Stochastic Orders of Past Lives based on Moment Generating Function Order

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Abstract: In this paper we study new notions of stochastic comparisons and aging classes based on the moment generating function order of past lives. We show that the mg-pl order is preserved under convolutions and mixtures. We introduced some examples of interested in reliability theory.

Keywords: Moment generating function order, Past lifetime, Stochastic order, Convolution , Mixtures

1 Introduction and Motivation

Notions of positive aging play an important role in reliability theory, survival analysis, economics applications, many different areas of applied probability and statistics, actuarial science and other fields. Therefore , an a abundance of classes of distributions describing have been considered in the literature, see e.g. *Barlow and Proschan* [1], *Lai and Xie* [2] , *Muller and Stoyan* [3] and *Shaked, and J.G. Shanthikumar* [4] For an overview. Throughout, the terms "increasing" and " decreasing" mean "non-decreasing" and "non-increasing" respectively. All expectations and integrals are implicitly assumed to exist whenever they are written. Let X and Y be two non-negative random variables, representing equipment lives with distributions F and G, and denote their survival functions by $\overline{F} = 1 - F$ and $\overline{G} = 1 - G$, respectively. The moment generating function of F and G is given by the following

$$\Psi_X(t) = \int_0^\infty e^{tx} f(x)dx \text{ and } \Psi_Y(t) = \int_0^\infty e^{ty} f(y)dy$$

also, define

$$\Phi_X(t) = \int_0^\infty e^{tx} \overline{F}(x)dx \text{ and } \Phi_Y(t) = \int_0^\infty e^{ty} \overline{G}(y)dy.$$

Let

$$X_t = [X - t | X > t], \quad t \in \{x : F(x) < 1\},$$

denote a random variable whose distribution is the same as the conditional distribution of $X - t$ given that $X > t$. When X is the lifetime of a device, X_t can be regarded as the residual lifetime of the device at time t, given that the device has survived up to time t. and

$$X_{(t)} = (t - X \mid t \geq X)$$

be the past (inactivity) time at time $t \in \{x : F_X(x) < 1\}$(see , for instance, Block et al. [5], Li and Lu [6] , Ahmad et al .[7] and Nanda et al. [8], and references therein. The corresponding survival functions of X_t and $X_{(t)}$ can be represented as

$\overline{F_t}(x) = P(X_t > x) = P(X - t \mid X > t)$

$$= \frac{\overline{F}(x+t)}{\overline{F}(t)},$$

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and

$$\overline{F_{(t)}}(x) = P(X_{(t)} > x) = P(t - X \mid t \geq X)$$

$$= \frac{F(t-x)}{F(t)}.$$

The mean residual lifetime (also known as the mean remaining life) (MRL) deals with the residual lifetime of X. In fact the MRL measures the expected value of the remaining lifetime $X - t$ given that X has survival at least t units of time. The mean residual life is defined formally as

$$\mu_X(t) = E(X - t \mid X > t)$$

$$= \frac{\int_t^\infty \overline{F}(x)dx}{\overline{F}(t)}.$$

Clearly, $\mu_X(0) = \mu = E(X)$.

Another measure of interest is the mean past lifetime (MPL) However , it is reasonable to presume that in many realistic situations the random variable is not necessarily related to the future but can also refer to the past. For instant, consider a system whose state is observed only at certain preassigned inspection times. If at time t the system is inspected for the first time and it is found to be "down" , then the failure relies on the past, i.e., on which instant in interval $(0,t)$ it has failed sometime before the time t. It thus seems natural to study a notion that is dual to the residual life, in the sense that it refers to past lifetime and not future (see Di Crescenzo and Longobardi [9]. The mean past lifetime (also known as mean inactivity time) (MPL) is thus

$$m_X(t) = E(t - X \mid t \geq X) = \frac{\int_0^t F(x)dx}{F(t)}.$$

The following de...nition is well known ,cf. *Muller and Stoyan* [3].

Definition 1.1.

Let X and Y be two non-negative random variables. The random variable X is said to be smaller than Y in the moment generating function order, denoted by

$$X \leq_{mgf} Y \text{ if } \Psi_X(t) \leq \Psi_Y(t) \text{ for all } t > 0.$$

Clearly from the relation between $\Psi_X(t)$ and $\Phi_X(t)$ we have (c.f. Karl and Muller [10]) showed that

$$X \leq_{mgf} Y \text{ if } \Phi_X(t) \leq \Phi_Y(t) \text{ for all } t > 0.$$

Applications, properties and interpretations of the moment generating function order can be found in Karl and Muller [10], Zhang and Li [11] , Li [12]. Actually, an equivalent condition for $mg - rl$ order is given in Wang and Ma [13]. Following this idea we introduce next a new aging classes following based on the moment generating function of past lives order.

Definition 1.2.

let X and Y be two non-negative random variables X is said to be smaller than Y in the moment generating function order of past lifetime , denoted by

$$X \leq_{mg-pl} Y \text{ if } X_{(t)} \leq_{mg} Y_{(t)} \text{ for all } t \in R^+ \tag{1}$$

Thus we have the following implications:

$$X \leq_{mpl} Y \Longleftarrow X \leq_{mg-pl} Y \Longrightarrow X \leq_{mg} Y.$$

In this paper, we present some preservation properties of the moment generating function ordering of past lives (mg-pl) order . In Section 2, we introduced two characterizations and new aging classes using the moment generating function order of residual lives. Some preservation results under the reliability operation of convolution and mixtures are discussed. Finally, we described some examples of applications in recognizing situations where the random variables are comparable according to the moment generating function of past lifetime order .

2 Definitions , Notation and Characterizations.

In this section we present definitions, notation and basic properties used throughout the paper. If

$$X_{(t)} = (t - X \mid t \geq X)$$

be the past (inactivity) time at time $t \in \{x : F_X(x) < 1\}$, given that the system or the unit , has survived up to time t, then we get the following

$$\Phi_{X_{(t)}}(s) = \int_0^t \frac{e^{su}F(u)du}{e^{st}F(t)}, s > 0 \tag{2}$$

and

$$\Phi_{Y_{(t)}}(s) = \int_0^t \frac{e^{su}G(u)du}{e^{st}G(t)}, s > 0 \tag{3}$$

observe that by definition of $\leq_{mg-pl}$ order, it holds, $X \leq_{mg-pl} Y$, i.e.

$$X_{(t)} \leq_{mg} Y_{(t)} \text{ if and only if } \Phi_{X_{(t)}}(s) \leq \Phi_{Y_{(t)}}(s) \text{ for all } t \in R^+, s \geq 0. \tag{4}$$

A necessary and sufficient condition for the $\leq_{mg-pl}$ order is given in the following proposition.

Proposition 2.1:

Let X and Y be two continuous non-negative random variables. Then

$$X \leq_{mg-pl} Y \text{ if and only if } \frac{\int_0^t e^{su}F(u)du}{\int_0^t e^{su}G(u)du} \text{ is decreasing in } t \text{ for } s \geq 0.$$

Proof:

Let us observe that

$$\Phi_{X_{(t)}}(s) = \int_0^t \frac{e^{su}F(u)du}{e^{st}F(t)} = \frac{\int_0^t e^{su}F(u)du}{\frac{\partial}{\partial t}\int_0^t e^{su}F(u)du} \tag{5}$$

therefore given $s > 0$, by definitions (4) and (5) , we have

$X_{(t)} \leq_{mg-pl} Y_{(t)}$ if and only if $\Phi_{X_{(t)}}(s) \leq \Phi_{Y_{(t)}}(s)$

$$\Longleftrightarrow \frac{\int_0^t e^{su}F(u)du}{\frac{\partial}{\partial t}\int_0^t e^{su}F(u)du} \leq \int_0^t \frac{e^{su}G(u)du}{e^{st}G(t)}$$

$$\Longleftrightarrow \frac{\int_0^t e^{su}F(u)du}{\int_0^t e^{su}G(u)du} \text{ is decreasing in } t \text{ for } s \geq 0.$$

In the following , we propose new aging classes following the previous procedures for the moment generating function order of past lives.

Definition 2.1.

The random variable X is said to have decreasing (increasing) past lives in the moment generating function order , denoted by

$$X \in DPL_{mg}(IPL_{mg}) \text{ if } X_{(t)} \leq_{mg} (_{mg}\geq)X_{(t_1)} \text{ for } 0 \leq t_1 < t.$$

Proposition 2.2.

Denote $\Psi(t) = \int_0^t e^{su}F(u)du$ then $X \in DPL_{mg}(IPL_{mg})$ if and only if $\Psi(t)$ is logconcave (logconvex) in t .

Proof:

$$X \in DPL_{mg}(IPL_{mg}) \Longleftrightarrow X_{(t)} \leq_{mg} (_{mg}\geq)X_{(t_1)}$$

thus

$$\frac{\int_0^t e^{su}F(u)du}{e^{st}F(t)} \leq (\geq)\frac{\int_0^{t_1} e^{su}F(u)du}{e^{st_1}F(t_1)}$$

$$\frac{e^{st}F(t)}{\int_0^t e^{su}F(u)du} \leq (\geq)\frac{e^{st_1}F(t_1)}{\int_0^{t_1} e^{su}F(u)du}$$

$$\Longleftrightarrow \Psi(t) \text{ is logconcave (logconvex) in } t$$

Definition 2.2.

A probability vector $\underline{\lambda} = (\lambda_1, \lambda_2, ..., \lambda_n))$ with $\lambda_i > 0$ for $i = 1, 2, ..., n$ is said to be smaller than the probability vector $\underline{\mu} = (\mu_1, \mu_2, ..., \mu_n)$ in the sense of discrete likelihood ratio order, denoted by $\underline{\lambda} \leq_{dlr} \underline{\mu}$, if and

$$\frac{\mu_i}{\lambda_i} \leq \frac{\mu_j}{\lambda_j}, \text{ for all } 1 \leq i \leq j \leq n.$$

Definition 2.3.

A function $g(x)$, $-\infty < x < \infty$, is said to be a *Polya function* of order 2 (PF_2) if

(*i*) $g(x) \geq 0$ for $-\infty < x < \infty$, and

(*ii*) $\begin{vmatrix} g(x_1 - y_1) & g(x_1 - y_2) \\ g(x_2 - y_1) & g(x_2 - y_2) \end{vmatrix} \geq 0$ for all $-\infty < x_1 < x_2 < \infty$ and $-\infty < y_1 < y_2 < \infty$, or equivalently,

(*iii*) $\log[g(x)]$ is concave on $(-\infty, \infty)$.

The equivalence of (*ii*) and (*iii*) is shown in Exercise 12, page 79, in *Barlow* and *Proschan* [1].

3 Preservation Properties

In this section we will develop preservation properties under some reliability operations such as convolution and mixtures based on the moment generating function of past lives order. The following theorem shows that the $(mg - pl)$ order is preserved under convolution.

Theorem 3.1.

Let X_1, X_2 and Y be three non-negative random variables, where Y is independent of both X_1 and X_2, and let Y have density function h. If $X_1 \leq_{mg-pl} X_2$ and h is logconcave then $X_1 + Y \leq_{mg-pl} X_2 + Y$.

Proof:

From Proposition 2.1, it is enough to show that for all $0 \leq t_1 < t_2$, and $x > 0$, we have the following

$$\frac{\int_0^\infty \int_0^{t_1} e^{sx}P(X_1 \leq x+u)h(t_1-u)dudx}{\int_0^\infty \int_0^{t_1} e^{sx}P(X_2 \leq x+u)h(t_1-u)dudx} \leq \frac{\int_0^\infty \int_0^{t_2} e^{sx}P(X_1 \leq x+u)h(t_2-u)dudx}{\int_0^\infty \int_0^{t_2} e^{sx}P(X_2 \leq x+u)h(t_2-u)dudx} \tag{6}$$

Since Y is non-negative then $h(t-u) = 0$ when $t < u$, hence the above inequality is equivalent to

$$\frac{\int_0^\infty \int_0^\infty e^{sx}F_{X_1}(x+u)h(t_1-u)dudx}{\int_0^\infty \int_0^\infty e^{sx}F_{X_2}(x+u)h(t_1-u)dudx} \leq \frac{\int_0^\infty \int_0^\infty e^{sx}F_{X_1}(x+u)h(t_2-u)dudx}{\int_0^\infty \int_0^\infty e^{sx}F_{X_2}(x+u)h(t_2-u)dudx} \tag{7}$$

for all $0 \leq t_1 < t_2$, or equivalently

$$\begin{vmatrix} \int_0^\infty \int_0^\infty e^{sx}F_{X_2}(x+u)h(t_2-u)dudx & \int_0^\infty \int_0^\infty e^{sx}F_{X_1}(x+u)h(t_2-u)dudx \\ \int_0^\infty \int_0^\infty e^{sx}F_{X_2}(x+u)h(t_1-u)dudx & \int_0^\infty \int_0^\infty e^{sx}F_{X_1}(x+u)h(t_1-u)dudx \end{vmatrix} \geq 0 \tag{8}$$

by the well known basic composition formula (Karlin [14]), the left side of 3.3 is equal to

$$\int_{u_1 < u_2} \begin{vmatrix} h(t_2 - u_1) & h(t_2 - u_2) \\ h(t_1 - u_1) & h(t_1 - u_2) \end{vmatrix} \begin{vmatrix} \int_0^\infty e^{sx}F_{X_2}(x+u_1)dx & \int_0^\infty e^{sx}F_{X_1}(x+u_1)dx \\ \int_0^\infty e^{sx}F_{X_2}(x+u_2)dx & \int_0^\infty e^{sx}F_{X_1}(x+u_2)dx \end{vmatrix} du_1 du_2.$$

The conclusion now follows if we note that the first determinant is non- positive since h is log- concave , and that the second determinant is non- positive since $X_1 \leq_{mg-pl} X_2$. This completes the proof.

Corollary 3.1.

If $X_1 \leq_{mg-pl} Y_2$ and $X_2 \leq_{mg-pl} Y_2$ where X_1 is independent of X_2 and Y_1 is independent of Y_2, then the following statements hold:

(a) If X_1 and Y_2 have log- concave densities, then $X_1 + X_2 \leq_{mg-pl} Y_1 + Y_2$.

(b) If X_2 and Y_1 have log- concave densities, then $X_1 + X_2 \leq_{mg-pl} Y_1 + Y_2$.

Proof.

The following chain of inequalities, which establish (a), follows by *Theorem* 3.1:

$$X_1 + X_2 \leq_{mg-pl} X_1 + Y_2 \leq_{mg-pl} Y_1 + Y_2.$$

The proof of (b) is similar.

Theorem 3.2.

If $X_1, X_2, ...$ and $Y_1, Y_2, ...$ are sequences of independent random variables with $X_i \leq_{mg-pl} Y_i$ and X_i, Y_i have log- concave densities for all i, then

$$\sum_{i=1}^{n} X_i \leq_{mg-pl} \sum_{i=1}^{n} Y_i , \quad (i = 1, 2, ...).$$

Proof.

Using the induction technique we prove the theorem. Clearly, the result is true for $n = 1$. Assume that the result is true for $p = n - 1$, i.e.,

$$\sum_{i=1}^{n-1} X_i \leq_{mg-pl} \sum_{i=1}^{n-1} Y_i. \tag{9}$$

Note that each of the two sides of (9) has a log-concave density (see, e.g., *Karlin*, [14], page128). Appealing to *Corollary 3*.1, the result follows.

Suppose that $X_i, i = 1, 2, ..., n$ be a collection of independent random variables. Suppose that F_i and $\overline{F}_i$ are the life distribution and survival function of X_i respectively. Let $\underline{\lambda} = (\lambda_1, \lambda_2, ..., \lambda_n)$ and $\underline{\mu} = (\mu_1, \mu_2, ..., \mu_n)$ be two probability vectors. Let now X and Y be two random variables having the distributions functions F , G defined by and

$$F(x) = \sum_{i=1}^{n} \lambda_i F_i(x) \quad \text{and } G(x) = \sum_{i=1}^{n} \mu_i G_i(x). \tag{10}$$

The following theorem gives conditions under which X and Y are comparable with respect $(mg - pl)$ order. Actually , this is a closure under mixture property of the moment generating function of past lives order.

Theorem 3.3.

Let $X_1, X_2, .., X_n$ be a collection of independent random variables with corresponding life distributions $F_1, F_2, ..., F_n$ such that $X_1 \leq_{mg-pl} X_2 \leq_{mg-pl} ..... \leq_{mg-pl} X_n$ and let $\underline{\lambda} = (\lambda_1, \lambda_2, ..., \lambda_n)$ and $\underline{\mu} = (\mu_1, \mu_2, ..., \mu_n)$ such that $\underline{\lambda} \leq_{dlr} \underline{\mu}$. Let X and Y have survival function $\overline{F}$, and $\overline{G}$ defined in (10). Then $X \leq_{mg-pl} Y$.

Proof:

Because of Proposition 2.1 , we need to establish that

$$\frac{\int_0^\infty e^{su} \sum\limits_{i=1}^{n} \mu_i F_i(x+u) du}{\int_0^\infty e^{su} \sum\limits_{i=1}^{n} \lambda_i F_i(x+u) du} \leq \frac{\int_0^\infty e^{su} \sum\limits_{i=1}^{n} \mu_i F_i(y+u) du}{\int_0^\infty e^{su} \sum\limits_{i=1}^{n} \lambda_i F_i(y+u) du} \quad \text{for all } 0 < y < x. \tag{11}$$

Multiblying by the denominators and canceling out equal terms it can shown that (11) is equivalent to

$$\sum_{\substack{i=1 \\ i \neq j}}^{n} \sum_{j=1}^{n} \mu_i \lambda_j \int_0^\infty e^{su} F_i(x+u) du \int_0^\infty e^{sv} F_j(y+v) dv$$

$$\leq \sum_{\substack{i=1 \\ i \neq j}}^{n} \sum_{j=1}^{n} \mu_i \lambda_j \int_0^\infty F_i(y+u) du \int_0^\infty e^{sv} F_j(x+v) dv$$

or

$$\sum_{\substack{i=1 \ j=1 \\ i<j}}^{n} \sum^{n} \left[\mu_i \lambda_j \int_0^\infty e^{su} F_i(x+u) du \int_0^\infty e^{sv} F_j(y+v) dv \right.$$

$$\left. + \mu_j \lambda_i \int_0^\infty e^{su} \overline{F_j}(u+x) du \int_0^\infty e^{sv} F_i(v+y) dv \right]$$

$$\leq \sum_{\substack{i=1 \ j=1 \\ i<j}}^{n} \sum^{n} \left[\mu_i \lambda_j \int_0^\infty e^{sv} F_i(v+y) dv \int_0^\infty e^{su} F_j(u+x) du \right.$$

$$\left. + \mu_j \lambda_i \int_0^\infty e^{sv} F_j(v+x) dv \int_0^\infty e^{su} F_i(u+y) du \right].$$

Now, for each fixed pair (i,j) with $i<j$ we have

$$\left[\mu_i \lambda_j \int_0^\infty e^{sv} F_i(v+y) dv \int_0^\infty e^{su} F_j(x+u) du \right.$$

$$\left. + \mu_j \lambda_i \int_0^\infty e^{sv} F_j(v+y) dv \int_0^\infty e^{su} F_i(u+y) du \right]$$

$$- \left[\mu_j \lambda_i \int_0^\infty e^{su} F_i(u+x) du \int_0^\infty e^{sv} F_j(v+y) dv \right.$$

$$\left. + \mu_j \lambda_i \int_0^\infty e^{su} F_j(u+x) du \int_0^\infty e^{sv} F_j(v+y) dv \right]$$

$$= (\mu_i \lambda_j - \mu_j \lambda_i) \left[\int_0^\infty e^{sv} F_i(v+y) dy \int_0^\infty e^{su} F_j(x+u) dx \right.$$

$$\left. - \mu_j \lambda_i \int_0^\infty e^{su} F_i(u+x) dx \int_0^\infty e^{sv} F_j(v+y) dy \right]$$

which is non- negative because both terms are non- negative by assumption. This completes the proof.

To demonstrate the usefulness of the above results in recognizing $(mg - pl)$ order random variables, we consider the following examples.

Example 3.1.

Let $X_{\underline{\alpha}}$ denote the convolution of n exponential distributions with parameters $\alpha_1, ..., \alpha_n$ respectively. Assume without loss of generality that $\alpha_1 \leq ... \leq \alpha_n$. Since exponential densities are log-concave, Theorem 3.3 implies that $X_{\underline{\alpha}} \leq_{mg-pl} Y_{\underline{\beta}}$ whenever $\alpha_i \geq \beta_i$ for $i = 1, ..., n$.

Example 3.2.

Let $X_i \sim Exp(\alpha_i)$, $i = 1, ..., n$ be independent random variables. Let X and Y be $\underline{\lambda}$ and $\underline{\mu}$ mixtures of X_i's. An application of Theorem 3.3, immediately $X \leq_{mg-pl} Y$ for every two probability vector $\underline{\alpha}$ and $\underline{\beta}$ such that $\underline{\alpha} \leq_{dlr} \underline{\beta}$.

Another application of Theorem 3.3 is contained in following example.

Example 3.3.

Let $X_{\underline{\alpha}}$ and $X_{\underline{\beta}}$ be as given in Example 3.1. For $0 \leq \theta \leq \phi \leq 1$ and $\theta + \phi = 1$, we have

$$\theta X_{\underline{\alpha}} + \phi X_{\underline{\beta}} \leq_{mg-pl} \phi X_{\underline{\alpha}} + \theta X_{\underline{\beta}}.$$

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