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The use of Power Normalization as a New Trend in the Order Statistics Limit Theory

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Abstract: In the last two decades E. Pancheva and her collaborators were investigating various limit theorems for extremes using a wider class of normalizing mapping than the linear ones to get a wider class of limit laws. This wider class of extreme limits can be used in solving approximation problems. This review and expository paper is about this new trend in the limit theory of order statistics. We focus on the use of the power normalizing mapping. The review is given covering the possible limit laws of extreme, central and intermediate order statistics under power normalization. The paper also traces the domains of attraction of these possible limits. The final section focuses on the statistical inference about the upper tail of a distribution function by using the power normalization. Moreover, two models for generalized Pareto distribution under power normalization are given.

Keywords: Weak convergence, power normalization, extreme value distributions, central order statistics, intermediate order statistics, domain of attractions, statistical modeling of extreme values, generalized extreme value distribution, generalized Pareto distributions, maximum likelihood method.
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1 General Introduction

Let $\{X_n\}$ be a sequence of i.i.d random variables (rv's) with common distribution function (df) $F \in \mathscr{F}$, where $\mathscr{F}$ is some class of df's. Let Y_n be a measurable function of $X_1, X_2, ..., X_n$ (e.g., $Y_n = X_{n:n} = \max\{X_1, X_2, ..., X_n\}$). We say that the df of Y_n weakly converges (written $\xrightarrow[n]{w}$) to a nondegenerate limit Φ if there exist normalizing constants $a_n > 0$ and b_n such that

$$P(Y_n \le G_n(x)) = P(Y_n \le a_n x + b_n) \xrightarrow[n]{w} \Phi(x) \in \mathscr{F}', \tag{1}$$

on all continuity points of Φ, where $\mathscr{F}' \subseteq \mathscr{F}$. Naturally, one may have some questions concerning the limit relation (1).
Question 1: Is the use of the normalizing constants in equation (1) is necessarily ?
The answer is yes, otherwise we get a degenerate limit, (or even no limit df), e.g. $P(X_{n:n} \le x) = F^n(x) \to 0$, if $x < r(F) = \sup\{x : F(x) < 1\}$ and $P(X_{n:n} \le x) = F^n(x) \to 1$, if $x \ge r(F)$.
Question 2: Do the change of the normalizing constants can lead to a big change in the limit Φ ?
The following essential theorem gives the answer, which is no.
Theorem 1.1 (The classical Khinchin's type theorem). Let $F_n(x)$ be a sequence of df's. Furthermore, let

$$F_n(G_n(x)) \xrightarrow[n]{w} F(x), \text{ is a nondegenerate df,}$$

with $G_n(x) = a_n x + b_n, a_n > 0$. Then, with $G_n^*(x) = c_n x + d_n,\ c_n > 0,$ we have

$$F_n(G_n^*(x)) \xrightarrow[n]{w} F^*(x),\ F^* \text{ is a nondegenerate df,}$$

if and only if $G_n^{-1}(G_n^*(x)) = G_n^{-1} o G_n^*(x) \longrightarrow g(x),\ \forall x,$ where $g(x) = ax + b,\ \frac{c_n}{a_n} \longrightarrow a,\ \frac{d_n - b_n}{a_n} \longrightarrow b$, as $n \to \infty$, and $F^*(x) = F(g(x))$.

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Theorem 1.1 leads to the following definition.
Definition 1.1. We say that the df's $F(x)$ and $F^*(x)$ are of the same type, under linear transformation, if there are real numbers $A>0$ and B such that $F^*(x)=F(Ax+B)$.

Clearly the relation between F and F^* in definition 1.1 is symmetric, reflexive and transitive. Hence it gives rise to equivalence classes of df's. Therefore, Khinchin's type theorem shows that the change of the normalizing constants can not cause any change in the nondegenerate type of $\Phi(x)$. This fact convinces us that the probability limit theory basically deals with the types of df's rather than the df's themselves.

Now, if $\mathscr{F}' \subset \mathscr{F}$, the limit relation (1) gives a sufficiently simple approximation to the df of Y_n. Namely,

$$P(Y_n \le x) \sim \Phi(G_n^{-1}(x)) = \Phi\left(\frac{x-b_n}{a_n}\right).$$

In this case we call $\Phi(G_n^{-1}(x)) = \Phi(\frac{x-b_n}{a_n})$ the asymptotic df of Y_n or the statistical model of the df of Y_n. For example, in view of extreme value theorem, due to Gnedenko [16], we have $\mathscr{F}' = \{\Phi_1(\frac{x-b}{a}), \Phi_2(\frac{x-b}{a}), \Phi_3(\frac{x-b}{a})\}$, where

$$\Phi_1(x) = \begin{cases} 0, & x \le 0, \\ \exp(-x^{-\alpha}), & x > 0, \end{cases}$$

$$\Phi_2(x) = \begin{cases} \exp(-(-x)^{\alpha}), & x \le 0, \\ 1, & x > 0, \end{cases}$$

$$\Phi_3(x) = \exp(-e^{-x}), \ \ \forall\, x. \tag{2}$$

For statistical purposes, the above three limit laws can be incorporated into the von Mises type representation $H_\gamma(x) = \exp(-(1+\gamma x)^{\frac{-1}{\gamma}})$, where γ is a given real number. When $\gamma=0$, $H_\gamma(x)$ is defined as $\lim_{\gamma\to 0} H_\gamma(x)$. A part from changes of origin and the unit on the x-axis the df $H_\gamma(x)$ yields the three laws Φ_1, Φ_2 and Φ_3, according as $\gamma>0, \gamma<0$ or $\gamma=0\ (\gamma\to 0)$, respectively. **Question 3:** Can we use in the relation (1) nonlinear normalization ?
The answer is yes, but under some conditions. Namely, Pancheva ([22],[23]) showed that the normalizing mapping $G_n(x)$ may be used in equation (1) if it is continuous strictly increasing and max-automorphism mapping (the max-automorphisms preserve the max-operation), which forms a group $\mathscr{G}$ w.r.t the composition. Choosing mappings from $\mathscr{G}$, for normalization in the limit theorems, we are imposed to change the notion of type(F) for a nondegenerate df F. We say a df F_1 belongs to type(F_2) if $\exists g \in \mathscr{G}$ such that $F_1 = F_2(g) = F_2 \circ g$. The convergence to type Khinchin's theorem is the main tool for proving limit theorems. In this case, a convergence to type takes place if both convergences $F_n \xrightarrow[n]{w} F_1$ and $F_n \circ G_n \xrightarrow[n]{w} F_2$, where $G_n \in \mathscr{G}$, imply $F_2 \in type(F_1)$. Pancheva, et al. [25] showed that, the compactness of the normalizing sequence $G_n(x)$ is necessary and sufficient for a convergence to type, i.e., the convergence to type Khinchin's theorem is applicable. Pancheva, et al. [25] showed that this theorem is applicable for the linear normalization, as well as the power normalization $G_n(x) = b_n|x|^{a_n}\mathrm{sign}(x), a_n, b_n > 0$, with $G_n^{-1}(x) = |\frac{x}{b_n}|^{\frac{1}{a_n}}\mathrm{sign}(x)$. The convergence to type Khinchin's theorem under power normalization takes the form (see Barakat and Nigm [6]).
Theorem 1.2 (Khinchin's type theorem under power normalization). Let

$$F_n(G_n(x)) \xrightarrow[n]{w} F(x), \text{ is a nondegenerate df},$$

with $G_n(x) = b_n|x|^{a_n}\mathrm{sign}(x)$. Then, with $G_n^*(x) = \beta_n|x|^{\alpha_n}\mathrm{sign}(x)$, we have

$$F_n(G_n^*(x)) \xrightarrow[n]{w} F^*(x),\ F^* \text{ is a nondegenerate df},$$

if and only if $G_n^{-1} o G_n^*(x) \longrightarrow g(x), \forall x$, where $g(x) = B|x|^A \mathrm{sign}(x)$, $\frac{\alpha_n}{a_n} \longrightarrow A$, $\left(\frac{\beta_n}{b_n}\right)^{\frac{1}{\alpha}} \longrightarrow B$, as $n\to\infty$, and $F^*(x) = F(g(x)) = F(B|x|^A\mathrm{sign}(x))$.

Clearly, the employment of a strictly monotone continuous transformation not cause any wastage of information, which is contained in the data under consideration (e.g., the sufficiency property is preserved under one to one transformation). Nevertheless, we may lose some flexibility when using the nonlinear normalization. For example under linear normalization all negative data can be transformed to positive numbers and vise versa, but this can not be done under the power normalization. Although, no one can claim that the employment of nonlinear normalization in general is preferable, but as Pancheva [23] showed in some cases of practical interest it is not only better to use nonlinear transformation, but we have to use it. For example, by using relatively non difficult monotone mappings, in certain cases, we can achieve a better rate of convergence (see Barakat et al. [10]). Another reason for using nonlinear normalization is to get a wider class of limit laws, which can be used in solving approximation problems.

2 The Possible Limit Laws of Extreme Order Statistics Under Power Normalization

Let $X_1, X_2, ..., X_n$ be independent rv's having a common left continuous df F, and denote by $X_{1:n} \le X_{2:n} \le ... \le X_{n:n}$ the corresponding order statistics (os's). Pancheva [21] considered the power normalization and derived all the possible limit df's of $X_{n:n}$ subjected to this normalization. These limit df's are usually called the power max stable df's ($p-$max-stable df's) and are given by $H_{i;\beta}(x) = \exp[-u_{i;\beta}(x)],\ i = 1,2,\ldots,6,\ \beta > 0,$ where

$$u_{1;\beta}(x) = \begin{cases} \infty, & x \le 1, \\ (\log x)^{-\beta}, & x > 1; \end{cases} \quad u_{2;\beta}(x) = \begin{cases} \infty, & x \le 0, \\ (-\log x)^{\beta}, & 0 < x \le 1, \\ 0, & x > 1; \end{cases}$$

$$u_{3;\beta}(x) = \begin{cases} \infty, & x \le -1, \\ (-\log(-x))^{-\beta}, & -1 < x \le 0, \\ 0, & x > 0; \end{cases} \quad u_{4;\beta}(x) = \begin{cases} (\log(-x))^{\beta}, & x \le -1, \\ 0, & x > -1; \end{cases}$$

$$u_5(x) = \begin{cases} \infty, & x \le 0, \\ \frac{1}{x}, & x > 0; \end{cases} \quad u_6(x) = \begin{cases} |\,x\,|, & x \le 0, \\ 0, & x > 0, \end{cases} \tag{3}$$

and we adopt the notation $u_{i;\beta}(x) = u_i(x),\ i = 5,6.$ The corresponding min-stable distribution can be easily written as

$$L_{i;\beta} = 1 - H_{i;\beta}(-x) = 1 - \exp\left(-u^*_{i;\beta}(x)\right) = 1 - \exp\left(-u_{i;\beta}(-x)\right),\ i = 1,2,\ldots,6,\ \beta > 0.$$

Mohan and Ravi [19] showed that the p-max-stable df's (3) attract more than linear stable df's (2), see also Subramanya [32]. Therefore, using the power normalization, we get a wider class of limit df's which can be used in solving approximation problems. In this way we can essentially extend the field of applications of the extreme value model. A unified approach to the results of Mohan and Ravi [19] and Subramanya [32] has been obtained by Christoph and Falk [15]. Barakat and Nigm [6] extended the result of Pancheva [21] to the extreme os's $X_{r:n}$ and $X_{n-r+1:n}$, where the rank r is fixed w.r.t n.

Theorem 2.1. For suitable normalizing constants $a_n, b_n > 0$, the df of the normalized upper extreme os $G_n^{-1}(X_{n-r+1:n}) = |\frac{X_{n-r+1:n}}{b_n}|^{\frac{1}{a_n}} \operatorname{sign}(X_{n-r+1:n})$ converges weakly to a nondegenerate df $\Psi_r(x)$, if and only if

$$n(1 - F(G_n(x)) = n\Big(1 - F(b_n|x|^{a_n}\operatorname{sign}(x))\Big) \longrightarrow u_{i;\beta}(x),\ \text{as } n \to \infty.$$

Moreover, $\Psi_r(x) = \Gamma_r(u_{i;\beta}(x))$, where $\Gamma_r(.)$ is the incomplete gamma function.

As in the linear normalization case, Nasry-Roudsari [20] has summarized the types (3) by the following von Mises type representations

$$H_{1;\gamma}(x;a,b) = \exp\left[-(1+\gamma\log ax^b)^{-\frac{1}{\gamma}}\right],\ x > 0,\ 1+\gamma\log ax^b > 0 \tag{4}$$

and

$$H_{2;\gamma}(x;a,b) = \exp\left[-(1-\gamma\log a(-x)^b)^{-\frac{1}{\gamma}}\right],\ x > 0,\ 1-\gamma\log a(-x)^b > 0. \tag{5}$$

Remark 2.1. The result of Christoph and Falk [15] reveals that the upper tail behaviour of F, might determine whether F belongs to the domain of attraction of $H_{1;\gamma}(x;a,b)$ or of $H_{2;\gamma}(x;a,b)$. In the first case, the right end-point $x^\circ = \sup\{x : F(x) < 1\}$ has to be positive, while for the second case necessarily $x^\circ \le 0$. It is worth mentioning that (4) and (5) can be incorporated into the unified formula by using the result of Christoph and Falk [15] and by adopting the notation $\mathscr{S}^-(x) = -1,$ if $x \le 0$ and $\mathscr{S}^-(x) = 1,$ if $x > 0$.

$$H_\gamma(x;a,b) = \exp\left[-(1+\mathscr{S}^-(x^\circ)\gamma\log a|x|^b)^{-\frac{1}{\gamma}}\right],\ 1+\mathscr{S}^-(x^\circ)\gamma\log a|x|^b > 0.$$

Barakat and Nigm [6] studied the weak convergence of extremes under power normalization assuming that the sample size is a rv. The continuation of the restricted convergence of the power normalized extremes on the half-line of real line to the whole line is proved in Barakat and Nigm [7]. Barakat et al. [9] proved that the restricted convergence of the power normalized extremes on an arbitrary nondegenerate interval implies the weak convergence. Recently, by using the theory of the second-order regular variation, Barakat et al. [10] studied the rates of convergence of the extremes under the power normalization to the each of the $p-$max-stable laws (3). Moreover, a comparison between the rates of convergence under linear normalization and the power normalization is done. Barakat and Nigm [8] derived a symmetric nonparametric measure of asymptotic dependence between the os's under power normalization. This nonparametric measure is based on the notion of the associated copula for os's and some L_1-distances. More recently, Chen, et al [13] studied the rates of convergence of extremes for a general error distribution type under power normalization. They have proved that, if F is of general error distribution type, then there exist two constants C_1, C_2 (depending on the parameters of F) and normalizing constants $a_n, b_n > 0$ (they have found these normalizing constants), for which $\frac{C_1}{\log n} \le \sup_{x>0} |F^n(b_n|x|^{a_n}\text{sign}(x)) - H_{5;\beta}(x)| \le \frac{C_2}{\log n}$ and $\sup_{x<0} |F^n(-b_n(-x)^{a_n})| \le 2^{-n}$. An interesting result has been proved by Ravi and Praveena [28] concerning the max domains of attraction under power normalization that every df in the max domain of attraction of the Frechéet $\Phi_{1:1}(x)$ $(H_{5;\beta}(x))$ law, as well as Weibull $\Phi_{2:1}(x)$ $(H_{6;\beta}(x))$ law, under power normalization is tail equivalent to some df satisfying the von-Mises type condition. More recently, Peng et al. [26] discussed two aspects of convergence of extremes under power normalization: convergence of moments and convergence of densities. A comprehensive survey of all the developments concerning the extremes under power normalization can be found in the recent book thesis by Barakat, et al [11].

3 The Possible Limit Laws of Central Order Statistics under Power Normalization

We consider the central os $X_{r_n:n}$ with variable rank sequences $\{r_n\}$, which satisfy the condition $\sqrt{n}(\frac{r_n}{n} - \lambda) \to t \in \Re$, as $n \to \infty$, and $\lambda \in (0,1)$. We begin with the normal $\lambda-$attraction case, in which $t = 0$ (the results when $t \ne 0$ follow as consequences). In the normal $\lambda-$attraction case, we have (c.f. Smirnov [30], and Leadbetter et al. [18], Pages 46-47)

$$F_{r_n:n}(G_n(x)) = I_{F(G_n(x))}(r_n, n - r_n + 1) \xrightarrow[n]{w} \Psi(x), \tag{6}$$

where Ψ is a nondegenerate limit df, if and only if $\sqrt{n}\left(\frac{F(G_n(x))-\lambda}{\sqrt{\lambda(1-\lambda)}}\right) \to \mathcal{N}^{-1}(\Psi(x))$, as $n \to \infty$, where $\mathcal{N}$ is the standard normal distribution and $I_u(a,b) = \frac{(a+b-1)!}{(a-1)!(b-1)!}\int_0^u t^{a-1}(1-t)^{b-1}dt$ is the incomplete beta function. Moreover, we have the following result.

Lemma 3.1 (Barakat and Omar [3]). Let the relation (6) be satisfied with a nondegenerate limit df Ψ. Then, for every sequence of integers $\{m_n\}$ such that $m_n < n$, $m_n \to \infty$ and $\frac{m_n}{n} \to \theta \in (0,1)$, we have

$$\sqrt{\theta}\,\mathcal{N}^{-1}((\Psi(x))) = \mathcal{N}^{-1}(\Psi(g_\theta(x))), \tag{7}$$

where $g_\theta(x) = \lim_{n\to\infty} G_{m_n}^{-1} \circ G_n(x)$ exists and satisfies the functional equation

$$g_{\theta\varphi}(x) = g_\theta \circ g_\varphi(x), \text{ with } \theta, \varphi \in (0,1). \tag{8}$$

If for each $G_n(x) \in \mathscr{G}$, then Pancheva ([21], [22], [23]) showed that the function $g_\theta(x)$, considered as a function of θ, is solvable (i.e., each equation of the form $g_\theta(x) = t$ for given x and t has a unique solution $\theta = \bar{g}(t,x)$. Moreover, the general solution of the functional equation (8) is given by (cf., Sreehari [31] and Barakat and omar [3])

$$g_\theta(x) = h^{-1}(h(x) - \mu \log\theta),\ \mu > 0, \theta \in (0,1), x \in \Re, \tag*{$(8)_1$}$$

$$g_\theta(x) = \ell^{-1}(\ell(x) + \acute{\mu} \log\theta),\ \acute{\mu} > 0, \theta \in (0,1), x \in \Re, \tag*{$(8)_2$}$$

$$g_\theta(x) = x,\ \forall\theta \in (0,1), \tag*{$(8)_3$}$$

where $h(x)$ and $\ell(x)$ are arbitrary continuous reversible functions in the sense that $h^{-1}(y)$ and $\ell^{-1}(y)$ are unique for $y \in \Re$ (see Pancheva [21] and Sreehari [31]).

Smirnov [30] solved the functional equation (7), when $G_n(x) = a_n x + b_n$, $a_n > 0$ and $b_n \in \Re$, which provides only four nondegenerate possible limits for the df of $X_{r_n:n}$ (each type of these limits has a domain of normal λ-attraction, i.e., $\sqrt{n}(\frac{r_n}{n} - \lambda) \to 0$, as $n \to \infty$). These limit types are:

$$\Psi_\ell^{(0)}(x) = \begin{cases} 0, & x \le -1, \\ \frac{1}{2}, & -1 < x \le 1, \\ 1, & x > 1; \end{cases} \quad \Psi_\ell^{(1)}(x) = \begin{cases} 0, & x \le 0, \\ \mathcal{N}(c_1 x^\alpha), & x > 0; \end{cases}$$

$$\Psi_\ell^{(2)}(x) = \begin{cases} 1, & x > 0, \\ \mathcal{N}(-c_2 \mid x \mid^\alpha), & x \le 0; \end{cases} \quad \Psi_\ell^{(3)}(x) = \begin{cases} \mathcal{N}(-c_2 \mid x \mid^\alpha), & x \le 0, \\ \mathcal{N}(c_1 x^\alpha), & x > 0. \end{cases}$$

Barakat and Omar [3] introduced another functional equation, which characterizes the possible nondegenerate limit df's of $X_{r_n:n}$ under transformations $G_n(x) \in \mathscr{G}$. Namely, let the relation (6) be satisfied with a nondegenerate df Ψ and the transformation $G_n(x)$. Then, as $n \to \infty$, clearly we have

$$m_n \left(\frac{F(G_n(x)) - \lambda}{\sqrt{\lambda(1-\lambda)}} \right)^2 \to (\mathscr{N}^{-1} \circ \Psi(g_\theta(x)))^2,$$

$$(n - m_n) \left(\frac{F(G_n(x)) - \lambda}{\sqrt{\lambda(1-\lambda)}} \right)^2 \to \left(\mathscr{N}^{-1} \circ \Psi(g_{1-\theta}(x))\right)^2$$

and

$$n \left(\frac{F(G_n(x)) - \lambda}{\sqrt{\lambda(1-\lambda)}} \right)^2 \to \left(\mathscr{N}^{-1} \circ \Psi(x)\right)^2.$$

Therefore, by adding the first two above relations and comparing the resulting sum with the third one, we get

$$(\mathscr{N}^{-1} \circ \Psi(g_\theta(x)))^2 + (\mathscr{N}^{-1} \circ \Psi(g_{1-\theta}(x)))^2 = (\mathscr{N}^{-1} \circ \Psi(x))^2.$$

By putting $J(x) = \mathscr{N}^{-1} \circ \Psi(x)$, we get the functional equation

$$J^2(g_\theta(x)) + J^2(g_{1-\theta}(x)) = J^2(x).$$

Since both the functions $\mathscr{N}$ and Ψ are nondecreasing in x, then $J(x)$ is also nondecreasing in x. Moreover, if $\overline{\rho} = \sup\{x : \Psi(x) < 1\}$ and $\underline{\rho} = \inf\{x : \Psi(x) > 0\}$ we have $J(\overline{\rho}) = \infty$ and $J(\underline{\rho}) = -\infty$. Barakat and Omar [3] obtained the following two general results.

Theorem 3.1. Let $\sqrt{n}(\frac{r_n}{n} - \lambda) \to 0$ and $G_n(x)$ be any strictly increasing continuous transformation for which (6) is satisfied. Then the possible nondegenerate types of $F_{r_n:n}(G_n(x))$ are

$$\Psi^{(0)}(x) = \mathscr{N}(J_0(x)) = \begin{cases} 0, & x \le \underline{\rho}, \\ \frac{1}{2}, & \underline{\rho} < x \le \overline{\rho}, \\ 1, & x > \overline{\rho}, \end{cases}$$

where $-\infty < \underline{\rho} < \overline{\rho} < \infty$, $g_\theta(\underline{\rho}) = \underline{\rho}$, $g_\theta(\overline{\rho}) = \overline{\rho}$;

$$\Psi^{(1)}(x) = \mathscr{N}(J_1(x)) = \begin{cases} 0, & x \le x_{01}, \\ \Phi(c_1 e^{\frac{\ell(x)}{2\mu}}), & x_{01} < x \le \overline{\rho}, \\ 1, & x > \overline{\rho}, \end{cases}$$

where $g_\theta(x_{01}) = x_{01} > -\infty$ $(\ell(x_{01}) = -\infty)$ and $g_\theta(x) < x, \forall x > x_{01}$. Moreover, $g_\theta(\overline{\rho}) = \overline{\rho} \le \infty$ $(\ell(\overline{\rho}) = \infty)$;

$$\Psi^{(2)}(x) = \mathscr{N}(J_2(x)) = \begin{cases} 0, & x \le \underline{\rho}, \\ 1 - \mathscr{N}(c_2 e^{-\frac{h(x)}{2\mu}}), & \underline{\rho} < x \le x_{02}, \\ 1, & x > x_{02}, \end{cases}$$

where $g_\theta(x_{02}) = x_{02} < \infty$, $(h(x_{02}) = \infty)$ and $g_\theta(x) > x, \forall x < x_{02}$. Moreover, $g_\theta(\underline{\rho}) = \underline{\rho} \ge -\infty$ $(h(\underline{\rho}) = -\infty)$, and finally,

$$\Psi^{(3)}(x) = \mathscr{N}(J_3(x)) = \begin{cases} 0, & x \le \underline{\rho}, \\ 1 - \mathscr{N}(c_2 e^{-\frac{h(x)}{2\mu}}), & \underline{\rho} < x \le x_{03}, \\ \mathscr{N}(c_1 e^{\frac{\ell(x)}{2\mu}}), & x_{03} < x \le \overline{\rho}, \\ 1, & x > \overline{\rho}, \end{cases}$$

where $g_\theta(x_{03}) = x_{03}$, $g_\theta(\underline{\rho}) = \underline{\rho}$, $g_\theta(\overline{\rho}) = \overline{\rho}$ and $-\infty \le \underline{\rho} < x_{03} < \overline{\rho} \le \infty$. Moreover, $g_\theta(x) > x, \forall x < x_{03}$ and $g_\theta(x) < x, \forall x > x_{03}$.

Theorem 3.2 (Barakat and Omar [3]). Let $r_n^\star \to \infty$, $\sqrt{n}(\frac{r_n^\star}{n} - \lambda) \to t \in \Re$ and $G_n(x) \in \mathscr{G}$ be any strictly monotone continuous transformation for which (6) is satisfied. Then the possible nondegenerate types of $F_{r_n^\star:n}(G_n(x))$ are

$$\Psi^{(i)}(x;t) = \mathscr{N}\left(\mathscr{N}^{-1}(\Psi^{(i)}(x)) - \frac{t}{\sqrt{\lambda(1-\lambda)}}\right) = \mathscr{N}\left(J_i(x) - \frac{t}{\sqrt{\lambda(1-\lambda)}}\right),\ i = 0,1,2,3.$$

The following theorem due to Barakat and Omar [3] shows an interesting fact that the possible nondegenerate weak limits of any central os with regular rank under the linear and power normalization are the same.
Theorem 3.3. Let $\sqrt{n}(\frac{r_n}{n} - \lambda) \to 0$ and $G_n(x)$ be any power transformation, i.e., $G_n(x) = b_n \mid x \mid^{a_n} \operatorname{sign}(x), a_n, b_n > 0$. Then the possible nondegenerate types of $F_{r_n:n}(G_n(x))$ are $\Psi_p^{(0i)}(x), i = 1,...,6$;

$$\Psi_p^{(1)}(x) = \begin{cases} 0, & x \le 0, \\ \mathscr{N}(x), & x > 0; \end{cases} \quad \Psi_p^{(2)}(x) = \begin{cases} 1, & x > 0, \\ \mathscr{N}(-\mid x \mid), & x \le 0; \end{cases}$$

and

$$\Psi_p^{(3)}(x) = \begin{cases} \mathscr{N}(-c_2 \mid x \mid), & x \le 0, \\ \mathscr{N}(c_1 x), & x > 0, \end{cases}$$

where $\Psi_p^{(01)}(x),...,\Psi_p^{(06)}(x)$ are all defined by $\Psi^{(0)}(x)$ ($\Psi^{(0)}(x)$ is defined in Theorem 3.1), with $\underline{\rho} < 0 < \overline{\rho}$, $\mid \underline{\rho} \mid \ne \overline{\rho}$; $0 < \underline{\rho} < \overline{\rho}$; $\underline{\rho} < \overline{\rho} < 0$; $\mid \underline{\rho} \mid = \mid \overline{\rho} \mid$; $0 = \underline{\rho} < \overline{\rho}$ and $0 = \overline{\rho} > \underline{\rho}$, respectively.

The type $\Psi^3(x)$ represents a family of two types for $c_1 \ne c_2$ and $c_1 = c_2 = 1$.
Remark 3.1. Theorem 3.3 states that the possible nondegenerate limit types for the central os's under linear and power normalization are the same. Although this fact is interesting, however it should not surprise us. This is because in the extremes case where the lower and upper extremes are sharply distinguished, Christoph and Falk [15] showed that the upper as well as the lower tail behaviour of the original df F might determine whether the df F belongs to one or another of the six possible power limit types. However, on one hand, neither the lower nor the upper tail behaviour has any influence on the weak convergence of the central os's. On the other hand, there are no sharp distinction between the lower and the upper central os's.

Theorem 3.3 incites us to make comparisons between the domains of attraction of each of these possible limit types under linear and power normalization.
Theorem 3.4 (Barakat and Omar [3].) Under linear normalization, let the df F belong to the domain of normal $\lambda-$attraction of the limit type $\Psi_\ell^{(3)}(x)$, such that $F^{-1}(\lambda) \ne 0$. Then, the domains of normal $\lambda-$attraction of the limit types $\Psi_p^{(1)}(x)$, $\Psi_p^{(2)}(x)$ and $\Psi_p^{(3)}(x)$, under the power normalization, do not contain the df F.
Remark 3.2. Theorem 3.4 shows that if any df F belongs to the domain of normal $\lambda-$attraction of the standard normal df under linear normalization, where $F^{-1}(\lambda) \ne 0$ (as happens in many cases, e.g., when F is absolutely continuous with finite positive probability density function at $F^{-1}(\lambda)$), then the domains of normal $\lambda-$attraction of the three limit types $\Psi_p^{(i)}(x), i = 1,2,3$, do not contain the df F.
Remark 3.3. Although, Theorem 3.4 gives some comparisons between the domains of attraction of the possible limit types under linear and power normalization, but determining the domains of attraction of the four limit laws $\Psi_p^{(i)}(x), i = 1,2,3,4$, is still open problem till now. Moreover, since the power and linear normalizations are leading to the same families of limit df's, the question of existence of a nonlinear normalization with a larger domain of attraction is still open for central os's.

4 The class of weak limits of intermediate Order Statistics under power normalization

We call $X_{k_n:n}$ and $X_{r_n:n}$ the upper and lower intermediate os's, respectively, if $k_n = n - r_n + 1, \frac{r_n}{n} \to 0$, as $n \to \infty$. A sequence $\{r_n\}$ is said to satisfy Chibisov's condition, if

$$\lim_{n\to\infty}\left(\sqrt{r_{n+z_n}} - \sqrt{r_n}\right) = \frac{\alpha l v}{2},\ l > 0, \tag{9}$$

for any sequence $\{z_n\}$ of integer-values, where $\frac{z_n}{n^{1-\frac{\alpha}{2}}} \to v$, as $n \to \infty$ ($0 < \alpha < 1$ and v is any arbitrary real number). As Chibisov [14] himself noted, the condition (9) implies that $\frac{r_n}{n^\alpha} \to \ell^2$, as $n \to \infty$. It is worth to mention that the latter condition implies Chibisov's condition (see, Barakat and Omar [5]), which means that the class of intermediate rank sequences which satisfy the Chibisov condition is a very wide class. Chibisov [14] showed that, whenever $\{r_n\}$ satisfies (9), the possible nondegenerate types of the limiting distribution of the lower intermediate term $X_{r_n:n}$, under linear normalization are

$$G_{1,\beta}(x) = \mathcal{N}(v_1(x;\beta)) = \mathcal{N}(\beta\log x),\ x>0;$$
$$G_{2,\beta}(x) = \mathcal{N}(v_2(x;\beta)) = \mathcal{N}(-\beta\log|x|),\ x\le 0;$$
$$G_3(x) = \mathcal{N}(v_3(x)) = \mathcal{N}(x). \tag{10}$$

The corresponding possible nondegenerate limiting distributions for the upper intermediate term $X_{k_n:n}$ are $\Upsilon_{i;\beta}(x) = 1-\mathcal{N}(v_i(-x,\beta)),\ i=1,2,3$ (note that $\Upsilon_{3;\beta}(x) = 1-\mathcal{N}(v_3(-x))$).

Remark 4.1. Note that $\Upsilon_{1;\beta} = G_{2;\beta},\ \Upsilon_{2;\beta} = G_{1;\beta}$ and $\Upsilon_{3;\beta} = G_{3;\beta}$. Therefore, we have $\{G_{i;\beta},\ i=1,2,3\} \equiv \{\Upsilon_{i;\beta},\ i=1,2,3\}$.

Barakat and Omar [5] showed that the possible nondegenerate types of the limit df's $\tilde{H}(x)$ and $\tilde{L}(x)$ of the df's of the upper and lower intermediate os's $X_{k_n:n}$ and $X_{r_n:n}$, respectively, under the power normalization are $\mathcal{N}(-\log u_{i;\beta}(x)) = 1-\mathcal{N}(\log u_{i;\beta}(x)), i=1,2,\ldots,6$, and $\mathcal{N}(\log u_{i;\beta}(-x)), i=1,2,\ldots,6$, respectively, where the functions $u_{i;\beta}(x), i=1,2,\ldots,6$, are defined in (3), i.e.,

$$\tilde{H}_{1;\beta}(x) = 1-\mathcal{N}(\beta\log((\log|x|))),\ x\le -1;$$
$$\tilde{H}_{2;\beta}(x) = \mathcal{N}(\beta\log(-\log|x|)),\ -1<x\le 0;$$
$$\tilde{H}_{3;\beta}(x) = 1-\mathcal{N}(\beta\log(-\log x)),\ 0<x\le 1;$$
$$\tilde{H}_{4;\beta}(x) = \mathcal{N}(\beta\log(\log x)),\ x>1;$$
$$\tilde{H}_{5;\beta}(x) = \tilde{H}_5(x) = \mathcal{N}(-\log|x|),\ x\le 0;$$
$$\tilde{H}_{6;\beta}(x) = \tilde{H}_6(x) = \mathcal{N}(\log x),\ x>0.$$

The corresponding types of the lower intermediate os are

$$\tilde{L}_{1,\beta}(x) = \mathcal{N}(\beta\log\log x),\ x>1;$$
$$\tilde{L}_{2,\beta}(x) = \mathcal{N}(-\beta\log(-\log x)),\ 0<x\le 1;$$
$$\tilde{L}_{3,\beta}(x) = \mathcal{N}(\beta\log(-\log|x|)),\ -1<x\le 0;$$
$$\tilde{L}_{4,\beta}(x) = \mathcal{N}(-\beta\log\log|x|),\ x\le -1;$$
$$\tilde{L}_{5;\beta}(x) = \tilde{L}_5(x) = \mathcal{N}(\log x),\ x>0;$$
$$\tilde{L}_{6;\beta}(x) = \tilde{L}_6(x) = \mathcal{N}(-\log|x|),\ x\le 0. \tag{11}$$

Remark 4.2. Although, in general, we have $\tilde{H}_{i;\beta} \ne \tilde{L}_{i;\beta},\ i=1,2,\ldots,6$, but a closer look at the two classes of possible limit laws of lower and upper intermediate os's under power normalization shows that they coincide, i.e., $\{\tilde{H}_{i;\beta},\ i=1,2,\ldots,6,\} \equiv \{\tilde{L}_{i;\beta},\ i=1,2,\ldots,6,\}$.

Remark 4.3. It is worth to mention that the possible limiting types of the intermediate os's under linear normalization, which are defined in (10), coincide with those of suitably linearly normalized record value (see Barakat [2]). This resemblance is due to the fact that both classes are governed by the same functional equation (see Barakat [2]). Therefore, it is not accidentally to find that the possible limiting power types (11) are coincide with the possible limiting types of record value under power normalization, which are obtained by Grigelionis [17].

Barakat and Omar [5] found the domains of attraction of all possible limit laws of the df of the lower intermediate os $X_{r_n:n}$ in the following theorem. Throughout this theorem, we write $F\in D_\ell(G)$ and $F\in D_p(L)$ to indicate that F belongs to the domain of attraction of the law G, under linear normalization $C_n(x) = a_nx+b_n$, and L, under power normalization $T_n(x) = \alpha_n|x|^{\beta_n}\operatorname{sign}(x)$, respectively. Also, we write $r(F) = \sup\{x : F(x)<1\}$ and $\ell(F) = \inf\{x : F(x)>0\}$ to denote the right and left end-points for the df F, respectively. Moreover, for any nondecreasing function F define $F^-(y) = \inf\{x : F(x)>y\}$.

Theorem 4.1. For any univariate continuous df F, we have the following implications:

1. If $0<\ell(F)<\infty$, then $F\in D_p(\tilde{L}_{1,\beta}) \Longleftrightarrow G\in D_\ell(G_{1,\beta})$, where

$$G(y) = \begin{cases} 0, & y\le \log\ell(F), \\ F(e^y), & y>\log\ell(F). \end{cases}$$

2.If $\ell(F)=0$, then $F\in D_p(\tilde{L}_{2,\beta}) \Longleftrightarrow G(y)=F(e^y)\in D_\ell(G_{2,\beta})$.
3.If $-\infty<\ell(F)<0$, then $F\in D_p(\tilde{L}_{3,\beta}) \Longleftrightarrow G(y)=\frac{F(-e^{-y})}{F(0)}\in D_\ell(G_{1,\beta^*})$, with rank sequence $k^*=[\frac{k}{F(0)}]$ and $\beta^*=\frac{\beta}{\sqrt{F(0)}}$, where $[\theta]$ denotes the integer part of θ.
4.If $\ell(F)=-\infty$, then $F\in D_p(\tilde{L}_{4,\beta}) \Longleftrightarrow G\in D_\ell(G_{2,\beta})$, where

$$G(y)=\begin{cases}F(-e^{-y}), & y\le 0,\\ 1, & y>0.\end{cases}$$

5.(i) If $\ell(F)=0$, then $F\in D_p(\tilde{L}_5) \Longleftrightarrow F(e^y)\in D_\ell(G_3)$.
(ii) If $\ell(F)>0$, then $F\in D_p(\tilde{L}_5) \Longleftrightarrow G(y)\in D_\ell(G_3)$, where

$$G(y)=\begin{cases}0, & y\le 0,\\ F(e^y), & y>0.\end{cases}$$

6.If $\ell(F)<0$, then $F\in D_p(\tilde{L}_6) \Longleftrightarrow G(y)=\frac{F(-e^{-y})}{F(0)}\in D_\ell(G_3)$, with rank sequence $k^*=[\frac{k}{F(0)}]$.

5 Statistical Inference under Power Normalization

In the literature, two classes of extreme value distributions are used in extreme value modeling. The first class, which is known as the generalized extreme value distribution under linear normalization (GEVL) H_γ, encompasses the three standard extreme value distributions (EVD), defined in (2): Fréchet, Weibull and Gumbel. The second class, which is called the generalized Pareto distribution (GPDL) under linear normalization introduced by Pickand [27], nests the Pareto, uniform and exponential distributions.

There are two main methods for modeling the extreme values, the Block Maxima (BM) method and the Peak Over Thresholds (POT) method.

In the BM method it is supposed to have observed maxima values of some quantities over a number of blocks. A typical example of the block is a year or a day and the observed quantities may be some environmental quantity such as the air pollutant. In this method, the block maxima is modeled by EVD. The choice of EVD is motivated by the facts: (i) The EVD are the only ones which can appear as the limit of linearly normalized maxima. (ii) They are the only ones which are “max-stable”, i.e., any change of the block size only leads to a change of location and scale parameters in the distribution.

In the POT method it is supposed to have all observed values, which are larger than some suitable threshold. These values are then assumed to follow the Generalized Pareto Family of Distributions (GPDL). The choice of GPDL is motivated by two characterizations: (i) The distribution of scale normalized exceedance over threshold asymptotically converges to a limit belonging to GPDL if and only if the distribution of BM converges (as the block length tends to infinity) to one of EVD. (ii) The distributions belonging to the GPDL are the only “stable”ones, i.e., the only ones for which, the conditional distribution of an exceedance is scale transformation of the original distribution.

Each of the families (4) and (5) is called generalized extreme value distribution under power normalization (GEVP). Clearly, both GEVP (4) and (5) satisfy the $p-\max-$stable property, i.e., for every n there exist power normalizing constants $a_n,b_n>0$, for which we have $H^n_{t;\gamma}(b_n|x|^{a_n}\text{sign}(x))=H_{t;\gamma}(x)$, $t=1,2$. Therefore, the two parametric models (4) and (5) enable us to apply the BM method under power normalization. For these models, the parametric approach to modeling extremes is based on the assumption that the data in hand form an i.i.d sample from an exact GEVP(γ,a,b) df in (4) or (5).

The main aim of this section is to develop the mathematical modeling of extremes under power normalization. The section material is quoted from Barakat [4] and Barakat, et al. [12], where the proofs of Theorems 5.1-5.4 can be found in Barakat et al. [12]. Firstly, we propose an estimator for the shape parameter γ within the model GEVP(γ,a,b). This estimator corresponds to a Dubey estimate in the GEVL model (cf. Reiss and Thomas [29], Page 111). Secondly, we derive the generalized Pareto distribution under power normalization (GPDP) for each of the models (4) and (5). Finally, we deal with estimators for the shape parameter within the derived GPDP.

Let $x_{1:n}\le x_{2:n}\le\ldots\le x_{n:n}$ be a given data of maximums of the given blocks. Clearly, in view of Remark 2.1, the modeling under power normalization can be applied only if all these maximums have the same sign. More specifically, if $0<x_{1:n}\le x_{2:n}\le\ldots\le x_{n:n}$, we select the model (4) and if $x_{1:n}\le x_{2:n}\le\ldots\le x_{n:n}<0$, we select the model (5). The estimate of the shape parameter γ corresponding to a Dubey estimate in the GEVL model is linear combinations of ratios of spacing

$$R_n=\frac{\log|x_{nq_2:n}|-\log|x_{nq_1:n}|}{\log|x_{nq_1:n}|-\log|x_{nq_0:n}|},$$

where $q_0 < q_1 < q_2$ and $q_i = \frac{i}{n}$. Note that this statistic is invariant under power transformation. We have the following theorem.

Theorem 5.1 (Statistical inference using BM method). Let q_0, q_1, q_2 satisfy the equation $(-\log q_1)^2 = (-\log q_2)(-\log q_0)$. Then,

$$\hat{\gamma} = \frac{2\log R_n}{\log(\log q_0/\log q_2)}.$$

On the other hand, if $q_0 = q, q_1 = q^a, q_2 = q^{a^2}$, for some $0 < q, a < 1$, we get the estimate family $\hat{\gamma} = \frac{\log R_n}{-\log a}$. By taking $a = \frac{1}{2}$, we get

$$\hat{\gamma} = \frac{\log R_n}{\log 2}.$$

Theorem 5.2 (GPDP). Let $F^{[u]}(x) = P(X \leq x | x > u)$.
(a) Let the df of the power normalized maximum os weakly converges to $H_{1;\gamma}(x; a, b)$. Then there exists $\alpha(u) > 0$ such that

$$F^{[u]}(ux^{\alpha(u)}) \xrightarrow{w} Q_{1;\gamma}(x; \overline{b}), \text{ as } u \uparrow r(F) > 0, \tag{12}$$

where $Q_{1;\gamma}(x; \overline{b}) = 1 + \log H_{1;\gamma}(x; 1, \overline{b})$, $\overline{b} = \frac{b}{c}$ and $c = 1 + \gamma \log a$.
(b) Let the df of the power normalized maximum os weakly converges to $H_{2;\gamma}(x; a, b)$. Then there exists $\alpha(u) > 0$ such that

$$F^{[u]}(u|x|^{\alpha(u)}) \xrightarrow{w} Q_{2;\gamma}(x; \underline{b}), \text{ as } u \uparrow r(F) \leq 0, \tag{13}$$

where $Q_{2;\gamma}(x; \underline{b}) = 1 + \log H_{2;\gamma}(x; 1, \underline{b})$, $\underline{b} = \frac{b}{c}$ and $\underline{c} = 1 - \gamma \log a$.

The following elementary theorem shows that the GPDP's (12) and (13) satisfy the peak over threshold stability property (POT).

Theorem 5.3 (The peak over threshold stability property). The left truncated GPDP yields again a GPDP. This means that, for every $0 < k < x$, we have $Q^{[k]}_{1;\gamma}(x; \sigma) = Q_{1;\gamma}(\frac{x}{k}; \overline{\sigma})$, where $\overline{\sigma} = \frac{\sigma}{c}$ and $c = 1 + \gamma \log k$. Moreover, for every $-1 < k < x < 0$, we have $Q^{[k]}_{2;\gamma}(x; \sigma) = Q_{2;\gamma}(\frac{x}{k}; \overline{\sigma})$, where $\overline{\sigma} = \frac{\sigma}{c'}$ and $c' = 1 - \gamma \log(-k)$.

In the following theorem we follow Pickand's method [27] to get estimates for the shape and the scale parameters, within (12) and (13).

Theorem 5.4 (Estimation of the shape and the scale parameters within GPDP model). Let n be the sample size and let $m = m(n)$ be an integer much smaller than n. Let $\{Y_i, i = 1, 2, \ldots, n\}$ be the descending os's, i.e., $Y_i = X_{n-i+1:n}$ is the ith largest observation in the sample. We treat the values $\frac{Y_i}{Y_{4m}}$, $i = 1, 2, \ldots, 4m-1$, as though they were the descending os's from a sample of size $4m-1$ from a population with a df of the form $Q_{1;\gamma}(x; \overline{\sigma})$ for some $\overline{\sigma}, \gamma, 0 < \overline{\sigma} < \infty, -\infty < \gamma < \infty$. The parameters γ and $\overline{\sigma}$ can be estimated by

$$\hat{\gamma} = (\log 2)^{-1} \log \frac{\log Y_m - \log Y_{2m}}{\log Y_{2m} - \log Y_{4m}}$$

and

$$\hat{\overline{\sigma}} = \frac{2^{\hat{\gamma}} - 1}{\hat{\gamma}(\log Y_{2m} - \log Y_{4m})}.$$

Moreover, the estimates of the shape and the scale parameters γ and $\underline{\sigma}$ in the GPDP (5) are given by

$$\hat{\gamma} = (\log 2)^{-1} \log \frac{\log |Y_m| - \log |Y_{2m}|}{\log |Y_{2m}| - \log |Y_{4m}|}$$

and

$$\hat{\underline{\sigma}} = \frac{1 - 2^{\hat{\gamma}}}{\hat{\gamma}(\log |Y_{2m}| - \log |Y_{4m}|)}.$$

The value $m = m(n)$ should satisfy the two conditions $\lim_{n\to\infty} m = \infty$ and $\lim_{n\to\infty} \frac{m}{n} = 0$.

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