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On covering of products of T-generalized state machines

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Abstract: We introduce the concepts of T-generalized state machines and coverings of products of them. Also some of algebraic properties of them are investigated. Finely some products such as direct sum and sum of T-generalized state machines are introduced. An interesting distributive property of cascade product over the sum of T-generalized state machines concern to covering of T-generalized state machines is stablished.

Keywords: T-generalized state machine; Covering; Cascade product; Wreath product; Sum of two T-generalized state machines; Direct sum.

1. Introduction

Since Wee (1967) introduced the concept of fuzzy automata following Zadeh (1965), fuzzy automata theory has been developed by many researchers. Recently, Malik et al. (1994, 1997) introduced the concepts of fuzzy state machines and fuzzy transformation semigroups based on Wee's concept of fuzzy automata and related concepts and applied algebraic techniques. Kim et al. (1998) introduced the notion of T-generalized state machine that is extension of fuzzy state machine . Even if $T=\wedge$ our notion of generalized state machine is different from the notion of Malik (1994). In this paper, we introduce On Covering of products of T- generalized state machines and investigate their algebraic structures. For the terminology in (crisp) algebraic automata theory, we refer to Holcombe (1982).

2. Preliminaries

Definition 2.1. (Kim et al. (1998)). A triple $M = (Q, X, \lrcorner)$ where Q and X are finite nonempty sets and $\lrcorner$ is a fuzzy subset of $Q \times X \times Q$, i.e. , $\lrcorner$ is a function from $Q \times X \times Q$ to $[0,1]$, is called a generalized state machine. If $\sum_{q \in Q} \tau(p, a, q) \leq 1$ for all $p \in Q$ and $a \in X$. If $\sum_{q \in Q} \tau(p, a, q) = 1$ for all $p \in Q$ and $a \in X$, then M is said to be complete. Note that the concept of generalized state machines is different from the concept of fuzzy finite state machines of Malik et al. (1994) which is also a fuzzification of the concept of state machines. Their notion is based on the concept of fuzzy automata introduced by Wee (1967). While a generalized state machine $(Q, X, \lrcorner)$ with $\tau(Q \times X \times Q) \subseteq \{0,1\}$ can always be regarded as a state machine, but a fuzzy finite state machine $(Q, X, \lrcorner)$ with $\tau(Q \times X \times Q) \subseteq \{0,1\}$ cannot be regarded as a state machine generally. So the concept of generalized state machines is a generalization of the concept of state machines, whereas the concept of fuzzy finite state machines of Maliket al. may not be considered as a generalization of the concept of state machines in a certain sense. This means that the concept of generalized state machines is a more adequate fuzzification of the concept of state machines than the concept of fuzzy finite state machines. Let $M = (Q, X, \lrcorner)$ be a generalized state machine. Then Q is called the set of states and X is called the set of input symbols. Let X^+ denote the set of all words of elements of X of finite length. Formally, every incomplete generalized state machine can be extended to a complete generalized state machine as follows:

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Definition 2.2. (Kim et al. (1998)). Let $M = (Q, X, \tau)$ be an incomplete generalized state machine. Let z be a state not in Q. The completion M^c of M is the complete generalized state machine (Q', X, τ') given by $Q' = Q \cup \{z\}$ and

$$\tau'(p', a, q') = \begin{cases} \tau'(p', a, q') & \text{if } p', q' \in Q, \\ 1 - \sum_{q \in Q} \tau(p', a, q') & \text{if } p' \in Q \text{ and } q' = z \\ 0 & \text{if } p' = z \text{ and } q' \in Q \\ 1 & \text{if } p', q' = z \end{cases}$$

For all $a \in X$. The new state z is called the sink state of M^c. If M is complete, then we take M itself as M^c.

Definition 2.3. (Schweizer and Sklar (1960)). A binary operation T on $[0,1]$ is called a t-norm if for all $a, b, c \in [0,1]$:

(1) $T(a, 1) = a$,

(2) $T(a, b) \le T(a, c)$ whenever $b \le c$,

(3) $T(a, b) = T(b, a)$,

(4) $T(a, T(b, c)) = T(T(a, b), c)$

The maximum and minimum will be written as $\vee$ and $\wedge$, respectively. T is clearly $\vee$-distributive, i.e., $T(a \vee b, c) = T(a, c) \vee T(b, c)$ for all $a, b, c \in [0,1]$. Define T_0 on $[0,1]$ by $T_0(a, 1) = a = T_0(1, a)$ and $T_0(a, b) = 0$ if $a \ne 1$ for all $a, b \in [0,1]$. Then $\wedge$ is the greatest t -norm on $[0,1]$ and T_0 is the least t -norm on $[0,1]$, i.e., for any t -norm T, $\wedge(a, b) \ge T(a, b) \ge T_0(a, b)$ for all $a, b \in [0,1]$. T will always mean a t –norm on $[0,1]$. By an abuse of notation we will denote $T\left(a_1, T\left(a_2, T(\ldots, T(a_{n-1}, a_n) \ldots)\right)\right)$ by $T(a_1, \ldots, a_n)$ where $a_1, \ldots, a_n \in [0,1]$. The legitimacy of this abuse is ensured by the associatively of T (Definition 2.3[4]).

Definition 2.4. (Kim et al. (1998)). Let $M = (Q, X, \tau)$ be a generalized state machine.
Define $\tau^+ : Q \times X^+ \times Q \to [0,1]$ by

$$\tau^+(p, a_1, \ldots, a_n, q) = \vee \{T(\tau(p, a_1, r_1), \tau(r_1, a_2, r_2), \ldots, \tau(r_{n-2}, a_{n-1}, r_{n-1}), \tau(r_{n-1}, a_n, q)) \mid r_i \in Q\},$$

where $p, q \in Q$ and $a_1, \ldots, a_n \in X$. When T is applied to M as above, M is called a T-generalized state machine. Hereafter a generalized state machine will always be written as a T-generalized state machine because a generalized state machine always induces a T-generalized state machine as in Definition 2.4.

Definition 2.5. (Kim et al. (1998)). Let (Q, X, τ) be a T-generalized state machine. Then

$$\tau^+(p, xy, q) = \vee \{T(\tau^+(p, x, r), \tau^+(r, y, q) \mid r \in Q\}$$

For all $p, q \in Q$ and $x, y \in X^+$.

Example 2.6. (Kim et al. (1998)).

$T = \wedge, Q = \{p_1, p_2, \ldots, p_7\}$ and $X = \{a\}$.Let Then (Q, X, τ) is a T-generalized state machine. However,

$$\sum_{q \in Q} \tau(p_1, aa, q) = \tau(p_1, aa, p_4) + \tau(p_1, aa, p_5) + \tau(p_1, aa, p_6)$$
$$\tau(p_1, aa, p_7) = 0.6 + 0.4 + 0.1 + 0.3 > 1.$$

3. Covering

Definition 3.1. (Kim et al. (1998)). Let $M_1 = (Q_1, X_1, \tau_1)$and $M_2 = (Q_2, X_2, \tau_2)$ be T-generalized state machines. If $\varepsilon : X_1 \to X_2$ is a function and $\eta : Q_2 \to Q_1$ is a subjective partial function such that $\tau_1^+(\eta(p), x, \eta(q)) \le \tau_2^+(p, \varepsilon(x), q)$ for all p, q in the domain of η and $x \in X_1^+$, then we say that (η, ε) is a covering of M_1 by M_2 and M_2 covers M_1 that denote by $M_1 \le M_2$ Moreover, if the inequality turns out equality whenever the left-hand side of the inequality is not zero (resp. the inequality always turns out equality), then we say that (η,ε) is a strong covering (resp. a complete covering) of M_1 by M_2 and that M_2 strongly covers (resp. completely covers) M_1 and denote by $M_1 \leq_s M_2$ (resp. $M_1 \leq_c M_2$). In Definition 3.1. we abused the function ε. We will write the natural semigroup homomorphism from ${X_1}^+$ to ${X_2}^+$ induced by ε Also for convenience sake. We give an example that is elementary and important.

Example 3.2. (Kim et al. (1998)). Let $M = (Q, X, \tau)$ be a T-generalized state machine. Define an equivalence relation $\sim$ on X by $a \sim b$ if and only if $\tau(p,a,q) = \tau(p,b,q)$ for all $p,q \in Q$. Construct a T-generalized state machine $M_1 = (Q, X/\sim, \tau^\sim)$ by defining $\tau^\sim(p,[a],q) = \tau(p,a,q)$. Now define $\varepsilon: X \to X/_\sim$ by $\varepsilon(a)=[a]$ and $\eta = 1_Q$. Then (η,ε) is a complete covering of M by M_1 clearly.

Proposition 3.3. (Cho. et al. (2001)). Let M_1, M_2 and M_3 be T-generalized state machines. If $M_1 \leq M_2$ (resp. $M_1 \leq_s M_2$, $M_1 \leq_c M_2$) and $M_2 \leq M_3$ (resp. $M_2 \leq_s M_3$, $M_2 \leq_c M_3$), then $M_1 \leq M_3$ (resp. $M_1 \leq_s M_3$, $M_1 \leq_c M_3$).

4. Products

In this section, we consider cascade products and wreath products of T-generalized state machines, where T is less than or equal to the ordinary product. We will always assume that T is less than or equal to the ordinary product.

Definition 4.1. (Cho. et al. (2001)). Let $M_1 = (Q_1, X_1, \tau_1)$ and $M_2 = (Q_2, X_2, \tau_2)$ be T-generalized state machines. The cascade product $M_1 \omega_T M_2$ of M_1 and M_2 with respect to $\omega: Q_2 \times X_2 \to X_1$ is the T-generalized state machine $(Q_1 \times Q_2, X_2, \tau_1 \, \omega_T \, \tau_2)$ with

$$(\tau_1 \, \omega_T \, \tau_2)((p_1,p_2), b, (q_1,q_2)) = T(\tau_1(p_1,\omega(p_2,b),q_1), \tau_2(p_2,b,q_2))$$

In Definition 4.1, $(Q_1 \times Q_2, X_2, \tau_1 \, \omega_T \, \tau_2)$ is clearly a T-generalized state machine. In fact, we have

$$\sum_{(q_1,q_2)\in Q_1\times Q_2} (\tau_1 \, \omega_T \, \tau_2)((p_1,p_2),b,(q_1,q_2)) = \sum_{(q_1,q_2)\in Q_1\times Q_2} T(\tau_1(p_1,\omega(p_2,b),q_1),\tau_2(p_2,b,q_2)) \leq \sum_{(q_1,q_2)\in Q_1\times Q_2} \tau_1(p_1,\omega(p_2,b),q_1),\tau_2(p_2,b,q_2)$$

Because for any t $-$norm T, $a \wedge b \geq T(a,b) \leq \sum_{q_1\in Q_1} \tau_1(p_1,\omega(p_2,b),q_1)$
and

$\sum_{q_2\in Q_2} \tau_2(p_2,b,q_2) \leq 1$ for all $p_1 \in Q_1, p_2 \in Q_2$ and $b \in X_2$.

Let $M_1=(Q_1,X_1,\tau_1)$ and $M_2=(Q_2,X_2,\tau_2)$ be T-generalized state machines and $\omega:Q_2\times X_2\to X_1$. Define $\omega^+:Q_2\times {X_2}^+\to {X_1}^+$ by $\omega^+(p_2,b_1b_2 \ldots b_n\}) = \omega(p_2,b_1)\omega(u_1,b_2) \ldots$

$\omega(u_{n-1},b_n)$, Where $p_2,u_1,u_2,\ldots,u_{n-1}\in Q_2$ and $b_1,\ldots,b_n\in X_2$ such that ${\tau_1}^+(p_1,\omega(p_2,b_1)\omega(u_1,b_2) \ldots \omega(u_{n-1},b_n),q_1) = \vee\{{\tau_1}^+(p_1,\omega(p_2,b_1)\omega(r_1,b_2) \ldots$
$\omega(r_{n-1},b_n),q_1) \mid r_1,r_2,\ldots,r_{n-1}\in Q_2\}$, Where $p_1,q_1\in Q_1$ and $b\in X_2$.

Lemma 4.2. (Cho. et al. (2001)). Let $M_1=(Q_1,X_1,\tau_1)$ and $M_2=(Q_2,X_2,\tau_2)$ be T-generalized state machines.Then $(\tau_1 \, \omega_T \, \tau_2)^+((p_1,p_2),x,(q_1,q_2))=T({\tau_1}^+(p_1,\omega^+(p_2,x),q_1),{\tau_2}^+(p_2,x,q_2))$, where $p_1,q_2\in Q_1$, $p_2,q_2\in Q_2$ and $x\in {X_2}^+$.

Definition 4.3. (Cho. et al. (2001)). Let $M_1=(Q_1,X_1,\tau_1)$ and $M_2=(Q_2,X_2,\tau_2)$ be T-generalized state machines. The wreath product $M_1 \circ_T M_2$ of M_1 and M_2 is the T-generalized state machine $(Q_1\times Q_2, {X_1}^{Q^2}\times X_2, \tau_1 \circ_T \tau_2)$ with $(\tau_1 \circ_T \tau_2)((p_1,p_2),(f,b),(q_1,q_2)) =$

$T(\tau_1(p_1,f(p_2),q_1),\tau_2(p_2,b,q_2))$, where $\{f \mid f:Q_2\to X_1\}$. In Definition 4.3 $(Q_1\times Q_2, {X_1}^{Q^2}\times X_2, \tau_1 \circ_T \tau_2)$ is clearly a T-generalized state machine.

In fact, we have

$\sum_{(q_1,q_2)\in Q_1\times Q_2} (\tau_1 \circ_T \tau_2)((p_1,p_2),(f,b),(q_1,q_2))=$
$\sum_{(q_1,q_2)\in Q_1\times Q_2} T(\tau_1(p_1,f(p_2),q_1),\tau_2(p_2,b,q_2)) \leq \sum_{(q_1,q_2)\in Q_1\times Q_2} \tau_1(p_1,f(p_2),q_1)\cdot$
$\tau_2(p_2,b,q_2)) \leq$
$\sum_{q_1\in Q_1} \tau_1(p_1,f(p_2),q_1)) \sum_{q_2\in Q_2} \tau_2(p_2,b,q_2)) \leq 1,$
for all $p_1\in Q_1$, $p_2\in Q_2$ and $b\in X_2$.

Theorem 4.4. (Cho. et al. (2001)). Let $M_1=(Q_1,X_1,\tau_1)$ and $M_2=(Q_2,X_2,\tau_2)$ be T-generalized state machines. Then $M_1\omega_T M_2\leq M_1 \circ_T M_2$.

Theorem 4.5. (Cho. et al. (2001)). Let $M_1=(Q_1,X_1,\tau_1)$ and $M_2=(Q_2,X_2,\tau_2)$ and $M=(Q,X,\tau)$ are T-generalized state machines. Let $M\leq M_1\, \omega_T M_2$. Then $M\leq M_1 \circ_T M_2$.

We now introduce two more ways of connecting T-generalized state machines.

Definition 4.6. (Kim et al. (1998)). Let $M=(Q,X,\tau)$ and $M'=(Q',X',\tau')$ be two T-generalized state machines such that $Q\cap Q'=\emptyset$ and $X\cap X'=\emptyset$. Then the directsum $M\oplus M'$ of M and M′ is T-generalized state machine $(Q\cup Q',X\cup X',\tau\oplus\tau')$ with

$$\tau\oplus\tau'(p,x,q)=\begin{cases}\tau(p,x,q) & \text{if } p,q\in Q,\ x\in X\\ \tau'(p,x,q) & \text{if } p,q\in Q',\ x\in X'\\ 1 & \text{if either } (p,x)\in Q\times X \text{ and } q\in Q',\\ & \text{or } (p,x)\in Q'\times X' \text{ and } q\in Q\\ 0 & \text{otherwise}\end{cases}$$

In Definition 4.6,$(Q\cup Q',X\cup X',\tau\oplus\tau')$ is clearly a T-generalized state machine.

Definition 4.7. (Kim et al. (1998)). Let $M=(Q,X,\tau)$ and $M'=(Q',X',\tau')$ be two T-generalized state machines such that $Q\cap Q'=\emptyset$ and $X\cap X'=\emptyset$. Then the sum $M+M'$ of M and M′ is T-generalized state machine $(Q\cup Q',X\cup X',\tau+\tau')$ with,

$$\tau+\tau'(p,x,q)=$$
$$\begin{cases}\tau(p,x,q) & \text{if } p,q\in Q,\ x\in X\\ \tau'(p,x,q) & \text{if } p,q\in Q',\ x\in X'\\ 0 & \text{otherwise}\end{cases}$$

5. Main Results

The following we proved that wreath product, sum and cascade products of T-generalized state machines are associative. However, it can easily be proved that the direct sum of T-generalized state machines is not associative.

Theorem 5.1 . If M,M' and M'' are T-generalized state machines, then

(i) $(M \circ_T M')\circ_T M''\cong M\circ_T (M\circ_T M'')$;

(ii) $(M+M')+M''\cong M+(M'+M'')$;

(iii) $(M\omega_1 M')\omega_2 M''\cong M\omega_3(M'\omega_4 M'')$

where ω_3 and ω_4 are determined by ω_1 and ω_2 in a natural way.

Proof . Let $M=(Q,X,\tau)$ and $M'=(Q',X',\tau')$ and $M''=(Q'',X'',\tau'')$

(i)Recall that $(M \circ_T M')\circ_T M''= ((Q\times Q') \times Q'',(X^{Q'} \times X')^{Q''} \} \times X'',(\tau \circ_T \tau')\circ_T\tau'')$ and $M\circ_T(M'\circ_T M'')=(Q\times(Q'\times Q''),(X^{Q'\times Q''})\times X'^{Q''} \times X'',\tau \circ_T(\tau' \circ_T\tau''))$.

Let α: $(Q\times Q')\times Q''\to Q\times(Q'\times Q'')$ be the natural maps and $p_1:X^{Q'} \times X'\to X^{Q'}$ and $p_2:X^{Q'} \times X'\to X'$ be the natural projection maps. Given a function $f: Q''\to X^{Q'} \times X'$ denote $f_1=p_1\circ f$ and $f_2=p_2\circ f$.

Define $f_1^-:Q'\times Q''\to X$ by $f_1^-[(q',q'')]=f_1(q'')(q')$ and

$\beta: (X^{Q'} \times X')^{Q''} \times X''\to X^{Q'\times Q''}\times((X')^{Q''}\times X'')$ by $\beta((f,x''))=\beta(f_1^-,(f_2,x''))$.

But

$\beta ((f,x''))=\beta((f_*,x_*''))$,

$\Rightarrow (f_1^-,(f_2,x''))=(f^-_{1*},(f_{2*}, x_*''))$,

$\Rightarrow f_1^-=f^-_{1*},f_2=f_{2*}$ and $x''=x_*''$.

Then

$f_1 (q'')(q')=f_{1*} (q'')(q')$, $f_2=f_{2*}$ and $x''=x_*''$.

$\Rightarrow f_1=f_{1*}$, $f_2=f_{2*}$ and $x''=x_*''$.

$\Rightarrow p_1\circ f = p_1\circ f_*$, $p_2\circ f = p_2 \circ_T f_*$ and $x'' = x_*''$,
$\Rightarrow f = f_*$, $x'' = x_*''$,
$\Rightarrow (f, x'') = (f_*, x_*'')$.
Therefore β is injective.
Let $(g, (h, x''))\in X^{Q'\times Q''}\times(X'^{Q''}\times X'')$
Define $f: Q''\to X^{Q'}\times X'$ by $f(q'')=(g_{q''}, h(q''))$ where $g_{q''}(q')=g(q', q'')$. Then
$\beta((f, x''))=(g,(h, x''))$.
Thus, β is onto.
It can be easily seen that, (α,β) is a required isomorphism.
(i) Let, $M+M'=(Q\cup Q', X\cup X', \tau+\tau')$, where,

$$\tau+\tau'(p,x,q)=\begin{cases} T(\tau(p,x,q),1) & \text{if } p,q\in Q,\ x\in X \\ T(\tau'(p,x,q),1) & \text{if } p,q\in Q',\ x\in X' \\ 0 & \text{otherwise}\end{cases}$$

and
$(M+M')+M''=((Q\cup Q')\cup Q'', (X\cup X')\cup X'', (\tau+\tau')+\tau'')$
and

$$\tau+\tau'+\tau''(p,x,q)=\begin{cases} T((\tau+\tau')(p,x,q),1) & \text{if } p,q\in Q\cup Q',\ x\in X\cup X' \\ T(\tau''(p,x,q),1) & \text{if } p,q\in Q'', x\in X'' \\ 0 & \text{otherwise}\end{cases}$$

It immediately follows that, $M'+M''=(Q'\cup Q'', X\cup X'', \tau+\tau'')$, and

$$\tau'+\tau''(p,x,q)=\begin{cases} T(\tau'(p,x,q),1) & \text{if } p,q\in Q',\ x\in X' \\ T(\tau''(p,x,q),1) & \text{if } p,q\in Q'',\ x\in X'' \\ 0 & \text{otherwise}\end{cases}$$

Moreover $M+(M'+M'')=(Q\cup(Q'\cup Q''), X\cup(X'\cup X''))$,

$$\tau+(\tau'+\tau'')(p,x,q)=\begin{cases} T(\tau(p,x,q),1) & \text{if } p,q\in Q,\ x\in X \\ T((\tau'+\tau'')(p,x,q),1) & \text{if } p,q\in Q'\cup Q'',\ x\in X'\cup X'' \\ 0 & \text{otherwise}\end{cases}$$

Set both α and β as identity mappings on $Q\cup Q'\cup Q''$ and $X\cup X'\cup X''$, respectively.
$(\tau+\tau')+\tau''(p,x,q)$

$$=\begin{cases} T((\tau+\tau')(p,x,q),1) & \text{if } p,q\in Q,\ x\in X\cup X' \\ T(\tau'')(p,x,q),1) & \text{if } p,q\in Q'', x\in X'' \\ 0 & \text{otherwise}\end{cases}$$

$$=\begin{cases} T\begin{cases} T(\tau(p,x,q),1) & \text{if } p,q\in Q,\ x\in X \\ T(\tau'(p,x,q),1) & \text{if } p,q\in Q',\ x\in X'\end{cases} \\ T(\tau'')(p,x,q),1) & \text{if } p,q\in Q'', x\in X'' \\ 0 & \text{otherwise}\end{cases}$$

$$=\begin{cases} T(\tau(p,x,q),1) & \text{if } p,q\in Q,\ x\in X \\ T\begin{cases} T(\tau'(p,x,q),1) & \text{if } p,q\in Q', x\in X' \\ T(\tau'')(p,x,q),1) & \text{if } p,q\in Q'', x\in X'' \\ 0 & \text{otherwise}\end{cases}\end{cases}$$

$$=\begin{cases} T(\tau(p,x,q),1) & \text{if } p,q\in Q,\ x\in X \\ T((\tau'+\tau'')(p,x,q),1) & \text{if } p,q\in Q'\cup Q'',\ x\in X'\cup X'' \\ 0 & \text{otherwise}\end{cases}$$

$=\tau+(\tau'+\tau'')(\alpha[p],\beta[x],\alpha[q])$.
Hence,
$(M+M')+M''\cong M+(M'+M'')$.

(ii) Consider, $M\,\omega_1 M'=(Q\times Q',X',\tau\,\omega_1\,\tau')$, where $\omega_1{:}Q'\times X'\rightarrow X$ and

$\tau\,\omega_1\,\tau'((p,p'),x',(q,q'))=,\ T(\tau(p,\omega_1(p',x'),q)\wedge\tau'(p',x',q'))$,
and
$(M\omega_1M')\omega_2M''=((Q\times Q')\times Q'',X'',(\tau\,\omega_1\,\tau')\omega_2\,\tau'')$, where $\omega_2{:}Q''\times X''\rightarrow X'$ and
$(\tau\,\omega_1\,\tau')\omega_2\,\tau''(((p,p'),p''),x'',((q,q'),q''))\ =$
$T(\tau\,\omega_1\,\tau'((p,p'),\omega_2(p'',x''),(q,q'))\wedge\tau''(p'',x'',q''))$
$=T(T(\tau(p,\omega_1(p',\omega_2(p'',x'')),q)\wedge\tau'(p',\omega_2(p'',x''),q')\wedge\tau''(p'',x'',q''))$.
Set $\omega_3{:}(Q'\times Q'')\times X''\rightarrow X$ by $\omega_3((p',p''),x'')=\omega_1(p',\omega_2(p'',x''))$ and $\omega_4=\omega_2$. It immediately follows that, $M'\omega_4M''=(Q'\times Q'',X'',\tau'\,\omega_4\,\tau'')$ and $\tau'\,\omega_4\,\tau''((p',p''),x'',(q',q''))=T(\tau'(p',\omega_4(p'',x''),q')\wedge\tau''(p'',x'',q''))$.
Moreover
$M\ \omega_3(M'\ \omega_4\ M'')=(Q\times(Q'\times Q''),\ X'',(\tau\ \omega_3\ \tau')\ \omega_4\ \tau'')$ and $\tau\ \omega_3(\tau'\omega_4\ \tau'')((p,(p',p'')),x'',(q,(q',q'')))=$
$T(\tau(p,\omega_3((p',p''),x''),q)\wedge\{\tau'\omega_4\ \tau''((p',p''),x''(q',q''))\})\ =$
$T(\tau(p,\omega_3((p',p''),x''),q)\wedge\{T(\tau'(p',\omega_4(p'',x''),q')\wedge\tau''(p'',x'',q''))\}$.
Let $\alpha{:}(Q\times Q')\times Q''\rightarrow(Q\times(Q'\times Q''))$ be the natural mapping and β be identity mapping on X''.
Then
$(\tau\,\omega_1\tau')\omega_2\,\tau''(((p,p'),p''),x'',((q,q'),q''))\ =$
$(T(\tau(p,\omega_1(p',\omega_2(p'',x'')),q)\wedge\tau'(p',\omega_2(p'',x''),q'))\wedge\tau''(p'',x'',q''))=$
$T(\tau(p,\omega_1(p',\omega_2(p'',x'')),q)\wedge\{T(\tau'(p',\omega_2(p'',x''),q')\wedge\tau''(p'',x'',q''))\})=$
$T(\tau(p,\omega_3((p',p''),x''),q)\wedge\{T(\tau'(p',\omega_4(p'',x''),q')\wedge\tau''(p'',x'',q''))\})=$
$\tau\,\omega_3(\tau'\omega_4\,\tau'')((p,(p',p'')),x'',(q,(q',q'')))=$
$\tau\,\omega_3(\tau'\omega_4\,\tau'')(\alpha[((p,p'),p'')],\beta[x''],\alpha[((q,q'),q'')])$.
Hence,
$(M\omega_1M')\omega_2M''\cong M\omega_3(M'\omega_4M'')$.

Example 5.2. In follow table M and M' are T-generalized state machine that satisfy in Theorem

M	Q={p,q, r}	X =	$\tau(p,a,q)=\tau(r,a,q)=(1/8)$, $\tau(p,a,q)=(3/4)$ otherwise 0
M′	Q′={ p′,q′}	X= {a}	$\tau'(p',a',q')=(4/5)$ otherwise 0

5.1 Coverings properties of products

The following theorem is a direct consequence of the definition of direct sum and sum of two T-generalized state machines.

Theorem 6.1. Let $M=(Q,X,\tau)$, $M'=(Q',X',\tau')$ be two T-generalized state machines.
Then
(i) $M\leq M+M'$;
(ii) $M'\leq M+M'$;
(iii) $M\leq M\oplus M'$;
(iv) $M'\leq M\oplus M'$.

Proof. Let $\eta{:}Q\cup Q'\rightarrow Q$ be a partial onto mapping defined by $\eta(q)=q$, for all $q\in Q$ and $\xi{:}X\rightarrow X\cup X'$ be the inclusion mapping.
(i) It is obvious that (η,ξ) is a required covering of M to $M+M'$.
Case (a). If $p,q\in Q$ and $x\in X$, then $T(\tau(\eta(p),x,\eta(q),1)=\tau+\tau'(p,\xi(x),q))\ =$
$T(\tau(p,x,q),1)$.
Case (b). If $p,q\in Q'$ and $x\in X'$, then $T(\tau(\eta(p),x,\eta(q),1)=\tau+\tau'(p,\xi(x),q))=T(\tau'(p,x,q),1)$.
In all other cases $T(\tau,1)=\tau+\tau'=0$.
(ii) Clearly.
(iii) Case (a). If $p,q\in Q$ and $x\in X$, then $T(\tau(\eta(p),x,\eta(q),1)=\tau\oplus\tau'(p,\xi(x),q))=T(\tau(p,x,q),1)$.
Case (b). If $p,q\in Q'$ and $x\in X'$, then $T(\tau(\eta(p),x,\eta(q),1)=\tau\oplus\tau'(p,\xi(x),q))=T(\tau'(p,x,q),1)$.
Case (c). If $(p,x)\in Q\times X$ and $q\in Q'$ or $(p,x)\in Q'\times X'$ and $q\in Q$ then $T(\tau(\eta(p),x,\eta(q),1)=0<\tau\oplus\tau'(p,\xi(x),q))=1$.
In all other cases $T(\tau,1)=\tau\oplus\tau'=0$.
(iv) clearly.

Theorem 6.2. Let $M=(Q,X,\tau)$, $M'=(Q',X',\tau')$ be two T-generalized state machines. Then
$M+M'\leq M\oplus M'$.

Proof. Set both (η,ξ) as identity maps on $Q\cup Q'$ and $X\cup X'$ respectively.

Case (a). If $p,q\in Q$ and $x\in X$, then $\tau+\tau'(\eta(p),x,\eta(q))=T(\tau(p,x,q),1)=\tau\oplus\tau'(p,\xi(x),q)$.
Case (b). If $p,q\in Q'$ and $x\in X'$ then $\tau+\tau'(\eta(p),x,\eta(q))=T(\tau'(\eta(p),x,\eta(q),1)=\tau\oplus\tau'(p,\xi(x),q)$.
Case (c). If $(p,x)\in Q\times X$ and $q\in Q'$ or $(p,x)\in Q'\times X'$ and $q\in Q'$ then $\tau+\tau'(\eta(p),x,\eta(q))=0<1=\tau\oplus\tau'(p,\xi(x),q)$.
In all other cases $\tau\oplus\tau'=\tau+\tau'=0$. all other cases $\tau\oplus\tau'=\tau+\tau'=0$.

Theorem 6.3. Let $M=(Q,X,\tau)$, $M'=(Q',X',\tau')$ and $M''=(Q'',X'',\tau'')$ be three T-generalized state machines such that $M\leq M'$. Then

(i) Given $\omega_1:Q''\times X''\to X$ there exists $\omega_2:Q''\times X''\to X'$ such that $M\ \omega_1\ M''\leq M'\ \omega_2\ M''$;

(ii) If (η,ξ) is a covering of M by M' and ξ is onto, then for each $\omega_1:Q\times X\to X''$ there exists $\omega_2:Q'\times X'\to X''$, such that $M''\ \omega_1\ M\leq M''\ \omega_2\ M'$;

(iii) $M\ {}^{\circ}_{T}\ M''\leq M'\ {}^{\circ}_{T}\ M''$;

(iv) $M+M''\leq M'+M''$;

(v) $M''+M\leq M''+M'$;

(vi) $M\oplus M''\leq M'\oplus M''$;

(vii) $M''\oplus M\leq M''\oplus M'$.

Proof. Since $M\leq M'$ there exist a partial onto mapping $\eta: Q'\to Q$ and a mapping $\xi:X\to X'$ such that $\tau(\eta(p'),x,\eta(q'))\leq\tau'(p',\xi(x),q')$

(i) Given $\omega_1:Q''\times X''\to X$, set $\omega_2=\xi\circ\omega_1$ and ξ' as an identity mapping on X'' and $\eta':Q'\times Q''\to Q\times Q''$

$M\ \omega_1 M''=(\tau\ \omega_1\ \tau'')(\eta'((p',p'')),x'',\eta'((q',q'')))$
$=T(\tau(p,\omega_1(p'',x''),q),\tau''(p'',x'',q''))$
$=T(\tau'(\eta(p'),\omega_1(p'',x''),\eta(q')),\tau''(p'',x'',q''))$
$\leq T(\tau'(p',\xi\circ\omega_1(p'',\xi'(x'')),q'),\tau''(p'',x'',q''))$
$=T(\tau'(p',\omega_2(p'',\xi'(x'')),q'),\tau''(p'',x'',q''))$
$=M'\omega_2 M''$.

(ii) Given $\omega_1:Q\times X\to X''$, construct $\omega_2:Q'\times X'\to X''$ such that $\omega_2(p',\xi(x))=\omega_1(\eta(p'),x)$.

Since ξ is onto and Σ is finite, such ω_2 exists. Clearly ω_2 is not unique.
Define $\eta':Q''\times Q'\to Q''\times Q$ by $\eta'(p'',p')=(p'',\eta(p'))$ and set $\xi'=\xi$.

$M''\omega_1 M=\tau''\omega_1\tau(\eta'((p'',p')),x,\eta'((q'',q')))$
$=\tau''\omega_1\tau((p'',\eta(p')),x,(q'',\eta(q')))$
$=T(\tau''(p'',\omega_1(\eta(p'),x),q''),\tau(p,x,q))$
$=T(\tau''(p'',\omega_2(p',\xi(x)),q''),\tau(\eta(p'),x,\eta(q')))$
$\leq T(\tau''(p'',\omega_2(p',\xi(x)),q''),\tau'(p',\xi(x),q'))$
$=\tau''\ \omega_2\ ((p'',p'),\xi(x),(q'',q'))=M''\omega_2 M'$.

(iii) Define $\eta':Q'\times Q''\to Q\times Q''$ by $\eta'(p',p'')=(\eta(p'),p'')$ and $\xi':X^{Q''}\times X''\to(X')^{Q''}\times X''$ by $\xi'(f,x'')=(\xi\circ f,x'')$.

$M\ {}^{\circ}_{T}\ M''=\tau\ {}^{\circ}_{T}\ \tau''(\eta'((p',p'')),(f,x'),\eta'((q',q'')))$
$=\tau\ {}^{\circ}_{T}\ \tau''((\eta(p'),p''),(f,x''),(\eta(q'),q''))$
$=T(\tau(\eta(p'),(f,x''),\eta(q')),\tau''(p'',x'',q''))$
$\leq T(\tau'(p',\xi\circ f(p''),q'),\tau''(p'',x'',q''))$
$=T(\tau'(p',\xi'(f,x''),q'),\tau''(p'',x'',q''))$
$=\tau'\ {}^{\circ}_{T}\ \tau''((p',p''),\xi'(f,x''),(q,q''))=M'\ {}^{\circ}_{T}\ M''$.

(iv) Recall that $M+M''=(Q\cup Q'',X\cup X'',\tau+\tau'')$, where,

$$\tau+\tau''(p,x,q)=\begin{cases}T(\tau(p,x,q),1) & \text{if } p,q\in Q, x\in X\\ T(\tau''(p,x,q),1) & \text{if } p,q\in Q'', x\in X''\\ 0 & \text{otherwise}\end{cases}$$

and

$M'+M''=(Q'\cup Q'',X'\cup X'',\tau'+\tau'')$, where,

$$\tau'+\tau''(p,x,q)=\begin{cases}T(\tau'(p,x,q),1) & \text{if } p,q\in Q', x\in X'\\ T(\tau''(p,x,q),1) & \text{if } p,q\in Q'', x\in X''\\ 0 & \text{otherwise}\end{cases}$$

Define $\eta':Q'\cup Q''\to Q\cup Q''$ by $\eta'(p')=\begin{cases}\eta(p') & if \quad p'\in Q'\\ p' & \text{otherwise}\end{cases}$

And $\xi':X\cup X''\to X'\cup X''$ by $\xi'(x)=\begin{cases}\xi(x) & \text{if} \quad x\in X\\ x & \text{otherwise}\end{cases}$

since η is a partial onto mapping so is η'.
We claim that $\tau+\tau''(\eta'(p),x,\eta'(q'))\leq\tau'+\tau''(p',\xi'(x),q')$.
If $p',q'\in Q'$ and $x\in X$ or $p',q'\in Q''$ and $x\in X''$, then obviously $\tau+\tau''(\eta'(p),x,\eta'(q'))\leq\tau'+\tau''(p',\xi'(x),q')$.
in all other caces $\tau+\tau''(\eta'(p),x,\eta'(q'))=0$. The proofs of (v),(vi) and (vii) are now obvious. It can be easily seen that $M''\circ M'$, in general, does not cover $M''\circ M$, even though $M\leq M'$. This fact makes it imperative to introduce a weaker notion of covering. Hence, the following definition:

Definition 6.4. Let $M=(Q,X,\tau),M'=(Q',X',\tau')$ be two T-generalized state machines. Let $\eta:Q'\to Q$ be a partial onto mapping and $\xi:X\to X'$ be a partial mapping. The ordered pair (η,ξ) is called weak covering, if $\tau(\eta(p'),x,\eta(q'))\leq\tau'(p',\xi(x),q')$, for all p',q' in the domain of η and x in the domain of ξ. Symbolically we denote this fact by $M\leq_w M'$.

Remark 6.5. a weak covering differs from a covering only in one sense. ξ in Definition 6.2 is a partial mapping, while ξ in Definition 3.1 is a function. Thus, every covering is a weak covering.

Theorem 6.6. If M,M' and M'' be Three T-generalized state machines and $M\leq M'$,then $M''\circ M\leq_w M''\circ M'$.

Proof. Since $M\leq M'$ there exist a partial onto mapping $\eta:Q'\to Q$ and a mapping $\xi:X\to X'$ such that $\tau(\eta(p'),x,\eta(q'))\leq\tau'(p',\xi(x),q')$.
we have, $M''\circ M=(Q''\times Q,(X'')^{Q}\times X,\tau''\circ\tau)$ and $M''\circ M'=(Q''\times Q',(X'')^{Q'}\times X',\tau''\circ\tau')$.
Define $\eta':Q''\times Q'\to Q''\times Q$ by $\eta'(p'',p')=(p'',\eta(p'))$ and $\xi':(X'')^{Q}\times X\to(X'')^{Q'}\times X'$ by $\xi'(f,x)=(f\circ\eta,\xi(x))$. Obviously η' is a partial onto mapping and ξ' is a partial mapping.
Consider $\tau''\circ\tau(\eta'(p'',p'),(f,x),\eta'(q'',q'))$
$=\tau''\circ\tau((p'',\eta(p')),(f,x),(q'',\eta q'))$
$=T(\tau''(p'',f(\eta(p')),q'')\wedge\tau(\eta(p'),x,\eta(q')))$.
However, since $\tau(\eta(p'),x,\eta(q'))\leq\tau'(p',\xi(x),q')$, it follows that
$\tau''\circ\tau(\eta'(p'',p'),(f,x),\eta'(q'',q'))\leq T(\tau''(p'',f(\eta(p')),q'')\wedge\tau'(p',\xi(x),q'))$.
$=\tau''\circ\tau'((p'',p'),(f\circ\eta,\xi(x)),(q'',q'))$
$=\tau''\circ\tau'((p'',p'),\xi'(f,x),(q'',q'))$.
The following theorems are easy consequences of the transitive property of coverings of T-generalized state machines and Theorem 6.1.

Theorem 6.7. If M, M' and M'' be three T-generalized state machines and $M\leq M'$, then
(i)$M\,\omega_T\,M''\leq M'\circ_T M''$;
(ii)$M''\omega_T M\leq_w M''\circ_T M$;
(iii)$M+M''\leq M'\oplus M''$;
(iv)$M''+M\leq M''\oplus M'$.

Proof. Since $M\leq M'$ there exist a partial onto mapping $\eta:Q'\to Q$ and a mapping $\xi:X\to X'$ such that $\tau(\eta(p'),x,\eta(q))\leq\tau'(p',\xi(x),q')$.
(i) Given $\omega_T:Q''\times X''\to X$ and $\eta':Q'\times Q''\to Q\times Q''$. ξ_2 as an identity mapping on X''. Let $\xi\circ\omega\,(p'',\xi_2(x''))=\xi_1(x'')(p'')$
that $\xi_1(x''):Q''\to X'$.
$M\,\omega_T M''=(\tau\,\omega_T\tau'')(\eta'((p,p')),x'',\eta'((q',q'')))$
$=T(\tau(p,\omega_T(p'',x''),q),\tau''(p'',x'',q''))$
$=T(\tau(\eta(p'),\omega_T(p'',x''),\eta(q')),\tau''(p'',x'',q''))$
$\leq T(\tau'(p',\xi\circ\omega_T(p'',\xi_2(x'')),q'),\tau''(p'',x'',q''))$
$=T(\tau'(p',(\xi_1(x'')(p''),q'),\tau''(p'',x'',q''))$
$=\tau'\circ_T\tau''((p',p''),(\xi_1(x''),x''),(q',q''))$
$=M'\circ_T M''$.
(ii) Define $\eta':Q''\times Q'\to Q''\times Q$ by $\eta'(p'',p')=(p'',\eta(p'))$ and $\omega_T:Q''\times X''\to X$. Let ξ' as an identity mapping on X and $\xi''(x)$ is a partial mapping defined by $\xi''(x)(p')=\omega_T(p,\xi'(x))$
$\tau''\,\omega_T\tau((p'',p),x,(q'',q))=\tau''\,\omega_T\tau(\eta'((p'',p)),x,\eta'((q'',q)))$

$=\tau'' \omega_T \tau((p'',\eta(p')),x,(q'',\eta(q')))$
$=T(\tau''(p'',\omega_T(\eta(p'),x),q''),\tau(p,x,q))$
$=T(\tau''(p'',\omega_T(\eta(p'),\xi'(x)),q''),\tau(\eta(p'),x,\eta(q')))$
$\leq T(\tau''(p'',\omega_T(\eta(p'),\xi'(x)),q''),\tau'(p',\xi(x),q'))$
$=T(\tau''(p'',\omega_T(p,\xi'(x)),q''),\tau'(p',\xi(x),q'))$
$=T(\tau''(p'',\xi''(x)(p'),q''),\tau'(p',\xi(x),q'))$
$=\tau'' \circ_T \tau'((p'',p'),(\xi''(x),\xi(x)),(q'',q'))= M'' \circ_T M.$

(iii) Let $\eta: Q'\cup Q''\to Q\cup Q''$, $\xi: X\cup X''\to X'\cup X''$.

Case (a). If $p,q\in Q$ and $x\in X$ then
$\tau+\tau'(\eta(p),x,\eta(q))=T(\tau(\eta(p),x,\eta(q),1)\leq T(\tau'(p,\xi(x),q),1)=\tau\oplus\tau'(p,\xi(x),q)$.

Case (b). If $p,q\in Q''$ and $x\in X''$ then
$\tau+\tau'(\eta(p),x,\eta(q))=T(\tau''(\eta(p),x,\eta(q),1)=\tau\oplus\tau'(p,\xi(x),q)$.

Case (c). If $(p,x)\in Q'\times X'$ and $q\in Q''$ or $(p,x)\in Q''\times X''$ and $q\in Q'$
Then $\tau+\tau'(\eta(p),x,\eta(q))=0<1=\tau\oplus\tau'(p,\xi(x),q)$. In all other cases $\tau\oplus\tau'=\tau+\tau'=0$.

(iv) The prove is now obvious. Finally we prove an interesting distributive property for covering.

Theorem 6.8. Let M,M' and M'' be three T-generalized state machines. Then
$M\circ(M'+M'')\leq(M\circ M')+(M\circ M'')$.

Proof. Let $M=(Q,X,\tau)$, $M'=(Q',X',\tau')$ and $M''=(Q'',X'',\tau'')$.
Define $\eta:(Q\times Q')\cup(Q\times Q'')\to Q\times(Q'\cup Q'')$ by $\eta(p,p')=(p,p')$ and
$\xi: X^{Q'\cup Q''}\times(X'\cup X'')\to(X^{Q'}\times X')\cup(X^{Q''}\times X'')$ by

$$\xi(g,x')=\begin{cases}(g|_{Q'},x') & \text{if } x'\in X',\\ (g|_{Q''},x') & \text{if } x'\in X''\end{cases} \quad (*)$$

$M\circ_T (M'+M'')=\tau \circ_T (\tau'+\tau'')(\eta((p,p')),(g,x'),\eta((q,q')))$
$=\tau\circ_T (\tau'+\tau'')((p,p'),(g,x'),(q,q')) =T(\tau(p,g(p'),q), (\tau'+\tau'')(p',x',q')))$.

Case (a). If $p',q'\in Q'$, $x'\in X'$, then
(*) $\xi(g,x')=T(\tau(p,g(p'),q),T(\tau'(p',x',q'),1))$
$=T(T[\tau(p,g(p'),q),\tau'(p',x',q')],1)$
$=T(\tau \circ_T \tau'((p,p'),(g|_{Q'},x'),(q,q')),1)$
$=T(\tau\circ_T \tau'((p,p'),\xi(g,x'),(q,q')),1)$
$=(\tau \circ_T \tau')+(\tau \circ_T \tau'')$.

Case (b). If $p',q'\in Q''$, $x'\in X''$, then
(*) $\xi(g,x')=T(\tau(p,g(p'),q),T(\tau''(p',x',q'),1))$
$=T(T[\tau(p,g(p'),q),\tau''(p',x',q')],1)$
$=T(\tau \circ_T \tau''((p,p'),(g|_{Q''},x'),(q,q')),1)$
$=T(\tau \circ_T \tau''((p,p'),\xi(g,x'),(q,q')),1)$
$=(\tau\circ_T \tau')+(\tau\circ_T \tau'')$. In all other cases $T(\tau\circ_T \tau''((p,p'),\xi(g,x'),(q,q')),1)=(\tau\circ_T \tau')+(\tau\circ_T \tau'')=0$.

6. Conclusions

The results of covering of T-generalized state machines have interesting points.

1. The direct sum product of T-generalized state machines covers their sum product. These results lead to several covering properties.
2. The wreath product of a T-generalized state machine with sum of T-generalized state machines is covered by the sum of wreath products of T-generalized state machines.
3. Like direct product of T-generalized state machines, some sort of associative laws hold in case of cascade product, wreath product and sum of T-generalized state machines.

In case of T-generalized state machines different types of products and their coverings play a crucial role in decomposition. In this paper we have established T-generalized analog of different products and their coverings.

Therefore, we expect these results to be useful in decomposition of T-generalized state machines.

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