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A second-order differentiable smoothing approximation lower order exact penalty function

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Abstract: In this paper, we give a smoothing approximation to the lower order exact penalty functions for inequality-constrained optimization problems. Error estimations are obtained among the optimal objective function values of the smoothed penalty problem, of the nonsmooth penalty problem and of the original optimization problem. An algorithm based on the smoothed penalty function is presented, which is shown to be globally convergent under some mild conditions. Numerical examples are given to illustrate the effectiveness of the present smoothing method.

Keywords: smoothing approximation, lower order exact penalty function, inequality-constrained optimization problem, global solution.

1. Introduction

Consider the following optimization problem (P):

$$\begin{array}{ll} minimize & f(x) \\ subject\ to & g_i(x) \leq 0, i = 1, \cdots, m, x \in \Re^n, \end{array}$$

where $f : \Re^n \to \Re$ and $g_i : \Re^n \to \Re, i = 1, \cdots, m$ are continuously differentiable.

In Zangwill [1], an exact penalty function was defined as follows:

$$\varphi_q(x) = f(x) + q \sum_{i=1}^{m} g_i^+(x), \quad (1)$$

where $g_i^+(x) = \max(0, g_i(x)), i = 1, \cdots, m$, which is the l_1 exact penalty function for problem (P). Let $I = \{1, \cdots, m\}$. After Zangwill's development, extensive research of exact penalty function methods has been carried out in the literature ([2–6]). l_1 exact penalty function has many nice properties. However, (1) is not a smooth function and causes some numerical instability problems in its implementation when the value of the penalty parameter q becomes larger. In practice, we only need to obtain an approximately optimal solution to problem (P). Thus the smoothing of the l_1 exact penalty function attracts much attention ([7–11]).

In recent years, the following lower order exact penalty function

$$\varphi_{q,v}(x) = f(x) + q \sum_{i=1}^{m} g_i^+(x)^v, v \in (0, 1), \quad (2)$$

has been introduced and investigated ([12, 13]). It is shown in [13] that the second-order sufficiency condition implies local exact penalty property for the lower order penalty function with any positive penalty parameter. The smoothing of the $\frac{1}{2}$-order penalty function, i.e. $f(x)+q\sum_{i=1}^{m} g_i^+(x)^{\frac{1}{2}}$ has been investigated in[13, 14]. The first-order differentiable smoothing of the lower order penalty function (2) has been discussed in [15]. In this paper, we aim to smooth of (2).

The rest of this paper is organized as follows. In Sect. 2, a second-order differentiable smoothing approximation to the lower order exact penalty function (2) is introduced, and some fundamental properties of the smoothing function are discussed. In Sect. 3, we propose an algorithm to compute an approximate global solution to (P) based on our smooth penalty function. Some numerical examples are given in Sect. 4 to illustrate the effectiveness of the present method.

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2. A second-order differentiable smoothing lower order exact penalty function

We say the pair (x^*,λ^*) satisfies the second-order sufficiency condition, if (see [16], p.169)

$$\begin{array}{l}\nabla_x L(x^*,\lambda^*)=0,\\ g_i(x^*)\le 0, i\in I,\\ \lambda_i^*\ge 0. i\in I,\\ \lambda_i^* g_i(x^*)=0. i\in I,\\ y^T\nabla^2 L(x^*,\lambda^*)y>0, for\ all\ y\in V(x^*),\end{array} \quad (3)$$

where $L(x,\lambda)=f(x)+\sum_{i=1}^m \lambda_i g_i(x)$, and

$$V(x^*)=\left\{y\in\Re^n | \begin{array}{l}\nabla^T g_i(x^*)y=0, i\in A(x^*)\\ \nabla^T g_i(x^*)y\le 0, i\in B(x^*)\end{array}\right\},$$

$$A(x^*)=\{i\in I|g_i(x^*)=0,\lambda_i^*>0\},$$

$$B(x^*)=\{i\in I|g_i(x^*)=0,\lambda_i^*=0\},$$

Assumption 1 *$f(x)$ satisfies the coercive condition:*

$$\lim_{\|x\|\to\infty} f(x)=+\infty.$$

Under Assumption 1, we know that there exists a box X such that $G(P)\subset X$, where $G(P)$ is a set of global solution to problem (P). As far as the global solutions are concerned, problem $(P)$is equivalent to the following optimization problem(P'):

$$\begin{array}{ll} minimize & f(x)\\ subject\ to & g_i(x)\le 0, i=1,\cdots,m, x\in X,\end{array}$$

i.e.$G(P)=G(P')$,where $G(P')$ is the set of global solutions of problem (P'). For any $v\in(0,1)$, consider the following lower order penalty problem$(LOP)_v$:

$$\begin{array}{ll} minimize & \varphi_{q,v}(x)=f(x)+q\sum_{i=1}^m (g_i^+(x))^v\\ subject\ to & x\in X.\end{array}$$

We have the following global exact penalization property:

Lemma 1.*(Corollary 2.3 in [13]) Suppose that the set $G(P)$ is a finite set, and Assumption 1 holds, furthermore, for any $x^*\in G(P)$, there exists a $\lambda\in\Re_+^m$ such that the pair (x^*,λ^*) satisfies the second-order sufficiency condition (2.1). Then, for any given $v\in(0,1)$, there exists $q^*>0$, such that when $q>q^*, G(P)=G((LOP)_v)$, where $G((LOP)_v)$ is the set of global solutions of problem$(LOP)_v$.*

Let $p_v(u)=(\max(0,u))^v$, that is,

$$p_v(u)=\begin{cases}0, & if\ u<0,\\ u^v, & if\ u\ge 0.\end{cases} \quad (4)$$

then

$$\varphi_{q,v}(x)=f(x)+q\sum_{i=1}^m p_v(g_i(x)). \quad (5)$$

For any $\varepsilon>0$, let

$$p_{\varepsilon,v}(u)=\begin{cases}0, & if\ u<0,\\ \frac{1}{\varepsilon^2}\frac{v^2}{(v+1)(v+2)}u^{(v+2)}, & if\ 0\le u<\varepsilon,\\ u^v+\frac{1}{v+1}\varepsilon u^{v-1}-\frac{4}{v+2}\varepsilon^v, & if\ \varepsilon\le u.\end{cases} \quad (6)$$

It is easy to see that is twice continuously differentiable on $\Re$.

Assume that f and $g_i, i\in I$ are twice continuously differentiable. Let

$$\varphi_{q,\varepsilon,v}(x)=f(x)+q\sum_{i=1}^m p_{\varepsilon,v}(g_i(x)). \quad (7)$$

Then $\varphi_{q,\varepsilon,v}(x)$ is twice continuously differentiable on $\Re^n$. Consider the following smoothed optimization problem (SP):

$$(SP) \quad \min_{x\in X}\varphi_{q,\varepsilon,v}(x).$$

Proposition 1.*For any $x\in X$ and $\varepsilon>0$, we have that*

$$0\le\varphi_{q,v}(x)-\varphi_{q,\varepsilon,v}(x)\le\frac{3v+2}{(v+1)(v+2)}mq\varepsilon^v. \quad (8)$$

*Proof.*Note that

$$p_v(g_i(x))-p_{\varepsilon,v}(g_i(x))=$$
$$\begin{cases}0, & g_i(x)<0,\\ (0\le)[g_i(x)]^v-\frac{1}{\varepsilon^2}\frac{v^2}{(v+1)(v+2)}g_i(x)^{(v+2)} \\ (\le\frac{3v}{(v+1)(v+2)}\varepsilon^v), & 0\le g_i(x)<\varepsilon,\\ (0\le)-\frac{1}{v+1}\varepsilon g_i(x)^{v-1}+\frac{4}{v+2}\varepsilon^v \\ =\frac{3v}{(v+1)(v+2)}\varepsilon^v, & \varepsilon\le g_i(x),\end{cases}$$

so for any $i\in I$, we have

$$\begin{array}{l}0\le\varphi_{q,v}(x)-\varphi_{q,\varepsilon,v}(x)\\ =q\sum_{i=1}^m(p_v(g_i(x))-p_{\varepsilon,v}(g_i(x)))\\ \le\frac{3v+2}{(v+1)(v+2)}mq\varepsilon^v.\end{array}$$

Proposition 2.*Let $x_{q,v}^*\in X$ be a global solution of problem $(LOP)_v$ and $\bar{x}_{q,\varepsilon,v}\in X$ be a global solution of problem (SP) for some $q>0, v\in(0,1)$ and $\varepsilon>0$. Then we have that*

$$0\le\varphi_{q,v}(x_{q,v}^*)-\varphi_{q,\varepsilon,v}(\bar{x}_{q,\varepsilon,v})\le\frac{3v+2}{(v+1)(v+2)}mq\varepsilon^v. \quad (9)$$

Proof. By Proposition 1, we have that

$$0 \le \varphi_{q,v}(x^*_{q,v}) - \varphi_{q,\varepsilon,v}(x^*_{q,v}) \le \varphi_{q,v}(x^*_{q,v}) - \varphi_{q,\varepsilon,v}(\bar{x}_{q,\varepsilon,v}) \le \varphi_{q,v}(\bar{x}_{q,\varepsilon,v}) - \varphi_{q,\varepsilon,v}(\bar{x}_{q,\varepsilon,v}) \le \frac{3v+2}{(v+1)(v+2)} mq\varepsilon^v.$$

Corollary 1. *Let $x^* \in X$ be a global solution of problem (P) and $\bar{x}_{q,\varepsilon,v} \in X$ be a global solution of problem (SP) for given v and ε. Then there exists $q^* > 0$ such that for any $q > q^*$, it holds that*

$$0 \le f(x^*) - \varphi_{q,\varepsilon,v}(\bar{x}_{q,\varepsilon,v}) \le \frac{3v+2}{(v+1)(v+2)} mq\varepsilon^v. \tag{10}$$

where q^* is defined in lemma 1.

Proof. By Lemma 1, we know that x^* is a global solution of problem $(LOP)_v$. Then it follows from Proposition 2 that

$$0 \le \varphi_{q,v}(x^*_{q,v}) - \varphi_{q,\varepsilon,v}(\bar{x}_{q,\varepsilon,v}) \le \frac{3v+2}{(v+1)(v+2)} mq\varepsilon^v.$$

Note that

$$f(x^*) - \varphi_{q,\varepsilon,v}(\bar{x}_{q,\varepsilon,v}) = (f(x^*)) + q\sum_{i=1}^m p_v(g_i(x^*)) - \varphi_{q,\varepsilon,v}(\bar{x}_{q,\varepsilon,v}) = \varphi_{q,v}(x^*_{q,v}) - \varphi_{q,\varepsilon,v}(\bar{x}_{q,\varepsilon,v}),$$

since $\sum_{i=1}^m p_v(g_i(x^*)) = 0$, we complete the proof.

Definition 1. *For $\varepsilon > 0$, a point $x \in X$ is said to be ε-feasible solution to problem (P'), if*

$$g_i(x) \le \varepsilon \ \ for\ any \ \ i \in I. \tag{11}$$

Theorem 2. *Let $x^*_{q,v} \in X$ be a global solution of problem $(LOP)_v$ and $\bar{x}_{q,\varepsilon,v} \in X$ be a global solution to problem (SP). Furthermore, let $x^*_{q,v}$ be a feasible solution to problem (P') and $\bar{x}_{q,\varepsilon,v}$ be an ε-feasible solution to problem (P'), then we have that*

$$0 \le f(x^*_{q,v}) - f(\bar{x}_{q,\varepsilon,v}) < \frac{3}{v+2} mq\varepsilon^v. \tag{12}$$

Proof. It is clear that $\sum_{i=1}^m p_v(g_i(\bar{x}_{q,v})) = 0$ and

$$\sum_{i=1}^m p_{q,v}(g_i(\bar{x}_{q,\varepsilon,v})) \le \sum_{i=1}^m \frac{v^2}{(v+1)(v+2)}\varepsilon^v = \frac{v^2}{(v+1)(v+2)} m\varepsilon^v < \frac{1}{(v+1)(v+2)} m\varepsilon^v.$$

By proposition 2, we have that

$$0 \le \varphi_{q,v}(x^*_{q,v}) - \varphi_{q,\varepsilon,v}(\bar{x}_{q,\varepsilon,v}) = (f(x^*_{q,v}) + q\sum_{i=1}^m p_v(g_i(x^*_{q,v}))) - (f(\bar{x}_{q,\varepsilon,v}) + q\sum_{i=1}^m p_v(g_i(\bar{x}_{q,\varepsilon,v}))) \le \frac{3v+2}{(v+1)(v+2)} mq\varepsilon^v.$$

which implies

$$q\sum_{i=1}^m p_{\varepsilon,v}(g_i(\bar{x}_{q,\varepsilon,v})) \le f(x^*_{q,v}) - f\bar{x}_{q,\varepsilon,v}) \le \sum_{i=1}^m p_{\varepsilon,v}(g_i(\bar{x}_{q,\varepsilon,v})) + \tfrac{3v+2}{(v+1)(v+2)} mq\varepsilon^v.$$

Then it follows that

$$0 \le f(x^*_{q,v}) - f\bar{x}_{q,\varepsilon,v}) < \frac{1}{(v+1)(v+2)} mq\varepsilon^v + \frac{3v+2}{(v+1)(v+2)} mq\varepsilon^v < \frac{3}{v+2} mq\varepsilon^v.$$

Remark. It is easy to see that when q is sufficiently large, $x^*_{q,v}$ is a global solution to problem (P'), that is, $x^*_{q,v}$ is feasible to problem (P'). It is also easy to see that for given $\varepsilon > 0$, when is sufficiently large, $\bar{x}_{q,\varepsilon,v}$ is an ε-feasible solution to problem (P'). Therefore, it follows from (10) that when q is sufficiently large,

$$0 \le f^* - f(\bar{x}_{q,\varepsilon,v}) < \frac{3}{v+2} mq\varepsilon^v.$$

where f^* is the optimal value of problem(P').

3. Algorithm

We propose the following algorithm to solve problem (P').

Step 1. Given $x^0, \varepsilon_0 > 0, q_0 > 0, 0 < \eta < 1$ and $N > 1$, let $k = 0$ and go to Step 2.

Step 2. Use x^k as the starting point to solve problem

$$(SP_k) \quad \min_{x \in X} \varphi_{q_k,\varepsilon_k,v}(x).$$

Let x^*_k be the optimal solution obtained.

Step 3. If If is ε-feasible to problem (P'), then stop and the algorithm has generated an approximate global solution x^*_k of problem (P'). Otherwise, let $q_{k+1} = Nq_k, \varepsilon_{k+1} = \eta\varepsilon_k, x^k = x^*_k$ and then go to step 2.

4. Numerical examples

In this section, we give several numerical examples to show the effective of the presented algorithm. The numerical results have been recorded in the tables following each example.

Example 1. (The Rosen-Suzki problem in [4,14])

$$\begin{aligned}&minimize \quad f(x)\\&= x_1^2+x_2^2+2x_3^2+x_4^2-5x_1-5x_2-21x_3+7x_4\\&subject\ to\\&g_1(x)=2x_1^2+x_2^2+x_3^2+2x_1+x_2+x_4-5\le 0\\&g_2(x)=x_1^2+x_2^2+x_3^2+x_4^2+x_1-x_2+x_3-x_4-8\le 0\\&g_3(x)=x_1^2+2x_2^2+x_3^2+2x_4^2-x_1-x_4-10\le 0\end{aligned}$$

The corresponding problem (SP) is as follows:

$$(SP)\quad \min_{x\in X}\varphi_{q,\varepsilon,v}(x)=f(x)+q\sum_{i=1}^{3}p_{\varepsilon,v}(g_i(x)),$$

where X can be taken as $[-100,100]^4$,and we take $v=\frac{1}{2}$, so

$$p_{\varepsilon,v}(g_i(x)) = \begin{cases}0, & if\ \ g_i(x)<0,\\ \frac{1}{15\varepsilon^2}[g_i(x)]^{\frac{5}{2}}, & if\ \ 0\le g_i(x)<\varepsilon,\\ [g_i(x)]^{\frac{1}{2}}+\frac{2}{3}\varepsilon[g_i(x)]^{-\frac{1}{2}}-\frac{8}{5}\varepsilon^{\frac{1}{2}}, & if\ \varepsilon\le g_i(x),\end{cases}$$

In this example, we take $\varepsilon=10^{-6}, N=2$, and $\eta=0.5$ in step 1.

Numerical results for Example1

	x_k
1	$(0\ 0\ 0\ 0)^T$
2	$(1.7936\ 1.8747\ 4.6440\ \ -2.7916)^T$
3	$(1.1632\ 1.2944\ 3.8558\ \ -2.0281)^T$
4	$(0.1888\ 0.7348\ 2.1767\ \ -1.2519)^T$
13	$(0.1694\ 0.8355\ 2.0087\ \ -0.9648)^T$

	x_k^*	ρ_k	η_k
1	1	1	$(1.7936\ 1.8747\ 4.6440\ \ -2.7916)^T$
2	2	0.5	$(1.1632\ 1.2944\ 3.8558\ \ -2.0281)^T$
3	4	0.25	$(0.1888\ 0.7348\ 2.1767\ \ -1.2519)^T$
4	8	0.125	$(0.1723\ 0.8374\ 2.0184\ \ -0.9740)^T$
13	4096	0.5^{12}	$(0.1694\ 0.8355\ 2.0087\ \ -0.9648)^T$

	$f(x_k^*)$	$g_1(x_k^*)$	$g_2(x_k^*)$	$g_3(x_k^*)$
1	-77.7486	29.1855	35.4458	38.3968
2	-70.5806	0.8414	19.7616	18.6624
3	-47.4728	0.2097	1.7635	0.0511
4	-44.4257	0.0426	0.0808	-1.7948
13	-44.2338	-1.7134e-004	-1.28666e-004	-1.8832

It is clear from this table that the obtained approximate global solution is $x^*=(0.1694,0.8355,2.0087,-0.9648)$, with objective value $f^*=-44.2338$. In [14], the obtained approximate global solution is (0.169234,0.835656,2.008690,-0.964901) with corresponding objective function value -44.233582.

Example 2.([10])

$$\begin{aligned}&minimize \quad f(x)=10x_2+2x_3+x_4+3x_5+4x_6\\&subject\ to\\&g_1(x)=x_1+x_2-10=0,\\&g_2(x)=-x_1+x_3+x_4+x_5=0,\\&g_3(x)=-x_2-x_3+x_5+x_6=0,\\&g_4(x)=10x_1-2x_3+3x_4-2x_5-16\le 0,\\&g_5(x)=x_1+4x_3+x_5-10\le 0,\\&0\le x_1\le 12,\\&0\le x_2\le 18,\\&0\le x_3\le 5,\\&0\le x_4\le 12,\\&0\le x_5\le 1,\\&0\le x_6\le 16.\end{aligned}$$

We take $v=\frac{1}{3}$, the corresponding smoothed penalty function problem (SP) can be constructed as

$$p_{\varepsilon,v}(g_i(x)) = \begin{cases}0, & if\ \ g_i(x)<0,\\ \frac{1}{28\varepsilon^2}[g_i(x)]^{\frac{7}{3}}, & if\ \ 0\le g_i(x)<\varepsilon,\\ [g_i(x)]^{\frac{1}{3}}+\frac{3}{4}\varepsilon[g_i(x)]^{-\frac{2}{3}}-\frac{12}{7}\varepsilon^{\frac{1}{3}}, & if\ \varepsilon\le g_i(x),\end{cases}$$

Let $x^0=(0,0,\cdots,0),\varepsilon=10^{-6},\varepsilon_0=0.1,\rho_0=1000,\eta=0.01,N=2$. We use the algorithm to solve the example. After the third iteration we can get the approximate global solution (1.806,8.194,0.498,0.308,1.000,7.692) with corresponding objective function value $f^*=117.010$, which is much better than that obtained in [10].

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