}$$

and

$$c(x) = \frac{1}{2}b'(x) + D, \tag{3.23}$$

where D is a constant.

The equations (3.21)-(3.23) give the pairs $\{\lambda(x),\ \mu(x)\}$ for which a Lie point symmetry exists.

To relate the expressions for $\lambda(x)$ and $\mu(x)$ obtained in this section with those obtained previously, we consider $\mu_0 = 1$, $A = \alpha$, $B = 2\beta$, $C = \beta^2/\alpha$ and $D = 0$, i.e.

$$b(x) = \frac{1}{\alpha}(\alpha x + \beta)^2.$$

Then by eq.(3.21) we obtain

$$\mu(x) = \frac{1}{\alpha x + \beta}, \quad \text{i.e.} \quad \left(\frac{1}{\mu(x)}\right)'' = 0 \tag{3.24}$$

and by eq.(3.17),

$$\lambda(x) = \frac{\alpha^2 \Lambda_0}{(\alpha x + \beta)^3}, \quad \text{i.e.} \quad \lambda(x) = a\mu^3(x), \tag{3.25}$$

where $a = \alpha^2 \Lambda_0$, which are the same expressions as obtained in Theorem 2.1.

Acknowledgments

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